Galois descent

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Student algebra seminar

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MICHIGAN STATE UNIVERSITY

Outline

1. Motivating example
2. Twisted forms
3. Cocycles
4. Correspondence between twisted forms and cocycles
5. Revisit example

Compare and contrast: $M_2(\mathbb{R})$ and $\mathbb{H}$

	$M_2(\mathbb{R})$	$\mathbb{H}$
Definition	$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$	$\mathbb{H} = \{a + bj + ck + djk\}$ $a, b, c, d \in \mathbb{R}$
Multiplication	Matrix multiplication	$j^2 = k^2 = -1, jk = -kj$
$\mathbb{R}$-basis	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$1, j, k, jk$
$\mathbb{R}$-dimension	4	4

Not isomorphic as $\mathbb{R}$-algebras

Proposition

$M_2(\mathbb{R}) \ncong \mathbb{H}$.

Proof.

We show $\mathbb{H}$ is a division algebra.

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$$\overline{q} = a - bj - ck - djk \qquad N(q) = q\overline{q} = a^2 + b^2 + c^2 + d^2$$

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Clearly $q = 0 \iff N(q) = 0$. Then $\frac{q\overline{q}}{N(q)} = 1$ so if $q \neq 0$, it has an inverse $q^{-1} = \frac{\overline{q}}{N(q)}$. Thus $\mathbb{H}$ is a division algebra.

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Clearly $q = 0 \iff N(q) = 0$. Then $\frac{q\overline{q}}{N(q)} = 1$ so if $q \neq 0$, it has an inverse $q^{-1} = \frac{\overline{q}}{N(q)}$. Thus $\mathbb{H}$ is a division algebra. On the other hand, $M_2(\mathbb{R})$ is clearly NOT a division algebra, since it has zero divisors such as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\square$

Extension to $\mathbb{C}$

By "extension to $\mathbb{C}$" I mean applying $(-) \otimes_{\mathbb{R}} \mathbb{C}$.

$$M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes z_1 + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \otimes z_2 + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \otimes z_3 + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \otimes z_4 \mapsto \begin{pmatrix} az_1 & bz_2 \\ cz_3 & cz_4 \end{pmatrix}$$

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Proof.

$1, j, k, jk$ is an $\mathbb{R}$-basis of $\mathbb{H}$, so $1 \otimes 1, j \otimes 1, k \otimes 1, jk \otimes 1$ is a $\mathbb{C}$-basis of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$.

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So to define a $\mathbb{C}$-linear map $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{M}_2(\mathbb{C})$, it suffices to define it on this basis.

$$1 \otimes 1 \mapsto 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad j \otimes 1 \mapsto \widehat{j} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$k \otimes 1 \mapsto \widehat{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad jk \otimes 1 \mapsto \widehat{\ell} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Since this takes a $\mathbb{C}$-basis of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ to a $\mathbb{C}$-basis of $\mathrm{M}_2(\mathbb{C})$, the map is bijective.

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Since this takes a $\mathbb{C}$-basis of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ to a $\mathbb{C}$-basis of $\mathrm{M}_2(\mathbb{C})$, the map is bijective. To verify this is a morphism of algebras, it's enough to check $\widehat{j}^2 = \widehat{k}^2 = -1$ and $\widehat{j}\widehat{k} = \widehat{\ell}$ and $\widehat{j}\widehat{k} = -\widehat{k}\widehat{j}$, which are quick matrix calculations. □

$$
\begin{tikzcd}
\mathbb{C}\text{-algebras} & \mathrm{M}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, phantom, "\cong"] & \mathrm{M}_2(\mathbb{C}) \arrow[r, phantom, "\cong"] & \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \\
\mathbb{R}\text{-algebras} \arrow[u, "(-)\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{M}_2(\mathbb{R}) \arrow[u, rightsquigarrow] \arrow[r, dash] & \not\cong \arrow[r, dash] & \mathbb{H} \arrow[u, rightsquigarrow]
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Definition

Let L/K be a field extension. Let A be a K-algebra. A **twisted L/K-form** of A is a K-algebra B such that $A \otimes_K L \cong B \otimes_K L$.

