Root systems

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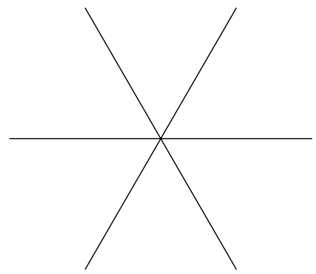
Contents

1 Definitions

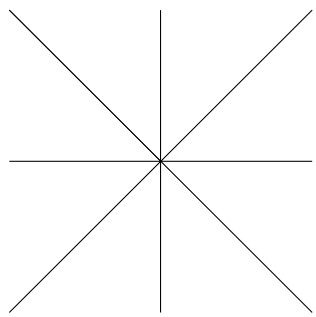
1.1 Intuitive definition

Definition 1.1. A **root system** is a "very symmetrical" set of vectors in $\mathbb{R}^n$.

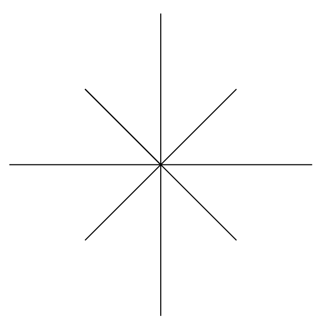
Example 1.2. Each of the following pictures takes place in $\mathbb{R}^2$, and the point where everything meets is the origin $(0, 0)$. Each line segment out from the origin represents a vector. Here is a picture of the A_2 root system.



Here is a picture of the B_2 root system.



Here is a picture of the C_2 root system.



In a minute I'll give a technical definition of "very symmetrical," but for now, just look at these and think about how you would describe the symmetries present. The main two things going on are:

1. For any vector v, $-v$ is also in the root system.
2. If you take any vector v , and look at the line perpendicular to v, and reflect the whole picture across that line, the reflection coincides with the original picture.

There is a third important feature that is not at all obvious from these pictures, which is that the dot product of any two vectors have nice algebraic properties.

1.2 Motivation for study of root systems

In this section I'm going to use a lot of words without defining them, but that's because they are just for motivation, not the focus of this seminar talk.

The main classical motivation for studying root systems is their role in the classification of semisimple Lie algebras over $\mathbb{C}$.

Theorem 1.3 (Cartan, Killing). *Every semisimple Lie algebra over $\mathbb{C}$ has an associated root system, and the root system determines the Lie algebra (up to isomorphism).*

From this, we can get a full classification of such Lie algebras, provided we can classify all the root systems. This was mostly worked out by Wilhelm Killing and Elie Cartan around 1880-1900.

More recently, in the 1940's and 1950's, Chevalley proved something similar for reductive algebraic groups.

Theorem 1.4 (Chevalley). *Every reductive algebraic group has associated root data, and the root data determines the group (up to isomorphism).*

I'm not going to go into what "root data" is, but basically it is a root system with some additional structure.

So the main takeaway is that classifying root systems is used in classifying some other important types of mathematical objects.

1.3 Precise technical definition

Everything that follows will take place in a fixed finite dimensional real vector space $E = \mathbb{R}^\ell$ with an inner product denoted by $(-,-)$. By $\mathrm{GL}(E)$ we mean invertible $\mathbb{R}$-linear endomorphisms of E, that need not preserve the inner product.

Definition 1.5. For $\alpha, \beta \in E$, we set $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. Note that $\langle -, - \rangle$ is linear in the first variable, but not the second.

Definition 1.6. For $\alpha \in E$, P_α denotes the hyperplane/subspace perpendicular to α.

$$P_\alpha = \{\beta \in E : (\alpha, \beta) = 0\}$$

Definition 1.7. For $\alpha \in E$, let $\sigma_\alpha \in \mathrm{GL}(E)$ denote the reflection through the P_α. One can check that a formula for σ_α is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta - \langle \beta, \alpha \rangle\, \alpha$$

Note that $\sigma_\alpha(\alpha) = -\alpha$, and that $\sigma_\alpha^2 = \mathrm{Id}$.

Definition 1.8. A **root system** in E is a finite set $\Phi \subset E$ such that

1. Φ spans E and does not contain zero.
2. For $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
3. *(Symmetry)* For $\alpha \in \Phi$, Φ is invariant under σ_α. (Hence σ_α permutes Φ.)
4. *(Integrality)* For $\alpha, \beta \in \Phi$, $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

An element $\alpha \in \Phi$ is a **root**.

