The norm torus

Understanding splitting of tori through examples

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These are notes I wrote for myself in order to try and understand the phenomena of split and non-split algebraic tori. Much of it was prepared and used for two student seminar talks at Michigan State University in November of 2019.

All external references are to Milne's notes on algebraic groups, found at `https://www.jmilne.org/math/CourseNotes/iAG200.pdf`.

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1 Affine algebraic groups

Throughout everything, we fix a field K which is perfect, so that the separable and algebraic closures are the same.

Definition 1.1. Let K be a field. The **category of K-algebras**, denoted Alg_K, is the category whose objects are unital, associative, commutative algebras over K, and whose morphisms are K-linear, multiplicative maps.

It is possible that for some things, we should assume our algebras are finitely generated. However, I don't want to do this in general, because something like K^{sep} may not be finitely generated as a K-algebra, but I still want to allow it as a K-algebra.

1.1 The algebraic group $\mathbb{G}_m$

Definition 1.2 ($\mathbb{G}_m$)**.** Let K be a field. We will define a (covariant) functor

$$\mathbb{G}_m : \mathrm{Alg}_K \to \mathrm{Gp}$$

Sometimes when dealing with multiple fields, we will denote this functor by $\mathbb{G}_m^K$ to clarify which field is being used. On objects, it is defined by

$$\mathbb{G}_m(A) = A^\times \cong \mathrm{Hom}_K(K[x, x^{-1}], A)$$

The isomorphism $A^\times \cong \mathrm{Hom}_K(K[x, x^{-1}], A)$ is given by

$$\mathrm{Hom}_K(K[x, x^{-1}], A) \to A^\times \qquad \phi \mapsto \phi(x)$$

Now we define $\mathbb{G}_m$ on morphisms. Given a morphism of K-algebras $f : A \to B$, there is an induced morphism of groups

$$\mathbb{G}_m(f) = f|_{A^\times} : A^\times \to B^\times \qquad a \mapsto f(a)$$

In terms of the isomorphism with $\mathrm{Hom}_K(K[x, x^{-1}], A)$, this is post-composition with f.

$$\mathrm{Hom}_K(K[x, x^{-1}], A) \to \mathrm{Hom}_K(K[x, x^{-1}], B) \qquad \phi \mapsto f \circ \phi$$

Remark 1.3. The definition above makes $\mathbb{G}_m$ into a representable covariant functor $\mathrm{Alg}_K \to \mathrm{Gp}$. The representing object is $K[x, x^{-1}]$. To be a bit more precise, $\mathbb{G}_m$ is representable because the isomorphisms $\mathrm{Hom}_K(K[x, x^{-1}], A) \cong A$ are natural, meaning that for any morphism $f : A \to B$ of K-algebras, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}_K(K[x, x^{-1}], A) & \xrightarrow{\cong} & A^\times \\ \big\downarrow{\scriptstyle \phi \mapsto \phi(x)} & & \big\downarrow{\scriptstyle f|_{A^\times}} \\ \mathrm{Hom}_K(K[x, x^{-1}], B) & \xrightarrow{\cong} & B^\times \end{array}$$

Definition 1.4. A representable covariant functor $G : \mathrm{Alg}_K \to \mathrm{Gp}$ is called an **affine algebraic K-group**. For a K-algebra A, the group $G(A)$ is called the **group of A-points** of G. The K-algebra representing G is denoted $\mathcal{O}(G)$ or $\mathcal{O}_G$.

Example 1.5. Another example of an affine algebraic K-group is GL_n. It is a functor

$$\mathrm{GL}_n : \mathrm{Alg}_K \to \mathrm{Gp} \qquad A \mapsto \mathrm{GL}_n(A)$$

$\mathbb{G}_m$ is just the special case of GL_n when $n = 1$.

Definition 1.6. The **trivial algebraic K-group** is the algebraic group G whose group of A-points is the trivial group for every A.

$$G(A) = \{1\}$$

It is represented by K itself viewed as a K-algebra (that is, $\mathcal{O}_G = K$), since for any K-algebra A, there is a unique K-algebra morphism $K \to A$.

$$G(A) = \{1\} \cong \mathrm{Hom}_K(K, A)$$

1.2 Morphisms

Definition 1.7. Let G, H be affine algebraic K-groups. A **morphism** of affine algebraic K-groups is a natural transformation $\eta : G \to H$. That is, for every K-algebra A, there is a group homomorphism

$$\eta_A : G(A) \to H(A)$$

which is compatible with K-algebra homomorphisms, in the sense that a K-algebra homomorphism $f : A \to B$ induces a commutative diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{\eta_A} & H(A) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ G(B) & \xrightarrow{\eta_B} & H(B) \end{array}$$

Example 1.8. The determinant is a morphism of affine algebraic K-groups,

$$\det : \mathrm{GL}_n \to \mathbb{G}_m$$

That is, for a K-algebra A, we have

$$\det_A : \mathrm{GL}_n(A) \to A^\times$$

which is compatible with K-algebra homomorphisms $f : A \to B$ in the manner above. In the case $n = 1$, this is just the identity map.

Definition 1.9. Fix a field K. There is a category whose objects are affine algebraic K-groups with morphisms as defined above. We denote this category by AlgGp_K.

Remark 1.10. Giving a morphism of affine algebraic K-groups $\eta : G \to H$ is equivalent to giving a morphism $\mathcal{O}_H \to \mathcal{O}_G$ of the representing algebras $\mathcal{O}_G, \mathcal{O}_H$, by Yoneda's lemma.

$$\mathrm{Hom}_{\mathrm{AlgGp}_K}(G, H) \cong \mathrm{Hom}_K\left(\mathcal{O}_H, \mathcal{O}_G\right)$$

Definition 1.11. Let G, H be affine algebraic K-groups. The **trivial morphism** $G \to H$ is the natural transformation $\phi : G \to H$ where $\phi_A : G(A) \to H(A)$ is the trivial group homomorphism for every K-algebra A. (It is immediate to check that the required naturality diagram commutes.) We denote the trivial homomorphism by 0.

1.3 Kernels

Definition 1.12. Let $\eta : G \to H$ be a morphism of algebraic K-groups, so for every K-algebra A we have a group homomorphism $\eta_A : G(A) \to H(A)$. For a K-algebra A, define

$$(\ker \eta)(A) = \ker(\eta_A)$$

This makes $\ker \eta$ into an algebraic K-group, for which we omit justification.

Remark 1.13. The "right" way to define the kernel is by the usual categorical universal property, and then to show that such objects exist in the category of algebraic groups, and then to show that that object is the one I just described.

Remark 1.14. According to Milne's notes, the representing object of the kernel of a homomorphism $G \to H$ is represented by $\mathcal{O}_G / I_H \mathcal{O}_G$, where I_H is the augmentation ideal of H, the kernel of the counit map $\mathcal{O}_H \to K$.

1.4 Extension of scalars

Definition 1.15. Let L/K be a field extension and A a K-algebra. Since we will so often have to write $A \otimes_K L$, we abbreviate this to A_L. We call A_L the algebra formed by **extending scalars** to L, or say that A_L is formed by **extension of scalars**.

Definition 1.16. Let L/K be a field extension, and let A be a K-algebra. Then A_L is a left A-module, via

$$A \times A_L \to A_L \qquad a \cdot (b \otimes \lambda) = (ab) \otimes \lambda$$

where $a, b \in A, \lambda \in L$. This only defines the action on simple tensors, so we extend by linearity.

Lemma 1.17. *Let L/K be a field extension, and let A be a K-algebra. Let $\mathcal{B}$ be a K-basis of L. Then*

$$\mathcal{B}_L = \{1 \otimes \lambda : \lambda \in \mathcal{B}\}$$

is an A-basis of A_L. In particular, A_L is a free A-module, and if L/K is finite of degree $d = [L : K]$, then A_L is a free A-module of rank d.

Proof. Let $\mathcal{V} = \{v_i : i \in I\}$ be a K-basis of A, and let $\mathcal{B} = \{\lambda_j : j \in J\}$ be a K-basis of L. Then we know from general theory that $\{v_i \otimes \lambda_j : i \in I, j \in J\}$ is a K-basis for A_L. That is, any $a \in A$ can be uniquely written as a sum of the type below with finitely many nonzero terms.

$$a = \sum_{i,j} a_{ij} (v_i \otimes \lambda_j)$$

with $a_{ij} \in K$. We want to show that $\mathcal{B}_L$ is an A-basis of A_L. Using the above, we can write $a \in A$ as

$$a = \sum_{i,j} a_{ij} (v_i \otimes \lambda_j) = \sum_{i,j} (a_{ij} v_i) \cdot (1 \otimes \lambda_j) = \sum_j \left(\left(\sum_i a_{ij} v_i \right) \cdot 1 \otimes \lambda_j \right)$$

where $a_{ij} \in K$. So $\mathcal{B}_L$ is an A-spanning set for A_L. Now we just need to verify linear independence. Suppose that for some $a_1, \ldots, a_d \in A$, we have

$$\sum_j a_j \cdot (1 \otimes \lambda_j) = 0$$

Using basis $\mathcal{V}$ for A, write a_j as

$$a_j = \sum_i a_{ij} v_i$$

for some $a_{ij} \in K$. Then

$$0 = \sum_j a_j \cdot (1 \otimes \lambda_j) = \sum_j \left(\sum_i a_{ij} v_i \right) \cdot (1 \otimes \lambda_j) = \sum_{i,j} a_{ij} (v_i \otimes \lambda_j)$$

By linear independence of the K-basis $\{v_i \otimes \lambda_j\}$, this implies that $a_{ij} = 0$ for all i, j, hence $a_j = 0$ for all i, j. Hence the A-spanning set $\mathcal{B}_L$ is linearly independent. □

Remark 1.18. Let L, K, A be as above. Consider the units of A_L, that is, $\mathbb{G}_m(A_L) = A_L^\times$. Note that there is an obvious map

$$A^\times \times L^\times \to (A \otimes_K L)^\times \qquad (a, \lambda) \mapsto a \otimes \lambda$$

which is a even a group homomorphism. This is one way to obtain units in A_L, but this generally not surjective.