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The set of (K-isomorphism classes of) twisted forms of A is denoted $\boldsymbol{TF_L(A)}$.

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$$\mathrm{Id} \otimes \sigma : A \otimes_K L \to A \otimes_K L \qquad a \otimes \ell \mapsto a \otimes \sigma(\ell)$$

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is an automorphism of $A_L = A \otimes_K L$ as an L-algebra. G acts on the set of L-algebra homomorphisms $A_L \to B_L$ via "conjugation."

$$G \times \mathrm{Hom}_L(A_L, B_L) \to \mathrm{Hom}_L(A_L, B_L) \qquad (\sigma, f) \mapsto {}^\sigma f = (\mathrm{Id}_B \otimes \sigma) \circ f \circ (\mathrm{Id}_A \otimes \sigma^{-1})$$

$$\begin{tikzcd} A \otimes_K L \arrow[r, "\mathrm{Id}_A \otimes \sigma^{-1}"] \arrow[rrr, bend right, "{}^\sigma f"'] & A \otimes_K L \arrow[r, "f"] & B \otimes_K L \arrow[r, "\mathrm{Id}_B \otimes \sigma"] & B \otimes_K L \end{tikzcd}$$

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G acts by group automorphisms of $\mathrm{Aut}_L(A_L)$. This means

$$^{\sigma}(f \circ g) = ({}^{\sigma}f) \circ ({}^{\sigma}g) \qquad ({}^{\sigma}f)^{-1} = {}^{\sigma}(f^{-1})$$

Fix a Galois extension L/K, and a K-algebra A. Let A be a K-algebra, and let B be a twisted L/K-form of A. Remember this means we have an L-algebra isomorphism

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We will associate to f a function $\mathrm{Gal}(L/K) \to \mathrm{Aut}_L(A_L)$, which will satisfy a “cocycle condition."

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$$a : \mathrm{Gal}(L/K) \to \mathrm{Aut}_L(A_L) \qquad \sigma \mapsto a_\sigma = f^{-1} \circ {}^\sigma f$$

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A \otimes_K L \arrow[r, "\mathrm{Id} \otimes \sigma^{-1}"] \arrow[rrr, bend right, "{}^\sigma f"'] \arrow[rrrr, bend right=40, "a_\sigma = f^{-1} \circ {}^\sigma f"'] & A \otimes_K L \arrow[r, "f"] & B \otimes_K L \arrow[r, "\mathrm{Id} \otimes \sigma"] & B \otimes_K L \arrow[r, "f^{-1}"] & A \otimes_K L
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The equation $a_{\sigma\tau} = a_\sigma {}^{\sigma} a_\tau$ is called the **cocycle condition**.

Definition

Let G be a group and let X be another group on which G acts by automorphisms.

$$G \times X \to X \qquad (\sigma, x) \mapsto {}^{\sigma}x$$

A **1-cocyle** or **crossed homomorphism** is a map

$$a : G \to X \qquad \sigma \mapsto a_\sigma$$

such that

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Definition

The set of all 1-cocycles from G to X is denoted

$$Z^1(G, X)$$

Remarks

1. A regular homomorphism $G \to X$ would satisfy $a_{\sigma\tau} = a_\sigma a_\tau$, so this is why we say "crossed" homomorphism; it has the extra σ instead, $a_{\sigma\tau} = a_\sigma {}^\sigma a_\tau$.

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2. If you try to write the cocycle condition $a_{\sigma\tau} = a_\sigma {}^\sigma a_\tau$ in more "traditional" notation, it looks something like

$$a(\sigma\tau) = a(\sigma) * \Big(\sigma \cdot a(\tau)\Big)$$

where $\cdot$ is the action of G on X and $*$ is the multiplication in X.