Definition 1.9. Since Φ must span E, the dimension $\ell = \dim_{\mathbb{R}} E$ is an invariant of Φ, called the **rank** of Φ, and denoted $\mathrm{rk}(\Phi)$.

Definition 1.10. Let Φ be a root system in E. For $\alpha \in \Phi$, the hyperplanes P_α divide up E into connected components. The connected components of $E \setminus \bigcup_\alpha P_\alpha$ are the **Weyl chambers** of Φ.

Definition 1.11. Let Φ, Φ' be root systems in E, E' respectively. A **morphism** of root systems is a linear map $T : E \to E'$ such that $T(\Phi) \subset \Phi'$ and T preserves the angle bracket. That is, for $\alpha, \beta \in \Phi$,

$$\langle \alpha, \beta \rangle = \langle T\alpha, T\beta \rangle$$

Endomorphism, isomorphisms, and automorphisms of root systems are defined in the usual way from this notion of morphism. We write $\mathrm{Aut}(\Phi)$ for the group of automorphisms of Φ, so and we view $\mathrm{Aut}(\Phi)$ as a subgroup of $\mathrm{GL}(E)$. Note that for $\alpha \in \Phi$, $\sigma_\alpha \in \mathrm{Aut}(\Phi)$.

2 Examples

Before any general facts or attempt at classification, I'll describe some concrete examples of root systems in low dimensions. All of these examples will take place in $\mathbb{R}^\ell$ with standard basis $e_1, \ldots, e_\ell$ with the dot product determined by

$$(e_i, e_j) = \delta_{ij}$$

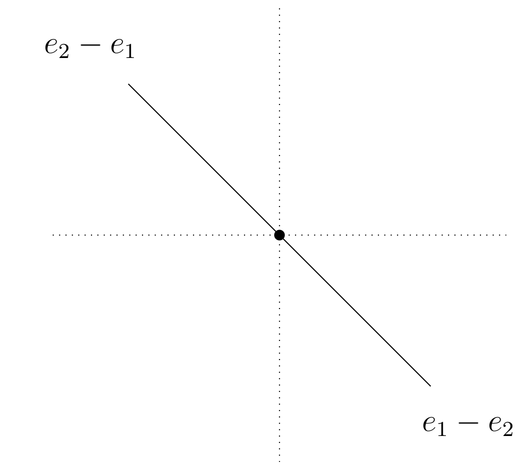
where δ_{ij} is the Kronecker delta.

2.1 The A_1 root system

Example 2.1 (A_1 root system)**.** Consider $\mathbb{R}^2$ with the usual inner product given by dot product, and standard basis e_1, e_2. Let

$$\Phi = \{e_1 - e_2, e_2 - e_1\}$$

We can draw this as below. The dotted lines represent the e_1, e_2 axes.



Let E be the span of $(1, -1)$. Then Φ is a root system in E. All the properties are obvious except integrality.

$$\langle e_1 - e_2, e_2 - e_1 \rangle = \frac{2(e_1 - e_2, e_2 - e_1)}{(e_2 - e_1, e_2 - e_1)} = \frac{2(-1-1)}{(1+1)} = -2$$

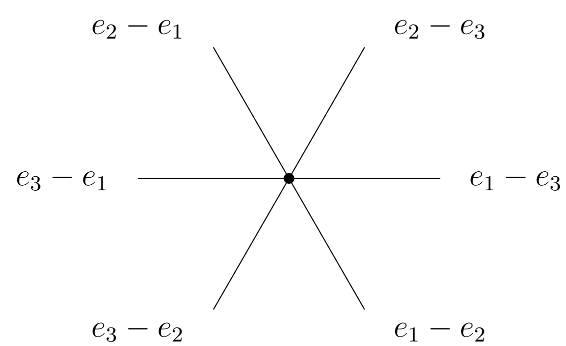
This is called the root system of type A_1. The 1 refers to the dimension of E. It is possible to draw this root system as a subset of $\mathbb{R}^1$, but I chose not to because this formulation generalizes a bit more cleanly.

2.2 The A_2 root system

Example 2.2 (A_2 root system)**.** Consider $\mathbb{R}^3$ with the usual inner product, given by dot product, and standard basis vectors e_1, e_2, e_3. Let

$$\Phi = \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\}$$

The span of Φ is the plane with normal vector $e_1 + e_2 + e_3$. Let E be this subspace. We claim Φ is a root system in E. By what I just said, Φ spans E. The other properties are most easily verified by drawing a picture (in the plane E).