Definition 1.19. Let L/K be a field extension, and let H be an affine algebraic K-group. The **extension of scalars** of H is the affine algebraic K-group H_L defined by

$$H_L : \mathrm{Alg}_L \to \mathrm{Gp} \qquad A \mapsto H(A)$$

where A is an L-algebra. Since A is an L-algebra, it is also a K-algebra, so we can plug it into H. This defines a functor

$$(-)_L : \mathrm{AlgGp}_K \to \mathrm{AlgGp}_L \qquad H \mapsto H_L$$

Example 1.20. Let L/K be an extension and $\mathbb{G}_m^K$ be the algebraic K-group defined previously. Then the extension of scalars to L, $(\mathbb{G}_m^K)_L$, is just $\mathbb{G}_m^L$.

$$(\mathbb{G}_m^K)_L(A) = \mathbb{G}_m^K(A) = A^\times = \mathbb{G}_m^L(A)$$

Proposition 1.21. *Let H be an affine algebraic K-group with representing object $\mathcal{O}_H$, and L/K a field extension. Then H_L is an affine algebraic L-group, with respresenting object $\mathcal{O}_H \otimes_K L$.*

Proof. We have a natural isomorphism

$$H(-) \cong \mathrm{Hom}_{\mathrm{Alg}_K}(\mathcal{O}_H, -)$$

which we just think of as an equality $H(A) = \mathrm{Hom}_{\mathrm{Alg}_K}(\mathcal{O}_H, A)$ for an arbitrary K-algebra A. We need to describe a natural isomorphism

$$\eta : H_L(-) \cong \mathrm{Hom}_{\mathrm{Alg}_L}(\mathcal{O}_H \otimes_K L, -)$$

Let B be an L-algebra, and recall $H_L(B) = H(B)$. Given a K-algebra homomorphism $\phi \in H_L(B)$, $\phi : \mathcal{O}_H \to B$, we get an L-algebra homomorphism

$$\eta_B(\phi) : \mathcal{O}_H \otimes_K L \to B \qquad x \otimes \lambda \mapsto \lambda\phi(x)$$

Define

$$\eta_B : H_L(B) \to \mathrm{Hom}_{\mathrm{Alg}_L}(\mathcal{O}_H \otimes_K L, B) \qquad \phi \mapsto \eta_B(\phi)$$

We claim that this is a natural isomorphism, but omit the details of the inverse map, and the proof of naturality. □

1.5 Tori

Definition 1.22. Let T be an affine algebraic K-group. T is a **K-torus (of finite rank)** if $T_{K^{\mathrm{sep}}} \cong (\mathbb{G}_m)^r$ for some $r \in \mathbb{Z}_{\geq 0}$. The number r is called the **rank** or **absolute rank** of the torus T. In particular, note that

$$T(K^{\mathrm{sep}}) \cong (K^{\mathrm{sep}\,\times})^r$$

However, it is not sufficient to have just this isomorphism on K^{sep}-points, the isomorphism $T_{K^{\mathrm{sep}}} \cong \mathbb{G}_m^r$ is an isomorphism of algebraic K^{sep}-groups, which means there are natural isomorphisms $T(A) \cong \mathbb{G}_m(A)^r$ for every K^{sep}-algebra A.

Example 1.23. $\mathbb{G}_m^K$ is a K-torus of rank 1. More generally, $(\mathbb{G}_m)^r$ is a K-torus of rank r.

Example 1.24. Let K be a field, and view GL_n as an algebraic K-group. Inside of GL_n is the diagonal subgroup.

$$\mathrm{D}_n : \mathrm{Alg}_K \to \mathrm{Gp} \qquad A \mapsto \mathrm{D}_n(A) = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : a_i \in A^\times \right\}$$

There is an isomorphism $\mathrm{D}_n \cong (\mathbb{G}_m)^n$, so D_n is a K-torus of rank n. On A-points, the isomorphism is

$$(\mathbb{G}_m)^n(A) \xrightarrow{\cong} \mathrm{D}_n(A) \qquad (a_1, \ldots, a_n) \mapsto \mathrm{diag}(a_1, \ldots, a_n)$$

In particular, it is a torus inside of GL_n.

Remark 1.25. In the study and classification of algebraic groups, subgroups of a given algebraic group which are tori play a very important role in understanding the structure of that group.

Definition 1.26. Let T be a K-torus of rank r. Let E/K be a field extension. T is **split over** E if $T_E \cong (\mathbb{G}_m^E)^r$. We say T is **split** if it is split over K. If T is not split (over K), it is called **non-split**.

Remark 1.27. It is immediate from the definitions that every torus is split over K^{sep}.

Theorem 1.28. *Let T be a K-torus of rank r. There is a unique minimal finite field extension E/K such that T is split over E.*

Proof. Omitted. □

Definition 1.29. The unique minimal finite extension E/K such that a given torus T splits over E is called the **splitting field** of T.

Example 1.30. It is not that easy to give examples of non-split tori. By the end of these notes, the goal is to see that the kernel of the norm map $N : R_{K^{\text{sep}}/K}\mathbb{G}_m^{K^{\text{sep}}} \to \mathbb{G}_m^K$ is a non-split torus, assuming $K \neq K^{\text{sep}}$. We will encounter some easier examples before then.

1.6 Weil restriction

Definition 1.31. Let L/K be a finite field extension, and let G be an affine algebraic L-group, with representing L-algebra $\mathcal{O}_G$. That is, G is (naturally isomorphic to) the covariant functor

$$G = \text{Hom}_L(\mathcal{O}_G, -) : \text{Alg}_L \to \text{Gp} \qquad B \mapsto \text{Hom}_L(\mathcal{O}_G, B)$$

where B is an L-algebra. The **Weil restriction** of G, denoted $R_{L/K}G$, is the affine algebriac K-group defined by

$$R_{L/K}G : \text{Alg}_K \to \text{Gp} \qquad A \mapsto G(A_L) = \text{Hom}_L(\mathcal{O}_G, A_L)$$

where A is a K-algebra. That is, $R_{L/K}G$ is basically just G, except that in order to evaluate it on a K-algebra, we first have to tensor it up to L first, to "extend scalars."

This describes how to compute $R_{L/K}G$ on objects (on K-algebras A) but not quite as clear how to evaluate it on morphisms of K-algebras. Let $f : A \to B$ be a morphism of K-algebras. Then the map induced by $R_{L/K}G$ is

$$R_{L/K}G(f) : \text{Hom}_L(\mathcal{O}_G, A_L) \to \text{Hom}_L(\mathcal{O}_G, B_L) \qquad \phi \mapsto (f \otimes 1) \circ \phi$$

where 1 means the identity map on L, $1 = \text{Id}_L$. This makes Weil restriction into a functor

$$R_{L/K} : \text{AlgGp}_L \to \text{AlgGp}_K \qquad G \mapsto R_{L/K}G$$

Lemma 1.32. *Let L/K be a finite field extension and G be an affine algebraic L-group with representing L-algebra $\mathcal{O}_G$. Then $R_{L/K}G$ is an affine algebraic K-group.*

Proof. See appendix to Milne's notes. It is phrased in somewhat different language, and unfortunately does not succinctly describe the representing object of $R_{L/K}$. □

Remark 1.33. The most important feature of Weil restriction is that the group of K-points of $R_{L/K}G$ is the group of L-points of G.

$$R_{L/K}G(K) \cong G(K \otimes_K L) \cong G(L)$$

Proposition 1.34. *Let L/K be a field extension. The functors $(-)_L$ and $R_{L/K}$ are an adjoint pair. That is, given an algebraic L-group G and an algebraic K-group H, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathrm{AlgGp}_K}(H, R_{L/K}G) \cong \mathrm{Hom}_{\mathrm{AlgGp}_L}(H_L, G)$$

Proof. Omitted. □

Remark 1.35. We can generalize the definition of Weil restriction $R_{L/K}G$ to the case where L is any algebra over K, not necessarily a field extension. To be more precise, let A be a K-algebra, and let G be an affine algebraic K-group. We can then define Weil restriction $R_{A/K}G = G_{A/K}$ by

$$R_{A/K}G(B) = G(B \otimes_K A)$$

where B is a K-algebra. This obviously has the property that the K-points are the A-points of G.