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where $\cdot$ is the action of G on X and $*$ is the multiplication in X.
3. There is a distinguished "unit cocycle"
$$G \to X \qquad \sigma \mapsto 1_X$$
so $Z^1(G, X)$ is never the empty set.

Definition

Let G be a group and X be a group on which G acts by automorphisms. Let $a, b \in Z^1(G, X)$ by 1-cocycles. They are **equivalent** or **cohomologous** if there exists $x \in X$ such that

$$b_\sigma = x^{-1} a_\sigma {}^\sigma x$$

for all $\sigma \in G$. This is an equivalence relation.

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Side note: X need not be abelian. This generalizes group cohomology H^1, but for non-abelian X, there is no H^2, H^3, etc. When X is abelian, H^1 has a group structure, but here H^1 is only a set with a distinguished “unit element."

Summary up to this point: L/K Galois, $G = \mathrm{Gal}(L/K)$, A, B are twisted L/K-forms of each other, $f : A_L \xrightarrow{\cong} B_L$, G acts on $X = \mathrm{Aut}_L(A_L)$. We described

$$\beta : \left\{ \text{isomorphisms } A_L \xrightarrow{\cong} B_L \right\} \to Z^1(G, A) \qquad f \mapsto a = \left(\sigma \mapsto f^{-1} \circ {}^{\sigma} f \right)$$

But: We want to capture information about B, not a particular isomorphism $A_L \to B_L$. The cohomology equivalence gives us a way to remove the dependence on the choice of f, and focus instead on B, using the following lemma.

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But: We want to capture information about B, not a particular isomorphism $A_L \to B_L$. The cohomology equivalence gives us a way to remove the dependence on the choice of f, and focus instead on B, using the following lemma.

Lemma

Let L/K be a Galois extension, let A be a K-algebra, and let B be a twisted L/K-form of A, and let β be the map described above.

1. If f, g are isomorphisms $A_L \to B_L$, then $\beta(f), \beta(g)$ are cohomologous. That is, $\mathrm{im}\,\beta$ is contained in a single equivalence class in $Z^1(G, X)$.
2. If $a, b \in Z^1(G, X)$ are cohomologous and $a \in \mathrm{im}\,\beta$, then $b \in \mathrm{im}\,\beta$. That is, $\mathrm{im}\,\beta$ is an entire equivalence class.

(1) Let f, g be isomorphisms $A_L \to B_L$, and let $a = \beta(f), b = \beta(g)$.

$$\begin{aligned} a : G \to A \qquad & \sigma \mapsto a_\sigma = f^{-1} \circ {}^\sigma f \\ b : G \to A \qquad & \sigma \mapsto b_\sigma = g^{-1} \circ {}^\sigma g \end{aligned}$$

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x^{-1} b_\sigma {}^\sigma x &= \left(g^{-1} f\right)^{-1} \left(b_\sigma\right) \left({}^\sigma(g^{-1} f)\right) \\
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Thus a, b are cohomologous.

Lemma

Let L/K be a Galois extension, let A be a K-algebra, and let B be a twisted L/K-form of A, and let β be the map described above.

1. If f, g are isomorphisms $A_L \to B_L$, then $\beta(f), \beta(g)$ are cohomologous. That is, $\operatorname{im}\beta$ is contained in a single equivalence class in $Z^1(G, X)$.
2. If $a, b \in Z^1(G, X)$ are cohomologous and $a \in \operatorname{im}\beta$, then $b \in \operatorname{im}\beta$. That is, $\operatorname{im}\beta$ is an entire equivalence class.

(2) Let $a, b \in Z^1(G, X)$ be cohomologous and suppose $a = \beta(f)$, so $a_\sigma = f^{-1} \circ {}^\sigma f$ for all $\sigma \in G$. Need to find $g : A_L \xrightarrow{\cong} B_L$ such that $\beta(g) = b$.