From the picture, it is clear that the only multiplies of any root that are in Φ are $\pm\alpha$. And we can see that Φ is closed under hyperplane reflections, meaning the hyperplanes perpendicular to each root. To verify the last condition regarding integrality, we just need to do some case checking. Let's just do one.

$$\langle e_1 - e_2, e_2 - e_3 \rangle = \frac{2(e_1 - e_2, e_2 - e_3)}{(e_1 - e_2, e_1 - e_2)} = \frac{2(-1)}{(2)} = -1$$

This is called the A_2 root system. The 2 refers to the dimension of the span of Φ.

2.3 The A_ℓ root system

Example 2.3 (A_ℓ root system)**.** Let $e_1, \ldots, e_\ell, e_{\ell+1}$ be the standard basis of $\mathbb{R}^{\ell+1}$. Let

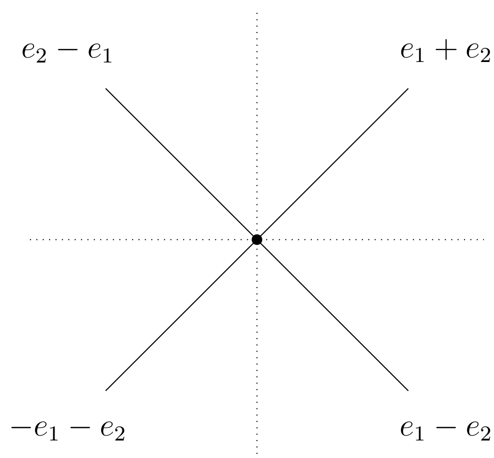
$$\Phi = \{\pm(e_i - e_j) : 1 \leq i < j \leq \ell + 1\}$$

Let $E \subset \mathbb{R}^{\ell+1}$ be the span of Φ, with the usual Euclidean inner product. Then Φ is a root system in E. Properties 1,2,4 are all clear (for #4, note that $(e_i, e_j) = \delta_{ij} \in \mathbb{Z}$). Property 3 requires a lot of tedious case-checking. This is the root system of type A_ℓ.

2.4 $A_1 \times A_1$

Example 2.4 ($A_1 \times A_1$ root system)**.** Consider $\mathbb{R}^2$ with the usual inner product (dot product), with standard basis e_1, e_2. We have two copies of the A_1 root system, one given by $\{e_1 - e_2, e_2 - e_1\}$ and the other given by $\{e_1 + e_2, -e_1 - e_2\}$. Let

$$\Phi = \{e_1 - e_2, e_2 - e_1, e_1 + e_1, -e_1 - e_1\}$$



Furthermore, the two copies of A_1 here do not interact, in the sense that the dot product (or $\langle , \rangle$ product) is zero between any vectors coming from different copies of A_1.

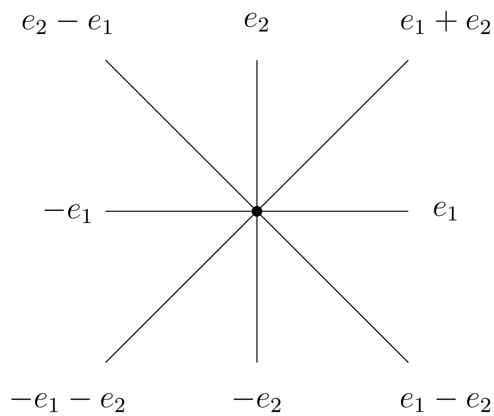
$$\langle e_1 + e_2, -e_1 - e_2 \rangle = \frac{2(e_1 + e_2, -e_1 - e_2)}{(-e_1 - e_2, -e_1 - e_2)} = \frac{2(-1-1)}{(1+1)} = -2$$

$$\langle e_1 + e_2, e_1 - e_2 \rangle = \frac{2(e_1 + e_2, e_1 - e_2)}{(e_1 - e_2, e_1 - e_2)} = 0$$

2.5 B_2

Example 2.5 (B_2 root system)**.** Consider $\mathbb{R}^2$ with inner product given by dot product with basis e_1, e_2. Let

$$\Phi = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$$

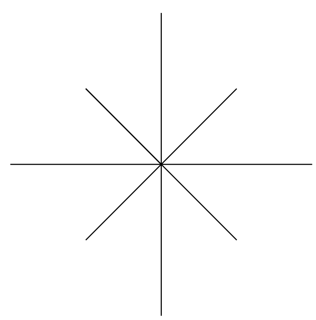


This is the root system of type B_2.