$$R_{A/K}G(K) = G(K \otimes_K A) = G(A)$$

1.7 Examples

Example 1.36 (Weil restriction for $\mathbb{G}_m$)**.** Let L/K be a finite field extension, and let $\mathbb{G}_m$ be the algebraic L-group defined above. Let's describe the Weil restriction $R_{L/K}\mathbb{G}_m$. For a K-algebra A, the group of K-points of $R_{L/K}\mathbb{G}_m$ is

$$R_{L/K}\mathbb{G}_m(A) \cong \mathbb{G}_m(A_L) = A_L^\times$$

In particular, following the remark above, the K-points of $R_{L/K}\mathbb{G}_m$ are the L-points of $\mathbb{G}_m$.

$$R_{L/K}\mathbb{G}_m(K) \cong \mathbb{G}_m(K_L) \cong \mathbb{G}_m(L) = L^\times$$

Also, given a morphism of K-algebras $f : A \to B$, we get an induced map

$$R_{L/K}\mathbb{G}_m(f) : R_{L/K}\mathbb{G}_m(A) \to R_{L/K}\mathbb{G}_m(B)$$

In terms of the characterization $R_{L/K}\mathbb{G}_m(A) = A_L^\times$, this map is

$$R_{L/K}\mathbb{G}_m(f) = (f \otimes 1)|_{A_L^\times} : A_L^\times \to B_L^\times \qquad \sum_i a_i \otimes \lambda_i \mapsto \sum_i f(a_i) \otimes \lambda_i$$

Lemma 1.37. *Let L/K be a finite separable extension of degree $d = [L : K]$, let E/L be the normal closure of L, and let A be an E-algebra. There are natural isomorphisms*

$$A \otimes_K L \cong A^d$$

Restricting to units, we obtain natural isomorphisms

$$(A \otimes_K L)^\times \cong (A^\times)^d$$

Proof. By the primitive element theorem, $L = K(\alpha)$ for some $\alpha \in L$. Let $g \in K[x]$ be the minimal polynomial of α, so that $\deg g = d$ and we have an isomorphism of K-algebras

$$L = K(\alpha) \cong \frac{K[x]}{(g)}$$

Since E is the normal closure of L, g splits completely in E, so we can write g as

$$g(x) = \prod_{i=1}^{d}(x - \beta_i) \in K^{\text{sep}}[x] \subset A[x]$$

where $\beta_i \in E \subset K^{\text{sep}} \subset A$, and the β_i are all the Galois conjugates of α. Viewing A as a K-algebra, we have

$$A \otimes_K L \cong A \otimes_K \frac{K[x]}{(g)} \cong \frac{A[x]}{(g)} \cong \frac{A[x]}{((x-\beta_1)\cdots(x-\beta_d))}$$

(These are all isomorphisms of K-algebras.) Since the elements $\beta_1, \ldots, \beta_d$ are distinct, the ideals $(x - \beta_1), \ldots, (x - \beta_d)$ are pairwise coprime, so by the Chinese Remainder Theorem the above we have

$$\frac{A[x]}{((x-\beta_1)\cdots(x-\beta_d))} \cong \frac{A[x]}{(x-\beta_1)} \times \cdots \times \frac{A[x]}{(x-\beta_d)}$$

Once again, this is an isomorphism of K-algebras. On the right side, each factor is isomorphic to A, hence

$$A \otimes_K L \cong A^d$$

We omit proof of the naturality of this isomorphism, since it requires carefully tracing through all the isomorphisms involved. □

Corollary 1.38. *Let L/K be a finite separable extension of degree d, and let E be the normal closure of L. Then $R_{L/K}\mathbb{G}_m^L$ is a K-torus of rank d, which is split over E.*

Proof. We need to show that $R_{L/K}\mathbb{G}_m^L \cong (\mathbb{G}_m^E)^d$ as algebraic K^{sep}-groups. Let A be an E-algebra. By the previous lemma, we have a natural isomorphism

$$R_{L/K}\mathbb{G}_m^L(A) = (A_L)^\times \cong (A^\times)^d$$

which is exactly what we need. □

Remark 1.39. One obvious consequence of the previous corollary is that if L/K is Galois, then the normal closure E is equal to L, so in this case $R_{L/K}\mathbb{G}_m^L$ is split over L.

2 Generalized norm map

2.1 Review of field norm

Definition 2.1. Let L/K be a field extension. For $x \in L$, we have a map

$$\ell_x : L \to L \qquad y \mapsto xy$$

This is a map of K-vector spaces, so we may think of ℓ_x as an element of $\text{End}_K(L)$, where here $\text{End}_K(L)$ is just endomorphisms of K-vector spaces, not of K-algebras. This gives the **left regular representation**

$$\ell : L \to \text{End}_K(L) \qquad x \mapsto \ell_x$$

Note that while ℓ_x is not a map of K-algebras, ℓ is a map of K-algebras. That is, ℓ is K-linear, and sends multiplication in L to composition in $\text{End}_K(L)$.

$$\ell_{xy} = \ell_x \circ \ell_y$$

In particular, if $x \in L^\times$, ℓ_x has inverse $\ell_{x^{-1}}$, so ℓ restricts to a group homomorphism

$$\ell : L^\times \to \text{GL}_K(L)$$

Definition 2.2. Let L/K be a finite field extension, and let $\ell : L \to \text{End}_K(L)$ be the left regular representation defined above. We also have the determinant map

$$\det : \text{End}_K(L) \to K$$

which is a group homomorphism. The **field norm** map is

$$N_K^L = \det \circ \ell : L^\times \to K^\times \qquad x \mapsto \det \ell_x$$

2.2 Goals for generalized norm

Let L/K be a field extension. We have the algebraic L-group $\mathbb{G}_m = \mathbb{G}_m^L$, and the Weil restriction $R_{L/K}\mathbb{G}_m$, which is an algebraic K-group. We also have the algebraic K-group $\mathbb{G}_m = \mathbb{G}_m^K$, not to be confused with $\mathbb{G}_m^L$ as an algebraic L-group. Taking K-points of $R_{L/K}\mathbb{G}_m$ and $\mathbb{G}_m$, we get

$$R_{L/K}\mathbb{G}_m(K) = L^\times \qquad \mathbb{G}_m(K) = K^\times$$

Between these, we have the usual field norm map.

$$N_K^L : L^\times \to K^\times$$

We would like to extend the norm map to a "generalized norm map" which is a morphism of algebraic K-groups $R_{L/K}\mathbb{G}_m \to \mathbb{G}_m$, that is, a natural transformation. That is, we want, for an arbitrary K-algebra A, to have a group homomorphism

$$N_A : R_{L/K}\mathbb{G}_m(A) = A_L^\times \to \mathbb{G}_m(A) = A^\times$$

satisfying the following properties. Properties (2),(3) encode the fact that N is a natural transformation.

1. $N_K : L^\times \to K^\times$ is the field norm.
2. N_A is a group homomorphism.

3. The morphisms N_A are natural, in the sense that for a morphism $f : A \to B$ of K-algebras, the following diagram commutes.

$$\begin{array}{ccc} R_{L/K}\mathbb{G}_m(A) & \xrightarrow{N_A} & \mathbb{G}_m(A) \\ {\scriptstyle R_{L/K}G(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ R_{L/K}\mathbb{G}_m(B) & \xrightarrow{N_B} & \mathbb{G}_m(B) \end{array}$$

Based on previous remarks, the above diagram can also be written

$$\begin{array}{ccc} (A \otimes_K L)^\times & \xrightarrow{N_A} & A^\times \\ {\scriptstyle (f\otimes 1)|_{(A\otimes_K L)^\times}}\downarrow & & \downarrow{\scriptstyle f|_{A^\times}} \\ (B \otimes_K L)^\times & \xrightarrow{N_B} & B^\times \end{array}$$

2.3 Construction of generalized norm

Definition 2.3. Let A be a K-algebra. For $a \in A$, we have a map

$$\ell_a : A \to A \qquad b \mapsto ab$$

which is a map of K-vector spaces. So we think of ℓ_a as an element of $\mathrm{End}_K(A)$ (not morphisms of algebras, just morphisms of vector spaces). We think of $\mathrm{End}_K(A)$ as a K-algebra with pointwise addition of maps and composition as multiplication. Then we have the **left regular representation**

$$\ell : A \to \mathrm{End}_K(A) \qquad a \mapsto \ell_a$$

As before in the special case where $A = L$ is a field extension of K, the map ℓ is a map of K-algebras.