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Then

$$b_\sigma = x^{-1} a_\sigma {}^\sigma x = x^{-1} f^{-1} \circ {}^\sigma f {}^\sigma x = (fx)^{-1} \circ {}^\sigma (fx)$$

so $b = \beta(g) = \beta(fx)$.

Interpretation of the lemma: Given a twisted form B of an algebra A, and $X = \mathrm{Aut}_L(A_L)$, the associated cohomology class in $H^1(G, X)$ which is the image of β above does not depend on the choice of isomorphism $f : A_L \to B_L$. Hence we have obtained a map

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To summarize, given a twisted form B:

1. Choose an L-isomorphism $f : A_L \to B_L$.
2. Associated to f is a cocycle $a \in Z^1(G, X)$, which is $a : G \to \mathrm{Aut}_L(X), a \mapsto f^{-1} \circ {}^{\sigma} f$. This cocycle does depend on the choice of f.
3. Take the equivalence class of a to get $[a] \in H^1(G, X)$. While the cocyle a depends on f, the class $[a]$ only depends on B.

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Theorem

The map $TF_L(A) \to H^1(G, X)$ above is a bijection.

Have: Galois extension L/K, $G = \mathrm{Gal}(L/K)$, K-algebra A, $X = \mathrm{Aut}_L(A_L)$, and a cohomology class $[a] \in H^1(G, X)$.

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- Define a "twisted action" of G on A_L, using the cocycle a. We denote A_L with the twisted action by ${}_aA_L$.

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- Even better, the K-algebra A does not depend on the choice of cocycle representative a, it depends only on the cohomology class $[a] \in H^1(G, X)$.

After some work: the process on previous slide gives a map

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Remarks

- If you twist the Galois action by the unit cocycle, you get the same action back.
- If you take the G-fixed points of the usual action on A_L, you get A. This is because G acts on only the L part, and by definition the fixed points of G are precisely K.

Theorem

The map $TF_L(A) \to H^1(G, X)$ above is a bijection.

$$\begin{tikzcd} L\text{-algebras} \\ K\text{-algebras} \arrow[u, "(-)\otimes_K L"] \end{tikzcd} \qquad \begin{tikzcd} & A_L \arrow[dl, rightsquigarrow, "?"'] \arrow[dr, rightsquigarrow, "?"] & \\ ? & A \arrow[u, rightsquigarrow] & ? \end{tikzcd}$$

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Given $x \in GL_2(\mathbb{C})$, conjugating by x gives an automorphism of $M_2(\mathbb{C})$.

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Not enough time today, but RHS can be computed: It has exactly 2 elements.

Corollary

$\mathrm{M}_2(\mathbb{R})$ and $\mathbb{H}$ are the only $\mathbb{C}/\mathbb{R}$-twisted forms of each other.

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Returning to general case, the same Skolem-Noether argument as above shows $\mathrm{Aut}_L(\mathrm{M}_n(L)) \cong \mathrm{PGL}_n(L)$, so

$$TF_L\Big(\mathrm{M}_n(K)\Big) \cong \mathrm{Br}(L/K) \cong H^1\Big(\mathrm{Gal}(L/K), \mathrm{PGL}_n(L)\Big)$$

References, sources, places to learn more

Chapter 2, especially section 2.3 of *Central Simple Algebras and Galois Cohomology* by Gille and Szamuely.

Notes on Galois descent by Keith Conrad at
`https://kconrad.math.uconn.edu/blurbs/galoistheory/galoisdescent.pdf`.

Appendix A section J of *Algebraic Groups: The theory of group schemes of finite type over a field* by J.S. Milne. (Note: make sure it's the 2017 edition, not the 2015 version which is on Milne's website, the section numbering in appendix A is significantly changed between editions.)

For an introduction to non-abelian group cohomology, see section 1.3 of *Algebraic Groups and Number Theory* by Platonov and Rapinchuk.

Notes I wrote for myself on Galois descent at `https://users.math.msu.edu/users/ruiterj2/math/Documents/Notes%20and%20talks/Galois%20descent.pdf`.