2.6 C_2

Example 2.6 (C_2 root system)**.** In $\mathbb{R}^2$ as before, consider the root system

$$\Phi = \{\pm 2e_1, \pm 2e_1, \pm e_1 \pm e_2\}$$



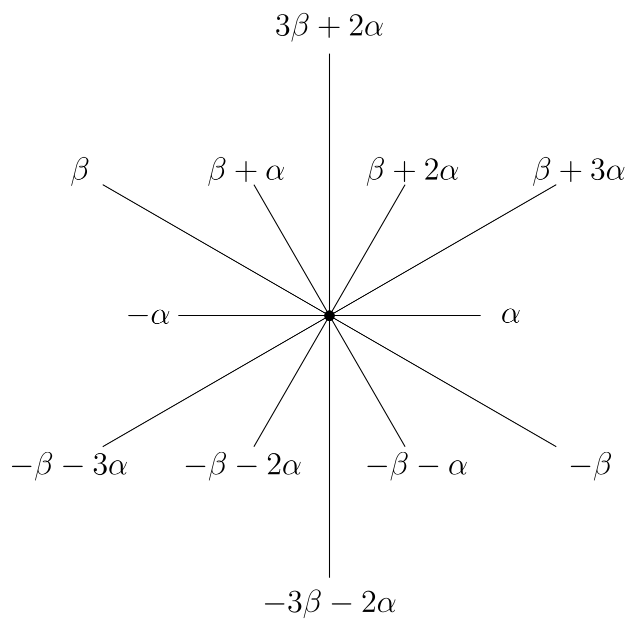
This is the root system of type C_2.

2.7 G_2

Example 2.7 (G_2 root system)**.** Consider $\mathbb{R}^3$ with basis e_1, e_2, e_3 and usual dot product. Let

$$\begin{aligned}\Phi = \{ &\pm (e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \\ &\pm (2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}\end{aligned}$$

The first six vectors is an exact copy of A_2 from earlier, which lives in the hyperplane perpendicular to $e_1 + e_2 + e_3$. Notice that the other six vectors also lie in this same plane, so we take E to be that plane. Instead of labelling all the roots in terms of e_i, instead we'll label them in terms of $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$.



One way to think about this picture is that it is two copies of A_2, with slightly differently scaled lengths. In the original copy involving α, the vectors have squared length 2. In the larger copy of A_2, the vectors have squared length 6. All of the angles between adjacent vectors are $\pi/6$.

Remark 2.8. Although it is far from obvious at this point, I have shown you all of the (irreducible) root systems of rank 2.

Remark 2.9. The B_2 and C_2 root systems generalize to infinite familes B_ℓ, C_ℓ of root systems in any dimension. Not so obviously, the G_2 root system does NOT generalize in this way, it is one of the exceptional root systems, along with F_4 and E_5, E_6, E_7, E_8.

3 Classification of root systems

Now that we've seen some examples, I'll outline the classification of root systems.

3.1 Restrictions on possible angles

The first and very immediate consequence of the definition of root system is that the angles between vectors appearing in a root system are very restricted, as a consequence of the integrality property.

Definition 3.1. For $\alpha, \beta \in E$, let $\theta_{\alpha\beta}$ denote the angle between α and β. Then

$$\cos\theta_{\alpha\beta} = \frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$$

Also note that

$$(\alpha,\beta)>0 \iff 0<\theta_{\alpha\beta}<\frac{\pi}{2} \iff \theta_{\alpha\beta} \text{ is acute} \iff \langle\alpha,\beta\rangle>0$$
$$(\alpha,\beta)<0 \iff \frac{\pi}{2}<\theta_{\alpha\beta}<\pi \iff \theta_{\alpha\beta} \text{ is obtuse} \iff \langle\alpha,\beta\rangle<0$$

Proposition 3.2. *Let Φ be a root system. For $\alpha,\beta\in\Phi$, with $\beta\neq\pm\alpha$,*

$$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle\in\{0,1,2,3\}$$

Proof. We know $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle\in\mathbb{Z}$. The bounds basically follow from the angle formula

$$\cos\theta_{\alpha\beta}=\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$$