Remark 2.4. Let L/K be a finite field extension, and let A be a K-algebra. Recall the abbreviation $A_L = A \otimes_K L$, and consider the left regular representation of A_L.

$$\ell : A_L \to \mathrm{End}_L(A_L) \qquad x \mapsto \ell_x$$

Recall that A_L has the structure of an A-module,

$$A \times A_L \to A_L \qquad (a, x) \mapsto a \cdot x$$

We know that ℓ_x is L-linear, but more than that, it is also A-linear. That is, for $x, y \in L$ and $a \in A$,

$$\ell_x(a \cdot y) = x(a \cdot y) = a \cdot (xy) = a \cdot \ell_x(y)$$

To justify this more fully, suppose $x = \sum_j c_j \otimes \gamma_j, y = \sum_i b_i \otimes \lambda_i$ with $c_j, b_i \in A$ and $\gamma_j, \lambda_i \in L$. Then

$$\ell_x(a \cdot y) = \ell_x\left(\sum_i ab_i \otimes \lambda_i\right) = \left(\sum_j c_j \otimes \gamma_j\right)\left(\sum_i ab_i \otimes \lambda_i\right) = \sum_{i,j} c_j ab_i \otimes \gamma_j \lambda_i$$

$$a \cdot \ell_x(y) = a \cdot (xy) = a \cdot \left(\left(\sum_j c_j \otimes \gamma_j\right)\left(\sum_i b_i \otimes \lambda_i\right)\right) = a \cdot \sum_{i,j} c_j b_i \otimes \gamma_j \lambda_i = \sum_{i,j} ac_j b_i \otimes \gamma_j \lambda_i$$

These are equal since A is commutative. All this to say ℓ_x is A-linear, so we may alternatively view the left regular representation as a map

$$\ell : A_L \to \mathrm{End}_A(A_L) \qquad x \mapsto \ell_x$$

As before, restricting to units gives invertible endormorphisms, so we have a group homomorphism

$$\ell : A_L^\times \to \mathrm{GL}_A(A_L) \qquad x \mapsto \ell_x$$

Definition 2.5. Let L/K be a finite field extension, and A a K-algebra. We have the determinant map [1]

$$\det : \mathrm{End}_A(A_L) \to A$$

which restricts to a group homomorphism

$$\det : \mathrm{GL}_A(A_L) \to A^\times$$

We then define the **generalized norm** map as

$$N_A = \det \circ \ell : A_L^\times \to A^\times \qquad x \mapsto \det \ell_x$$

Remark 2.6. In this example, we work out a matrix of $N_A(x)$ in terms of a fixed K-basis of L, and structure constants for L as a K-algebra in terms of the given basis.

Let L/K be a finite extension of degree $d = [L : K]$, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_d\}$ be a K-basis of L. Following previous work, we then have an A-basis for A_L,

$$\mathcal{B}' = \{1 \otimes \alpha_i\}$$

Given $x \in A_L^\times$, we want to write the matrix of ℓ_x in the basis $\mathcal{B}'$. We can do this in terms of the structure constants for L as a K-algebra, so let us give those a name. That is, for $1 \le i, j, k \le d$, define $b_{ij}^k \in K$ by

$$\alpha_i \alpha_j = \sum_{k=1}^{n} b_{ij}^k \alpha_k$$

Now consider $x \in A_L^\times$, and write it uniquely in terms of the basis $\mathcal{B}'$.

$$x = \sum_{i=1}^{d} x_i \cdot (1 \otimes \alpha_i) = \sum_{i=1}^{d} x_i \otimes \alpha_i$$

[1] Here it is important that we have required our K-algebras such as A to be commutative, since otherwise defining the determinant is complicated.

where $x_i \in A$. Then ℓ_x acts on a basis element $1 \otimes \alpha_j$ by

$$\ell_x(1 \otimes \alpha_j) = \sum_i x_i \otimes \alpha_i \alpha_j = \sum_i x_i \otimes \sum_k b_{ij}^k \alpha_k = \sum_{i,k} x_i b_{ij}^k \otimes \alpha_k = \sum_k \left(\sum_i x_i b_{ij}^k \otimes \right) \alpha_k$$

Here we are using the fact that $b_{ij}^k \in K$ and that the tensor is over K in order to pass b_{ij}^k through the tensor. Thus the (j,k) entry of the matrix of ℓ_x in the basis $\mathcal{B}'$ is $\sum_i x_i b_{ij}^k$. Thus

$$N_A(x) = \det \left(\sum_i x_i b_{ij}^k \right)$$

Example 2.7. Let $K = \mathbb{R}, L = \mathbb{C}$. We choose the usual $\mathbb{R}$-basis for $\mathbb{C}$, namely $\mathcal{B} = \{1, i\}$. The structure constants for this basis are not very hard to work out.

$$\begin{aligned} 1^2 &= 1 & b_{11}^1 &= 1, b_{11}^i = 0 \\ 1i &= i & b_{1i}^1 &= 0, b_{1i}^i = 1 \\ i1 &= i & b_{i1}^1 &= 0, b_{i1}^i = 1 \\ i^2 &= -1 & b_{ii}^1 &= -1, b_{ii}^i = 0 \end{aligned}$$

Now let A be a (unital, commutative, associative, finite dimensional) $\mathbb{R}$-algebra. Then we have the A-basis of A_L,

$$\mathcal{B}' = \{1 \otimes 1, 1 \otimes i\}$$

Let $x \in A_L^\times$, and write in terms of the basis $\mathcal{B}'$.

$$x = x_1 \otimes 1 + x_i \otimes i = \begin{pmatrix} x_1 \\ x_i \end{pmatrix}_{\mathcal{B}'}$$

Now consider $\ell_x \in \mathrm{GL}_A(A_L)$ acting on each element of $\mathcal{B}'$.

$$\ell_x(1 \otimes 1) = x = x_1 \otimes 1 + x_i \otimes i = \begin{pmatrix} x_1 \\ x_i \end{pmatrix}_{\mathcal{B}'}$$

$$\ell_x(1 \otimes i) = x_1 \otimes i - x_i \otimes 1 = \begin{pmatrix} -x_i \\ x_1 \end{pmatrix}_{\mathcal{B}'}$$

So the matrix of ℓ_x in the basis $\mathcal{B}'$ is

$$\begin{pmatrix} x_1 & -x_i \\ x_i & x_1 \end{pmatrix}$$

Thus we can compute $N_A(x)$ as

$$N_A(x) = \det \ell_x = x_1^2 + x_i^2$$

which is precisely the usual complex norm in the case $A = \mathbb{C}$.

$$N_{\mathbb{R}}^{\mathbb{C}} : \mathbb{C}^\times \to \mathbb{R}^\times \qquad x + iy \mapsto x^2 + y^2$$

Remark 2.8. Suppose L/K is Galois of degree $d = [L : K]$, and let K^{sep} be the separable closure, with a fixed embedding $\iota : K \hookrightarrow K^{\text{sep}}$. Let A be a K^{sep}-algebra. In a previous lemma, we fully described the unit group of $A \otimes_K L$.

$$A \otimes_K L \cong A^d \qquad (A \otimes_K L)^\times \cong (A^\times)^d$$

Using naturality of the generalized norm applied to ι, we get the following commutative diagram.

$$\begin{array}{ccc} L^\times & \xrightarrow{N_K^L} & K^\times \\ {\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota} \\ (A^\times)^d & \xrightarrow{N_A} & A^\times \end{array}$$

The vertical map on the left is somewhat mysterious, since it involves several isomorphisms, include isomorphisms coming from the Primitive Element Theorem and the Chinese Remainder Theorem. If follow through all the isomorphisms, is can be described very nicely as

$$L^\times \to ({K^{\text{sep}}}^\times)^d \hookrightarrow (A^\times)^d \qquad \lambda \mapsto (\sigma_1\lambda, \ldots, \sigma_d\lambda)$$

where $\sigma_1, \ldots, \sigma_d$ are the distinct elements of the Galois group $\text{Gal}(L/K)$. So really the image lands in $(K^{\text{sep}\,\times})^d$.

$$\begin{array}{ccc} L^\times & \xrightarrow{N_K^L} & K^\times \\ {\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota} \\ (K^{\text{sep}\,\times})^d & \xrightarrow{N_{K^{\text{sep}}}} & K^{\text{sep}\,\times} \\ \downarrow & & \downarrow \\ (A^\times)^d & \xrightarrow{N_A} & A^\times \end{array}$$

What can we say about N_A? The left regular representation of $(A^\times)^d$ is just

$$(A^\times)^d \to \text{GL}_d\left((A)^d\right) \qquad (a_1, \ldots, a_d) \mapsto \text{diag}(a_1, \ldots, a_d)$$

Taking the determinant of a diagonal matrix is just taking the product, so the norm map N_A is just given by

$$N_A : (A^\times)^d \to A^\times \qquad (a_1, \ldots, a_d) \mapsto \prod_{i=1}^{d} a_i$$

In the special case where $A = K^{\text{sep}}$, the commutative diagram above says that

$$N_{K^{\text{sep}}}(\sigma_1\lambda, \ldots, \sigma_d\lambda) = \prod_{i=1}^{d} \sigma_i(\lambda) = N_K^L(\lambda)$$

which is a fact we already knew about the regular field norm, when L/K is Galois.

2.4 Naturality of generalized norm

Proposition 2.9. *The maps N_A satisfy properties (1),(2),(3) of section 2.2.*

Proof. Let L/K be a finite field extension of degree $d = [L : K]$. It is clear that N_A is a group homomorphism, since it is a composition of two group homomorphisms, so we have property (1). It is also basically immediate from the definitions that $N_K = N_K^L$, so we have (2). All that remains is to verify naturality. Let $f : A \to B$ be a morphism of K-algebras. We need to check that the following diagram commutes.

$$\begin{array}{ccc} A_L^\times & \xrightarrow{N_A} & A^\times \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow{\scriptstyle f} \\ B_L^\times & \xrightarrow{N_B} & B^\times \end{array}$$

Fix a K-basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_d\}$. Let $\mathcal{B}'_A, \mathcal{B}'_B$ be the corresponding A- and B-bases of A_L, B_L respectively.

$$\mathcal{B}'_A = \{1_A \otimes \alpha_i\} \subset A_L \qquad \mathcal{B}'_B = \{1_B \otimes \alpha_i\} \subset B_L$$

where 1_A denotes the unit in A and 1_B denotes the unit in B. Since f is a K-algebra homorphism, $f(1_A) = 1_B$, so $f \otimes 1$ maps $\mathcal{B}'_A$ to $\mathcal{B}'_B$. Let $x \in A_L^\times$, and write it in terms of the basis $\mathcal{B}'_A$.

$$x = \sum_i x_i \otimes \alpha_i$$

Let $\widetilde{x} = (f \otimes 1)(x) \in B_L$. Then $\widetilde{x}$ is written in terms of $\mathcal{B}'_B$ as

$$\widetilde{x} = \sum_i f(x_i) \otimes \alpha_i$$

As in remark 2.6, for $1 \le i, j, k \le d$, define $b_{ij}^k \in K$ by

$$\alpha_i \alpha_j = \sum_k b_{ij}^k \alpha_k$$

By that same remark,

$$N_A(x) = \det\left(\sum_i b_{ij}^k x_i\right) \qquad N_B(\widetilde{x}) = \det\left(\sum_i b_{ij}^k f(x_i)\right)$$

A key fact about determinants is that they commute with ring homomorphisms. Also, since f is a homomorphism of K-algebras, b_{ij}^k is fixed by f, so

$$f \circ N_A(x) = f\left(\det\left(\sum_i b_{ij}^k x_i\right)\right) = \det\left(f\left(\sum_i b_{ij}^k x_i\right)\right) = \det\left(\sum_i b_{ij}^k f(x_i)\right) = N_B(\widetilde{x})$$

Hence the diagram commutes, and the proof is complete □

3 The norm torus

3.1 Generalized norm map on K^{sep}-points

Recall that every extension of a perfect field is separable.

Proposition 3.1. *Let K be a perfect field, and let L/K be a finite Galois extension of degree $d = [L : K] > 1$, and let $N : R_{L/K}\mathbb{G}_m \to \mathbb{G}_m$ be the norm. Let T be the kernel of N. Then T is a torus of rank $d-1$. In particular, $T(K^{\mathrm{sep}}) \cong (K^{\mathrm{sep}\,\times})^{d-1}$.*

Proof. We need to show that $T_{K^{\mathrm{sep}}} \cong (\mathbb{G}_m)^{d-1}$, which is to say we need natural isomorphisms $T(A) \cong \mathbb{G}_m(A)^{d-1}$ for any K^{sep}-algebra A. Let A be a K^{sep}-algebra. By definition, $T(A) = \ker N_A$. By the remark 2.8, the map N_A is just

$$N_A : (A^\times)^d \to A^\times \qquad (a_1, \dots, a_d) \mapsto \prod_{i=1}^{d} a_i$$

hence

$$\begin{aligned} T(A) &= \left\{(a_1, \dots, a_d) \in (A^\times)^d : a_1 \cdots a_d = 1\right\} \\ &= \left\{(a_1, \dots, a_{d-1}, a_d) \in (A^\times)^d : a_d = a_1^{-1} \cdots a_{d-1}^{-1}\right\} \\ &\cong \left\{(a_1, \dots, a_{d-1}) \in (A^\times)^{d-1}\right\} \\ &= (A^\times)^{d-1} \end{aligned}$$

Hence T is a torus of degree $d-1$. We omit the details of naturality. □

Definition 3.2. We will call the kernel of the norm the **norm torus** from now on.

Remark 3.3. The next major goal is to develop a criterion for when the norm torus is split (over the base field K). First, we'll work out some more concrete examples for particular field extensions.

3.2 Examples of norm tori

Now we give several worked examples of the norm torus. We describe the points of T over the base field, and determine whether T is split when possible.

Example 3.4 (Norm torus for $\mathbb{C}/\mathbb{R}$)**.** Let $L/K = \mathbb{C}/\mathbb{R}$, which is Galois of degree 2. We have the norm map

$$N : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m^{\mathbb{C}} \to \mathbb{G}_m^{\mathbb{R}}$$

and as before we let $T = \ker N$ be the norm torus, so T is an algebraic $\mathbb{R}$-group. We already know that T is a torus, but let's just do a partial verification by taking $\mathbb{C}$-points.

$$\begin{aligned} R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m^{\mathbb{C}}(\mathbb{C}) &= (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong (\mathbb{C}^\times)^2 \\ \mathbb{G}_m^{\mathbb{R}}(\mathbb{C}) &= \mathbb{C}^\times \\ N_{\mathbb{C}} : \mathbb{C}^\times \to \mathbb{C}^\times & \qquad (z, w) \mapsto zw \\ T(\mathbb{C}) = \ker N_{\mathbb{C}} &= \left\{(z, z^{-1}) : z \in \mathbb{C}^\times\right\} \cong \mathbb{C}^\times \end{aligned}$$

Let's take a moment to be explicit about the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C}^\times)^2$, since it is not quite as simple as one would like. One explicit isomorphism is given by

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to (\mathbb{C}^\times)^2 \qquad z \otimes w \mapsto (zw, z\overline{w})$$

Now let's take $\mathbb{R}$-points of the torus T, where we get the usual field norm.

$$\begin{aligned} R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m^{\mathbb{C}}(\mathbb{R}) &= (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{C}^\times \\ \mathbb{G}_m^{\mathbb{R}}(\mathbb{R}) &= \mathbb{R}^\times \\ N_{\mathbb{R}} : \mathbb{C}^\times \to \mathbb{R}^\times & \qquad z = x + iy \mapsto N_{\mathbb{R}}^{\mathbb{C}}(z) = |z| = x^2 + y^2 \\ T(\mathbb{R}) = \ker N_{\mathbb{R}} &= \left\{ z \in \mathbb{C}^\times : |z| = 1 \right\} \cong S^1 \end{aligned}$$

So if T is split over $\mathbb{R}$, then it would be the case that $\mathbb{R}^\times \cong S^1$. These are clearly not isomorphic groups, since S^1 has 4-torsion ($i^4 = 1$) but $\mathbb{R}^\times$ has no 4-torsion (the only torsion in $\mathbb{R}^\times$ is order 2, ± 1). Hence T is non-split.

Remark 3.5. My best guess is that the previous example is the primary motivation for using the word "torus" when talking about these special algebraic groups. In particular, if we take two copies of the previous example,

$$N_{\mathbb{R}} \times N_{\mathbb{R}} : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{R}^\times \times \mathbb{R}^\times$$

the kernel is $S^1 \times S^1$, which is what algebraic topologists mean by the word "torus."

Example 3.6 (Norm torus for quadratic extension)**.** We can generalize the previous example to an arbitrary quadratic extension of a perfect field. Let K be a perfect field, and L/K a quadratic extension. So we can write L as $L = K(\sqrt{d})$ where d is some non-square in K.

The previous example was just the case $d = -1 \in \mathbb{R}$, but now we're considering things like quadratic extensions of $\mathbb{Q}$ or $\mathbb{F}_q$ at the same time. Note that every quadratic extension is Galois.

We already know what happens when we take K^{sep} points in general, so let's just focus on taking K-points. Fix the K-basis $\mathcal{B} = \left\{1, \sqrt{d}\right\}$ of L, and let us describe the left regular representation of L in terms of this basis.

$$\begin{aligned} \ell_1(a + b\sqrt{d}) &= a + b\sqrt{d} = \begin{pmatrix} a \\ b \end{pmatrix}_{\mathcal{B}} \\ \ell_{\sqrt{d}}(a + b\sqrt{d}) &= a\sqrt{d} + bd = \begin{pmatrix} bd \\ a \end{pmatrix}_{\mathcal{B}} \end{aligned}$$

So we can describe the left regular representation of L by

$$\ell : L^\times \to \mathrm{GL}_2(K) \qquad a + b\sqrt{d} \mapsto \begin{pmatrix} a & bd \\ b & a \end{pmatrix}$$

And composing with determinant we get the K-points of the norm map N.

$$N_K = N_K^L : L^\times \to K^\times \qquad a + b\sqrt{d} \mapsto a^2 + db^2$$

In this scenario, the left regular representation of L is an embedding, so we may identify $L^\times$ with its image in $\mathrm{GL}_2(K)$.

$$L^\times = (K(\sqrt{d})^\times \cong \left\{ \begin{pmatrix} a & bd \\ b & a \end{pmatrix} : a, b \in L, \text{not both zero} \right\} \subset \mathrm{GL}_2(K)$$

Using this identification, K-points of the norm torus (the kernel of N_K^L) is just the intersection of this subgroup with $\mathrm{SL}_2(K)$.

$$T(K) = \ker N_K^L = \left\{ \begin{pmatrix} a & bd \\ b & a \end{pmatrix} : a, b \in L, \text{not both zero}, a^2 + bd^2 = 1 \right\} \subset \mathrm{SL}_2(K)$$

It's not clear from this how to to determine if T is split in this situation.