Rearranging this,

$$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle=4\cos^2\theta_{\alpha\beta}\leq 4$$

If it is equal to 4, then $\theta=\pi$, which would make $\beta=\pm\alpha$. □

Remark 3.3. Using the previous result, one can make a table of all the possible values of $\langle\alpha,\beta\rangle$, and a list of all possible angles (α,β) between roots. We can also list all the possible ratios of squared lengths, except for the case where α,β make a right angle. Without loss of generality, assume $(\alpha,\alpha)\leq(\beta,\beta)$.

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	$\theta_{\alpha\beta}$	$\frac{(\beta,\beta)}{(\alpha,\alpha)}=\frac{\lvert\beta\rvert^2}{\lvert\alpha\rvert^2}=\left(\frac{\lvert\beta\rvert}{\lvert\alpha\rvert}\right)^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

So from this table, there are only six different possible acute angles between roots, and there are only three possible square length ratios between roots of different lengths which don't make a right angle. We discuss this briefly for each example we have considered.

In the A_ℓ root system, all roots have the same length, and all the angles are integer multiples of $\frac{\pi}{3}$.

In the B_2 and C_2 root systems, there are two root lengths with squared ratio 2, and all the angles are integer multiples of $\frac{\pi}{4}$.

In the G_2 root system, there are two root lengths with squared ratio 3, and all the angles are integer multiples of $\frac{\pi}{6}$.

After seeing this example of how restrictive the integrality condition is on the geometry of a root system, hopefully you aren't that surprised that these things can be totally classified.

3.2 Irreducibility

Definition 3.4. A root system Φ is **reducible** if it can be written as a disjoint union of nonempty sets $\Phi = \Phi_1 \sqcup \Phi_2$ which are orthogonal, meaning $(\alpha, \beta) = 0$ for $\alpha \in \Phi_1, \beta \in \Phi_2$. It is **irreducible** if it is not reducible.

Proposition 3.5. *A root system Φ in E can be written as a disjoint union of irreducible root systems*

$$\Phi = \bigsqcup_i \Phi_i$$

such that

$$E = \bigoplus_i E_i = \bigoplus_i \operatorname{span}(\Phi_i)$$

is a direct sum decomposition into pairwise orthogonal subspaces $E_i = \operatorname{span}(\Phi_i)$.

Remark 3.6. Due to this, in order to classify all root systems, it suffices to classify irreducible root systems.

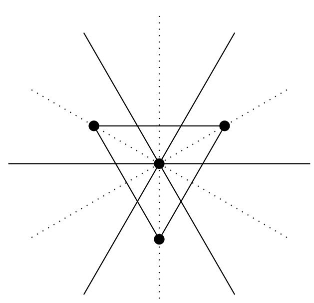
3.3 Weyl group

Definition 3.7. Let Φ be a root system in E. Recall that for $\alpha \in E$, $\sigma_\alpha \in \mathrm{GL}(E)$ is the reflection through the hyperplane perpendicular to α. The **Weyl group** of Φ is the subgroup of $\mathrm{GL}(E)$ generated by σ_α for $\alpha \in \Phi$.

$$W(\Phi) = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$$

Since Φ is a root system, each σ_α preserves Φ, so $W(\Phi)$ may also be viewed as a subgroup of the permutation group of Φ. In particular, $W(\Phi)$ is finite.

Example 3.8 (Weyl group of A_2)**.** What is the Weyl group of the A_2 root system? It is the subgroup of the symmetry group of A_2 generated by reflections. This is the symmetry group of an equilateral triagle, which is S_3.



Example 3.9 (Weyl group of A_ℓ)**.** The Weyl group of A_ℓ is $S_{\ell+1}$.

Example 3.10 (Weyl group of G_2)**.** The Weyl group of G_2 is the dihedral group of order 12.