Example 3.7. Let $K = \mathbb{F}_q$ be the finite field with q elements, and let $L = \mathbb{F}_{q^d}$ be the unique extension of K of degree d. Let T be the associated norm torus, and let us describe the K-points of T. Recall that the multiplicative group of a finite field is cyclic, and that the usual field norm map between finite fields is always surjective.

$$\begin{aligned} \mathbb{G}_m(\mathbb{F}_q) &= \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \\ R_{\mathbb{F}_{q^d}/\mathbb{F}_q}\mathbb{G}_m(\mathbb{F}_q) &= (\mathbb{F}_{q^d})^\times \cong \mathbb{Z}/(q^d-1)\mathbb{Z} \\ N_{\mathbb{F}_q} &: \mathbb{Z}/(q^d-1)\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(q-1)\mathbb{Z} \end{aligned}$$

Just from knowing we have a surjective homomorphism of cyclic groups, we know that the kernel must be cyclic of order $\frac{q^d-1}{q-1} = 1 + q + \cdots + q^{d-1}$. Thus

$$T(\mathbb{F}_q) \cong \mathbb{Z}/(1 + \cdots + q^{d-1})\mathbb{Z}$$

Hence if T is split, then $1 + q + \cdots q^{d-1} = q - 1$. This is clearly impossible for any q, since it is equivalent to $1 + q^2 + \cdots + q^{d-1} = -1$. Hence T is non-split.

4 Characters

Definition 4.1. Let G be an affine algebraic K-group. A **character** of G is a homomorphism (of affine algebraic groups) $\chi : G \to \mathbb{G}_m$. The characters of G themselves form an abelian group, and the **character group** $X(G)$ is the group of such characters.

$$X(G) = \mathrm{Hom}_{\mathrm{AlgGp}_K}(G, \mathbb{G}_m)$$

Note that by Yoneda's lemma, this is isomorphic to

$$X(G) \cong \mathrm{Hom}_{\mathrm{Alg}_K}(K[x, x^{-1}], \mathcal{O}_G) \cong \mathcal{O}_G^\times$$

Let's say a little more about how the characters form a group. Given two characters (natural transformations) $\chi, \chi;' : G \to \mathbb{G}_m$, we need to describe their sum $\chi + \chi' : G \to \mathbb{G}_m$. Given a K-algebra A, we have homomorphisms

$$\chi_A, \chi'_A : G(A) \to \mathbb{G}_m(A)$$

so we just define $(\chi + \chi')_A$ by

$$(\chi + \chi')_A := \chi_A + \chi'_A : G(A) \to \mathbb{G}_m(A) \qquad (\chi + \chi')_A(x) = \chi_A(x) \cdot \chi'_A(x)$$

where $\cdot$ denotes multiplication in $\mathbb{G}_m(A) = A^\times$. Given a morphism $f : A \to B$ of K-algebras, it is clear that the following diagram commutes, so $\chi + \chi'$ a morphism of algebraic groups.

$$\begin{array}{ccc} G(A) & \xrightarrow{(\chi+\chi')_A} & \mathbb{G}_m(A) = A^\times \\ \big\downarrow{\scriptstyle G(f)} & & \big\downarrow{\scriptstyle f|_{A^\times}} \\ G(B) & \xrightarrow[(\chi+\chi')_B]{} & \mathbb{G}_m(B) = B^\times \end{array}$$

Remark 4.2. It is clear that X respects binary products in the sense that

$$X(G \times H) \cong X(G) \oplus X(H)$$

Remark 4.3. We can think of $X(-) = \mathrm{Hom}_{\mathrm{AlgGp}_K}(-, \mathbb{G}_m)$ as a contravariant functor from the category of algebraic K-groups to the category of abelian groups. Later we will consider module structures on this, but for now the target category is just abelian groups.

Remark 4.4. Let G be an algebraic K-group and L/K a field extension. There is an embedding

$$X(G) \hookrightarrow X(G_L)$$

described as follows. Using the Yoneda lemma,

$$\begin{aligned} X(G_L) &= \mathrm{Hom}_{\mathrm{AlgGp}_L}(G_L, \mathbb{G}_m^L) \cong \mathrm{Hom}_L(L[x, x^{-1}], \mathcal{O}_G \otimes_K L) \cong (\mathcal{O}_G \otimes_K L)^\times \\ X(G) &= \mathrm{Hom}_{\mathrm{AlgGp}_K}(G, \mathbb{G}_m^K) \cong \mathrm{Hom}_K(K[x, x^{-1}], \mathcal{O}_G) \cong \mathcal{O}_G^\times \end{aligned}$$

In these terms, the embedding is simply

$$X(G) = \mathcal{O}_G^\times \hookrightarrow (\mathcal{O}_G \otimes_K L)^\times = X(G_L) \qquad \alpha \mapsto \alpha \otimes 1$$

In terms of varieties and geometry, this is simply the fact that every morphism "defined over K" is also "defined over L." Using this canonical embedding, we view $X(G)$ as a subgroup of $X(G_L)$.

Definition 4.5. Just to simplify notation, define

$$\widetilde{X}(G) = X(G_{K^{\mathrm{sep}}})$$

Remark 4.6. Let G be an algebraic K-group. Extending the previous remark, for any separable extension L/K, there is an embedding

$$X(G_L) \hookrightarrow \widetilde{X}(G) = X(G_{K^{\mathrm{sep}}})$$

Lemma 4.7 (Character group of $\mathbb{G}_m$). *Let K be a field, and consider the algebraic K-group $\mathbb{G}_m$. The character group is*

$$X(\mathbb{G}_m) \cong \mathbb{Z}$$

Explicitly, the isomorphism is given by

$$\mathbb{Z} \to \mathrm{Hom}_{\mathrm{AlgGp}_K}(\mathbb{G}_m, \mathbb{G}_m) \qquad n \mapsto \phi_n$$

where $\phi_n : \mathbb{G}_m \to \mathbb{G}_m$ is the natural transformation described on A-points by

$$\phi_n(A) : \mathbb{G}_m(A) = A^\times \to \mathbb{G}_m(A) = A^\times \qquad x \mapsto x^n$$

for any K-algebra A.

Proof. By the Yoneda lemma,

$$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathrm{Hom}_{\mathrm{Alg}_K}(K[x, x^{-1}], K[x, x^{-1}])$$

Let $\alpha : K[x, x^{-1}] \to K[x, x^{-1}]$ be a K-algebra homomorphism, so it is determined by the value of x. So let

$$\alpha(x) = \frac{f(x)}{x^n}$$

where $f \in K[x]$ is some polynomial and $n \in \mathbb{Z}$. Since x is a unit in $K[x, x^{-1}]$, $\alpha(x)$ is also a unit in $K[x, x^{-1}]$, hence

$$\alpha(x)^{-1} = \frac{x^n}{f(x)} \in K[x, x^{-1}]$$

so $f(x)$ cannot have any nonzero roots in K. Hence $f(x) = ax^m$ for some $a \in K^\times, m \in \mathbb{Z}$. Thus $\alpha(x) = ax^{m-n} = ax^t$ for some $a \in K, t \in \mathbb{Z}$. Since α is a K-algebra homomorphism, $\alpha(1) = 1$, so $a = 1$. Hence $\alpha(x) = x^t$ for some $t \in \mathbb{Z}$. Hence

$$\mathbb{Z} \mapsto \mathrm{Hom}_{\mathrm{Alg}_K}(K[x, x^{-1}], K[x, x^{-1}]) \qquad t \mapsto (x \mapsto x^t)$$

is a group isomorphism, so

$$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

□

Corollary 4.8. *Let T be a K-torus of rank r. Then $\widetilde{X}(T) \cong \mathbb{Z}^r$.*

Proof.

$$\widetilde{X}(T) = X(T_{K^{\mathrm{sep}}}) \cong X(\mathbb{G}_m^r) \cong X(\mathbb{G}_m)^r \cong \mathbb{Z}^r$$

□

Remark 4.9. We may interpret the previous corollary as follows. Consider the contravariant functor

$$\widetilde{X}(-) = \mathrm{Hom}_{\mathrm{AlgGp}_{K^{\mathrm{sep}}}}\left((-)_{K^{\mathrm{sep}}}, \mathbb{G}_m^{K^{\mathrm{sep}}}\right) : \mathrm{AlgGp}_K \to \mathrm{AbGp}$$

Let $\mathcal{A} \subset \mathrm{AlgGp}_K$ be the full subcategory of algebraic K-tori (of finite rank). Let $\mathcal{B} \subset \mathrm{AbGp}$ be the full subcategory of finitely generated free abelian groups. The previous corollary says that the restriction of $\widetilde{X}((-))$ to $\mathcal{A}$ lands in $\mathcal{B}$.

$$\widetilde{X}(-) : \mathcal{A} \to \mathcal{B}$$

It is clear that this functor is essentially surjective, since given a free abelian group of rank r, choose the split K-torus $T = (\mathbb{G}_m)^r$, and then $\widetilde{X}(T) \cong \mathbb{Z}^r$.

Eventually, we would like to say that $\widetilde{X}(-)$ is an equivalence of categories. However, this is not yet true, until we add additional structure to the character group. In particular, we need to endow it with a module structure in order to obtain this equivalence.

4.1 Character modules

Definition 4.10. Let K be a perfect field, and G be an algebraic K-group. Let L/K be a Galois extension, with Galois group $\Gamma = \mathrm{Gal}(L/K)$. Then we define the **Galois action** of Γ on $X(G_L)$ as follows. Recall that

$$X(G_L) = \mathrm{Hom}_{\mathrm{AlgGp}_L}(G_L, \mathbb{G}_m^L) \cong \mathrm{Hom}_{\mathrm{Alg}_L}(L[x, x^{-1}], \mathcal{O}_G \otimes_K L) \cong (\mathcal{O}_G \otimes_K L)^\times$$

Γ acts on L as automorphisms, so we just let Γ act on the L-part of the tensor on the far right. In terms of simple tensors, the action can be described as

$$\Gamma \times (\mathcal{O}_G \otimes_K L)^\times \to (\mathcal{O}_G \otimes_K L)^\times \qquad \sigma \cdot (\alpha \otimes \beta) = \alpha \otimes \sigma(\beta)$$

Remark 4.11. Let G be an algebraic K-group and L/K be Galois with Galois group Γ as above, with Γ acting on $X(G_L)$ as defined above. Let's trace back this action through the isomorphisms and see if we can understand the Γ action on homomorphisms of algebraic L-groups $G_L \to \mathbb{G}_m$. Let $y \in (\mathcal{O}_G \otimes_K L)^\times$ and $\sigma \in \Gamma$. We can write y as

$$y = \sum_i \alpha_i \otimes \beta_i$$

where $\alpha_i \in \mathcal{O}_G$ and $\beta_i \in L$, and having written y in this way, we can apply the definition to compute $\sigma \cdot y$ as

$$\sigma \cdot y = \sigma \cdot \sum_i \alpha_i \otimes \beta_i = \sum_i \sigma \cdot (\alpha_i \otimes \beta_i) = \sum_i \alpha_i \otimes \sigma(\beta_i)$$

Under the isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}_L}(L[x, x^{-1}], \mathcal{O}_G \otimes_K L) \cong (\mathcal{O}_G \otimes_K L)^\times$$

the element y corresponds to the morphism

$$\phi_y : L[x, x^{-1}] \to \mathcal{O}_G \otimes_K L \qquad x \mapsto y$$

So pulling back the Γ action to $\mathrm{Hom}(L[x, x^{-1}], \mathcal{O}_G \otimes_K L)$, we can write that

$$\sigma \cdot (\phi_y) = \phi_{\sigma \cdot y}$$

Now consider the isomorphism

$$\mathrm{Hom}_{\mathrm{AlgGp}_L}(G_L, \mathbb{G}_m^L) \cong \mathrm{Hom}_{\mathrm{Alg}_L}(L[x, x^{-1}], \mathcal{O}_G \otimes_K L)$$

which comes from the Yoneda lemma. Under this isomorphism, ϕ_y corresponds to the morphism ψ_y of algebraic L-groups which we now describe. Let A be an L-algebra. Since G_L is representable with representing object $\mathcal{O}_{G_L} = \mathcal{O}_G \otimes_K L$, there are natural isomorphisms

$$G_L(A) \cong \mathrm{Hom}_{\mathrm{Alg}_L}(\mathcal{O}_G \otimes_K L, A)$$

Now we can to describe the natural transformation (morphism of algebraic L-groups) ψ_y.

$$\psi_y : G_L \to \mathbb{G}_m^L$$

Given an L-algebra A, on A-points ψ_y is given by

$$\psi_{y,A} : G_L(A) = \mathrm{Hom}_{\mathrm{Alg}_L}(\mathcal{O}_G \otimes_K L, A) \to \mathrm{Hom}_{\mathrm{Alg}_L}(L[x, x^{-1}], A) \qquad f \mapsto f \circ \phi_y$$

Then finally pulling back the action of Γ to act on ψ_y, we have some sort of understanding of how Γ acts on $X(G_L)$. Given a character $\psi \in X(G_L)$, there exists $y \in \mathcal{O}_G \otimes_K L$ such that on A-points, $\psi_A = \psi_{y,A}$, and

$$\sigma \cdot \psi = \psi_{\sigma \cdot y}$$

where $\psi_{\sigma \cdot y}$ acts as $\psi_{\sigma \cdot y, A}$ on A-points.

Remark 4.12. The previous module structure makes sense for any Galois extension L/K, but it is primarily of interest in the case $L = K^{\mathrm{sep}}$, where $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is the absolute Galois group. That is, we are most interested in the action of Γ on $\widetilde{X}(G)$, for a K-group G.

Example 4.13 (Character module of $\mathbb{G}_m^{\mathbb{R}}$)**.** Let $L = \mathbb{C}, K = \mathbb{R}$, and consider the algebraic $\mathbb{R}$-group $G = \mathbb{G}_m^{\mathbb{R}}$, which as we know is a split torus. Note that $\mathcal{O}_G = \mathbb{R}[x, x^{-1}]$. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}\,\langle\sigma\rangle$, where σ denotes complex conjugation. Then $\widetilde{X}(G)$ is

$$\widetilde{X}(\mathbb{G}_m^{\mathbb{R}}) = X(\mathbb{G}_m^{\mathbb{C}}) \cong \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{C}}}(\mathbb{C}[x, x^{-1}], \mathbb{R}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C}) \cong (\mathbb{R}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C})^{\times}$$

which we also know is isomorphic to $\mathbb{Z}$, since the only $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x, x^{-1}]$ are $x \mapsto x^n$, where $n \in \mathbb{Z}$. So the correspondence with $(\mathbb{R}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C})^{\times}$ is given by

$$\mathbb{Z} \to (\mathbb{R}[x, x^{-1}] \otimes_{\mathbb{C}} \mathbb{C})^{\times} \qquad n \mapsto x^n \otimes 1$$

The action of Γ on such elements is trivial, since $1 \in \mathbb{C}$ is fixed by Γ. Thus the character module of $\mathbb{G}_m^{\mathbb{R}}$ is $\mathbb{Z}$ with trivial Γ action.

$$\Gamma \times \widetilde{X}(G) \to \widetilde{X}(G) \qquad \sigma \cdot \chi = \chi$$

The next example is the same, but in more generality - it turns out that the character module for $\mathbb{G}_m$ is always trivial, regardless of the base field and the extension field.

Example 4.14 (Character module of $\mathbb{G}_m$)**.** Let K be any field and L/K a Galois extension, and consider $G = \mathbb{G}_m$ as an algebraic K-group. Note that $\mathcal{O}_G = K[x, x^{-1}]$. Let $\Gamma = \mathrm{Gal}(L/K)$. Then $G_L = \mathbb{G}_m^L$, so $X(G_L) = X(\mathbb{G}_m^L) \cong \mathbb{Z}$, with the isomorphism arising from the fact that the only L-algebra endomorphisms of $L[x, x^{-1}]$ are power maps $x \mapsto x^n$. Let's write out the isomorphisms in more detail.

$$\begin{array}{rcccc} X(G_L) \cong & \mathrm{Hom}_{\mathrm{Alg}_L}(L[x,x^{-1}], K[x,x^{-1}] \otimes_K L) \cong & (K[x,x^{-1}] \otimes_K L)^\times \cong & (L[x,x^{-1}])^\times \cong & \mathbb{Z} \\ & \phi_n = (x \mapsto x^n \otimes 1) & x^n \otimes 1 & x^n & n \end{array}$$

So what is the action of $\sigma \in \Gamma$ on $n \in \mathbb{Z}$? The action was defined in terms of how Γ acts on the L part of the tensor in the form $x^n \otimes 1$. Since $1 \in K$ is fixed by σ, the action is trivial. That is to say, $X(G_L) \cong \mathbb{Z}$ is a trivial Γ-module.

$$\Gamma \times X(\mathbb{G}_m^L) \to X(\mathbb{G}_m^L) \qquad \sigma \cdot \chi = \chi$$

Example 4.15 (Character module of $R_{L/K}\mathbb{G}_m^L$)**.** Let L/K be finite Galois, and consider the algebraic K-torus $G = R_{L/K}\mathbb{G}_m^L$ of rank $d = [L : K]$. Let $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be the Galois group. We have a short exact sequence of groups

$$1 \to \mathrm{Gal}(K^{\mathrm{sep}}/L) \to \Gamma \to \mathrm{Gal}(L/K) \to 1$$

Consider the character modules $X(G_L)$ and $\widetilde{X}(G)$. As G is split over L, we have isomorphisms of abelian groups

$$X(G_L) \cong \mathbb{Z}^d$$

and we have defined a $\mathrm{Gal}(L/K)$ action on this. Since G also splits over K^{sep}, we have isomorphisms of abelian groups

$$\widetilde{X}(G) = X(G_{K^{\mathrm{sep}}}) \cong \mathbb{Z}^d$$

and this has an action from Γ. We also have an embedding $X(G_L) \hookrightarrow \widetilde{X}(G)$, so we can restrict the action of Γ to $X(G_L)$. Actually, the embedding $X(G_L) \hookrightarrow \widetilde{X}(G)$ is just the identity map, from the calculations above.