3.4 Bases for root systems

Definition 3.11. A **base** for a root system Φ is a subset $\Delta \subset \Phi$ such that Δ is a basis for E, and so that for every $\beta \in \Phi$ when written uniquely in terms of this basis, all the coefficients are integers, and have the same sign. That is, for $\beta \in \Phi$,

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \qquad k_\alpha \in \mathbb{Z}$$

with $k_\alpha \in \mathbb{Z}$ all having the same sign. That is, all the k_α are ≥ 0 or ≤ 0. After fixing a base Δ for Φ, set

$$\begin{aligned} \textbf{simple roots} &= \Delta \\ \textbf{positive roots} &= \Phi^+ = \left\{ \sum k_\alpha \alpha : k_\alpha \geq 0 \right\} \\ \textbf{negative roots} &= \Phi^- = \left\{ \sum k_\alpha \alpha : k_\alpha \leq 0 \right\} \end{aligned}$$

Note that $\Delta \subset \Phi^+$. Also note that as Δ is a basis for E, $|\Delta| = \mathrm{rk}(\Phi)$.

Example 3.12 (Bases for A_2)**.** A base for A_2 is given by $e_1 - e_2$ and $e_3 - e_1$.

Example 3.13 (Base for A_ℓ)**.** Consider again the A_ℓ root system $\Phi = \{\pm(e_i - e_j) : 1 \leq i < j \leq \ell + 1\}$. One base for this root system is $\{e_1 - e_2, \ldots, e_\ell - e_{\ell+1}\}$.

Example 3.14 (Base for G_2)**.** When we described G_2, we described it in such a way that $\alpha = e_1 - e_2$ and $\beta = 2e_2 - e_1 - e_3$ together form a base $\Delta = \{\alpha, \beta\}$ for G_2. Looking back to our picture, all of the roots are integral combinations of α and β with coefficients of the same sign. We can also look at the picture we drew of G_2 and see that the positive and negative roots are divided by a single hyperplane.

Theorem 3.15. *Every root system has a base (actually several).*

Theorem 3.16. *The Weyl group $W(\Phi)$ acts simply transitively on the set of bases for Φ.*

3.5 Cartan matrices

Definition 3.17. Let Φ be a root system, with base Δ. Fix an ordering of $\Delta = \langle \alpha_1, \ldots, \alpha_\ell \rangle$. The **Cartan matrix** associated to this is the matrix

$$C = (c_{ij}) = (\langle \alpha_i, \alpha_j \rangle) \in \mathrm{Mat}(\ell, \mathbb{Z})$$

Changing the ordering of Δ just acts by permuting rows and columns of C. Since W acts simply transitively on the set of bases, changing the basis just means applying an element of W to each α_i, which does not affect the bracket product, so C does not depend on the choice of basis.

Remark 3.18. The diagonal entries of the Cartan matrix are always 2, since $\langle \alpha, \alpha \rangle = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)} = 2$.

Example 3.19 (Cartan matrix for A_2)**.** Let's compute the Cartan matrix for A_2. We use the base $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_3 - e_1$. Then we compute a few brackets.

$$\begin{aligned}
\langle \alpha_1, \alpha_2 \rangle &= \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2(e_1 - e_2, e_3 - e_1)}{(e_3 - e_1, e_3 - e_1)} = \frac{2(-1)}{2} = -1 \\
\langle \alpha_2, \alpha_1 \rangle &= -1
\end{aligned}$$

So the Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Example 3.20 (Cartan matrix for A_ℓ)**.** Using the base previous mentioned for A_ℓ, the Cartan matrix associated to the root system of type A_ℓ is

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}$$

Example 3.21 (Cartan matrix for G_2)**.** We use the basis $\alpha = e_1 - e_2, \beta = 2e_2 - e_1 - e_3$ for G_2. The relevant bracket products are

$$\begin{aligned}
\langle \alpha, \beta \rangle &= \frac{2(e_1 - e_2, 2e_2 - e_1 - e_3)}{(2e_2 - e_1 - e_3, 2e_2 - e_1 - e_3)} = \frac{2(-1-2)}{1+4+1} = \frac{-6}{6} = -1 \\
\langle \beta, \alpha \rangle &= \frac{2(2e_2 - e_1 - e_3, e_1 - e_2)}{(e_1 - e_2, e_1 - e_2)} = \frac{2(-1-2)}{2} = -3
\end{aligned}$$

So the Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

3.6 Dynkin diagrams/Coxeter graphs

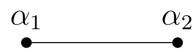
Definition 3.22. Given a root system Φ with base Δ, the associated **Dynkin diagram** is a graph, which can have multi-edges and directed edges. The vertices are elements of Δ, and between two vertices α, β, there are

$$\text{edge}(\alpha, \beta) = \max\Big(|\langle \alpha, \beta \rangle|, |\langle \beta, \alpha \rangle| \Big)$$

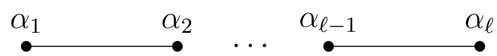
edges. If one of α, β is longer (if $(\alpha, \alpha) \neq (\beta, \beta)$ and $\langle \alpha, \beta \rangle > 1$, then we direct the multiple edges pointing toward the longer root. The Dynkin diagram depends only on the Cartan matrix, and it is clear that the Dynkin diagram does not depend on the order of the base, so it depends only on the root system Φ.

The **Coxeter graph** of Φ is just the underlying undirected multigraph of the Dynkin diagram.

Example 3.23 (Dynkin diagram for A_2)**.** There are two roots in the base, so the Dynkin diagram for A_2 has two vertices α_1, α_2. Between them there is one edge. They have the same length, so the edge is not directed.



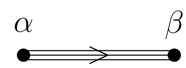
Example 3.24 (Dynkin diagram for A_ℓ)**.** We use the same base as before for A_ℓ, where the Cartan matrix is as before. There are ℓ roots $\alpha_1, \ldots, \alpha_\ell$. From the Cartan matrix, we read off that the Dynkin diagram is



Example 3.25 (Dynkin diagram for G_2)**.** For G_2, we use the base α, β as before, with Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Since $|\langle \beta, \alpha \rangle| = 3$, there are three edges between the two vertices. We add an arrow pointing at the longer root, β.



Proposition 3.26. *A root system Φ is irreducible iff the Dynkin diagram is connected.*

Proof. If $\Phi = \Phi_1 \sqcup \Phi_2$ is reducible with $(\Phi_1, \Phi_2) = 0$, then the Cartan matrix has blocks, so the Dynkin diagram has multiple components. Conversely, if the Dynkin diagram has two components, then there are disjoint orthogonal subsets of Φ, making it reducible. $\square$

Proposition 3.27. *Let Φ, Φ' be root systems in E, E' respecitvely. Then $\Phi \cong \Phi'$ if and only if they have the same Dynkin diagram.*

Proof. If $\Phi \cong \Phi'$, then they clearly have the same Cartan matrix and same Dynkin diagram. The other direction is the challenge.

If the Dynkin diagrams are the same, the Cartan matrices are the same up to permutating rows and columns, so we can choose bases $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ and $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$ of Φ' with $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for all i, j. Define $T : E \to E'$ by $T(\alpha_i) = \alpha'_i$. As Δ, Δ' are bases, T is an isomorphism of vector spaces, and it preserves the angle bracket. We have $T(\Delta) = \Delta'$, we just need to check $T(\Phi) = \Phi'$.

Let $\beta \in \Phi$ and $\alpha_i \in \Delta$. Then

$$\begin{aligned} T\sigma_{\alpha_i}\beta &= T(\beta - \langle \beta, \alpha_i \rangle \alpha_i) = T\beta - \langle \beta, \alpha_i \rangle \alpha'_i = T\beta - \langle T\beta, T\alpha_i \rangle \alpha'_i \\ &= T\beta - \langle T\beta, \alpha'_i \rangle \alpha'_i = \sigma_{\alpha'_i} T\beta \end{aligned}$$

For $\sigma \in W(\Phi)$, write it as a product of simple reflections σ_{α_i}, and let $\sigma' \in W(\Phi')$ be the analogous product of $\sigma_{\alpha'_i}$. Since the simple reflections σ_{α_i} generate the Weyl group $W(\Phi)$, the previous equation shows for any $\sigma \in W(\Phi)$ that

$$T\sigma\beta = \sigma' T\beta$$

Since $W(\Phi)$ acts transitively on Φ, choose σ so that $\sigma\beta \in \Delta$. Then $T\sigma\beta = \sigma' T\beta \in T(\Delta) = \Delta'$. So $T\beta$ is in the $W(\Phi')$-orbit of Δ', which is exactly Φ'. Thus $T(\Phi) \subset \Phi'$. Applying the same argument to T^{-1} gives the reverse inclusion, so $T(\Phi) = \Phi'$. Hence T is an isomorphism of root systems. $\square$

3.7 Classification of Coxeter graphs

Theorem 3.28. *The only possible Dynklin diagrams arising from root systems are the following. Conversely, for each of these, there exists a root system with the given Dynkin diagram.*