Using Theorem 4.22, since G splits over L, the subgroup $\mathrm{Gal}(K^{\mathrm{sep}}/L) = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ acts trivially on $X(G_L)$, so the action of Γ factors through $\Gamma/\mathrm{Gal}(K^{\mathrm{sep}}/L) \cong \mathrm{Gal}(L/K)$. This doesn't tell us anything new, since we already know that $\mathrm{Gal}(L/K)$ acts on $X(G_L)$. It should be relatively clear that these actions are the same.

The upshot of the above is that $X(G_L) = \widetilde{X}(G)$, and that the action of Γ or its quotient $\mathrm{Gal}(L/K)$ are essentially the same. Concretely, this action is as follows: label the d generators of $\widetilde{X}(G) \cong \mathbb{Z}^d$ corresponding to the elements of the Galois group $\mathrm{Gal}(L/K)$, then the action is simply by permutation. That is, to act on a generator $\sigma \in \widetilde{X}(G)$ by an element $\tau \in \mathrm{Gal}(L/K)$, simply multiply in $\mathrm{Gal}(L/K)$ to get another generator $\tau\sigma$.

I still do not understand why exactly this should be the action, but it feels right, at least.

Remark 4.16. If the previous example is to be believed, $R_{L/K}\mathbb{G}_m^L$ is a non-split torus, provided that L/K is a nontrivial extension, since the action is not trivial.

4.2 Equivalence of categories

Definition 4.17. Let M be a finitely generated abelian group, written multiplicatively. Fix a field K. Let $K[M]$ be the K-vector space with basis M, so every element can be written uniquely as

$$\sum_i a_i m_i$$

where $a_i \in K, m_i \in M$. Define multiplication in $K[M]$ by

$$\left(\sum_i a_i m_i\right)\left(\sum_j b_j n_j\right) = \sum_{i,j} a_i b_j (m_i n_j)$$

where the multiplication $(m_i n_j)$ takes place inside M. This makes $K[M]$ into a commutative, unital, associated K-algebra. It is finitely generated as a K-algebra because any choice of generators for M as an abelian group will also generate $K[M]$ as an algebra.

Definition 4.18. Let M be a finitely generated abelian group. Define the functor

$$D(M)(-) : \mathrm{Alg}_K \to \mathrm{Gp} \qquad A \mapsto D(M)(A) = \mathrm{Hom}_{\mathrm{Gp}}(M, A^\times)$$

Proposition 4.19. *Let $D(M)(-)$ be the functor above. $D(M)(-)$ is an affine algebraic group, with representing object $K[M]$.*

Proof. We need to show that $D(M)(A)$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Alg}_K}(K[M], A)$. Giving a K-linear map $K[M] \to A$ is the same as giving a map of sets $M \to A$, and the further requirement that $K[M] \to A$ is a K-algebra homomorphism is just the requirement that the set map $M \to A$ is also a group homomorphism $M \to A^\times$. So these sets are naturally identified. □

Definition 4.20. Let $\mathcal{C}$ be the category of finitely generated abelian groups. Define the functor

$$D : \mathcal{C} \to \mathrm{AlgGp}_K \qquad M \mapsto D(M)(-)$$

Note that this is contravariant.

Lemma 4.21. *Let T be an algebraic K-torus of rank r, and let $\widetilde{X}(G)$ be the character module with $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ action. The Γ action makes $\widetilde{X}(G)$ into a continuous Γ-module.*

Proof. See Milne notes page 231. □

Theorem 4.22. *Let K be a perfect field with separable closure K^{sep} and absolute Galois group $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. In the following diagram, the horizontal rows are equivalences of categories, with the opposing arrows being quasi-inverses.*[2]

$$\begin{array}{ccc}
\{\text{diagonalizable algebraic } K\text{-groups}\} & \underset{D}{\overset{\widetilde{X}(-)}{\rightleftarrows}} & \{\text{f.g. abelian groups}\} \\
\uparrow & & \uparrow \\
\{\text{split algebraic } K\text{-tori}\} & \underset{D}{\overset{\widetilde{X}(-)}{\rightleftarrows}} & \{\text{f.g. free abelian groups}\}
\end{array}$$

[2] f.g. stands for finitely generated, and cts stands for continuous.

$$
\begin{array}{ccc}
\{\text{algebraic } K\text{-groups of multiplicative type}\} & \underset{D}{\overset{\widetilde{X}(-)}{\rightleftarrows}} & \{\text{f.g. abelian groups w/ cts } \Gamma\text{-action}\} \\
\uparrow & & \uparrow \\
\{\text{algebraic } K\text{-tori}\} & \underset{D}{\overset{\widetilde{X}(-)}{\rightleftarrows}} & \{\text{f.g. free abelian groups w/ cts } \Gamma\text{-action}\} \\
\uparrow & & \uparrow \\
\{\text{ algebraic } K\text{-tori split over } K\} & \underset{D}{\overset{\widetilde{X}(-)}{\rightleftarrows}} & \{\text{f.g. free abelian groups w/ trivial } \Gamma\text{-action}\}
\end{array}
$$

Proof. See Milne notes page 231. □

Remark 4.23. In the previous equivalence, note that trivial Γ action corresponds to tori which are split *over* K, so being split over a nontrivial extension E/K does not determine whether the action is trivial.

Remark 4.24. In the equivalence above between tori and free abelian groups, note that the rank is preserved. That is, a torus of rank r corresponds to a free abelian group of rank r.

Remark 4.25. One significant consequence of the result above is that $\widetilde{X}(-)$ is an exact functor.

Example 4.26. Let $K = \mathbb{R}$, with separable closure $K^{\text{sep}} = \mathbb{C}$ and absolute Galois group $\Gamma = \mathbb{Z}/2\mathbb{Z}\langle\sigma\rangle$, where σ denotes complex conjugation. Let $M = \mathbb{Z}$. There are only two possible actions of Γ on M: a trivial action, and a nontrivial action. First, consider the trivial action.

$$\Gamma \times M \to M \qquad \sigma \cdot m = m$$

Let M_1 denote M with this trivial action. There is also a nontrivial action,

$$\Gamma \times M \to M \qquad \sigma \cdot m = -m$$

Let M_2 denote M with this nontrivial action. Both are continuous actions.

Under the above equivalence of categories, M_1 corresponds to the split torus $\mathbb{G}_m^{\mathbb{R}}$. On the other hand, M_2 corresponds to some non-split torus of rank 1, so it must be the norm torus we exhibited earlier. We already knew this torus was non-split, but this serves as further confirmation.

4.3 Non-splitting of norm torus

Proposition 4.27. *Let K be a perfect field with separable closure K^{sep} and absolute Galois group $\Gamma = \text{Gal}(K^{\text{sep}}/K)$. Let L/K be a finite Galois extension of degree $d = [L : K] \geq 2$, and let T be the associated norm torus. Then T has nontrivial Γ-action, hence T is non-split.*

Proof. Consider the exact sequence of algebraic K-groups

$$0 \to T \xrightarrow{\iota} R_{L/K}\mathbb{G}_m^L \xrightarrow{N} \mathbb{G}_m^K$$

Then apply the contravariant exact functor $\widetilde{X}(-)$, to obtain an exact sequence of continuous Γ-modules.

$$\widetilde{X}(\mathbb{G}_m^K) \xrightarrow{N^*} \widetilde{X}(R_{L/K}\mathbb{G}_m) \to \widetilde{X}(T) \to 0$$

These are respectively free abelian of rank 1, d, $d-1$.

$$\mathbb{Z} \xrightarrow{N^*} \mathbb{Z}^d \to \mathbb{Z}^{d-1} \to 0$$

It remains to determine the Γ-action on each term. On the left term, the action is trivial, since $\mathbb{G}_m$ is split over K. On the middle term, following example 4.15, we have a generator for each element of $\mathrm{Gal}(L/K)$, and $\mathrm{Gal}(L/K) \cong \Gamma/\mathrm{Gal}(K^{\mathrm{sep}}/L)$ acts to permute the generators simply using left multiplication in $\mathrm{Gal}(L/K)$.

The map N^* is analogous to the norm map in group cohomology, so it sends $1 \in \mathbb{Z}$ to the sum over all generators, the element $(1, \dots, 1)$. By the first isomorphism theorem, $\widetilde{X}(T) \cong \mathbb{Z}^d/\operatorname{im} N^*$ as Γ-modules, so

$$\widetilde{X}(T) \cong \mathbb{Z}^d/\mathbb{Z}\langle 1, \dots, 1\rangle$$

where $\mathrm{Gal}(L/K)$ acts by permuting the basis of $\mathbb{Z}^d$, which is to say, Γ acts in this way and the subgroup $\mathrm{Gal}(K^{\mathrm{sep}}/L)$ acts trivially. In particular, Γ acts nontrivially as long as $d \geq 2$, so in that case T is not split. (In the case $d = 2$, $\widetilde{X}(T)$ is $\mathbb{Z}^2/\mathbb{Z}\langle 1, 1\rangle$ with permutation action, which corresponds to $\mathbb{Z}$ with the nontrivial action coming from multiplication by -1.) □