Lie algebras and their root systems
A case study in the classification of Lie algebras

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Calvin College Mathematics Colloquium
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Historical background

The matrix algebra $\mathfrak{sl}(3, \mathbb{C})$ is the set of $3 \times 3$ matrices with complex entries and trace zero.

$$\mathfrak{sl}(3, \mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a + e + i = 0 \right\}$$

This is a vector space, because:

- The zero matrix has trace zero.
- A scalar multiple of a traceless matrix is traceless.
- The sum of two traceless matrices is traceless.
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Basis matrices

Definition

The matrix $e_{ij}$ is a matrix with a one in the $ij$th place and zeroes elsewhere.

Usual basis for $\mathfrak{sl}(3, \mathbb{C})$:

$$\{e_{11} - e_{22}, e_{22} - e_{33}\} \cup \{e_{ij} : i \neq j\}$$

Direct sum expression for $\mathfrak{sl}(3, \mathbb{C})$:

$$\mathfrak{sl}(3, \mathbb{C}) = \text{span}\{e_{11} - e_{22}, e_{22} - e_{33}\} \oplus \bigoplus_{i \neq j} \text{span}\{e_{ij}\}$$
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Is it a ring?

We just determined that $\mathfrak{sl}(3, \mathbb{C})$ is closed under matrix addition. Is it closed under matrix multiplication?

\[
\begin{pmatrix}
-2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
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A new binary operation

Definition

Let $x, y$ be matrices. The **bracket** of $x$ and $y$ is $[A, B] := AB - BA$.

Proposition

Let $A, B$ be matrices. Then $[A, B]$ has trace zero.

Proof.

From linear algebra, we know that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}$ is linear. Hence $\text{tr}[A, B] = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$. \qed

Corollary

$A, B \in \text{sl}(3, \mathbb{C}) \implies [A, B] \in \text{sl}(3, \mathbb{C})$
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**Corollary**

\( A, B \in \text{sl}(3, \mathbb{C}) \iff [A, B] \in \text{sl}(3, \mathbb{C}) \)
Proposition (Bracket is **alternating**)

*Let $A$ be a matrix. Then $[A, A] = 0$.***

**Proof.**

$$[A, A] = A^2 - A^2 = 0$$

---

Proposition (Bracket is **antisymmetric**)

*Let $A, B$ be matrices. Then $[A, B] = -[B, A]$.***

**Proof.**

$$[A, B] = AB - BA = -BA + AB = -(BA - AB) = -[B, A]$$

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**Corollary**

*Let $A, B \in \text{sl}(3, \mathbb{C})$. Then $[A, B] = [B, A] \iff [A, B] = 0$.***
Properties of bracket (1)

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Corollary  
Let $A, B \in \text{sl}(3, \mathbb{C})$. Then $[A, B] = [B, A] \iff [A, B] = 0$. 
Proposition (Bracket is \textit{bilinear})

Let $A$, $B$, $C$ be matrices and let $\lambda$ be a scalar. Then

$$[A + B, C] = [A, C] + [B, C]$$

$$[C, A + B] = [C, A] + [C, B]$$

$$[\lambda A, C] = [A, \lambda C] = \lambda [A, C]$$

Proof.

Apply the fact that matrix multiplication distributes over matrix addition and commutes with scalar multiplication.
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Proposition (Jacobi identity)

Let $A, B, C$ be matrices. Then


Proof.

Remember that matrix multiplication distributes over matrix addition. Expand all the terms out, get a lot of terms like $ABC$, and see that they all cancel in pairs.
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Brackets of basis matrices

**Definition**

The *Kronecker delta* function is the function

\[ \delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases} \]

**Lemma**

\[ [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} \]
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**Definition**

A **Lie algebra** is a vector space $L$ over a field $F$ with a bilinear form $[\cdot, \cdot] : L \times L \to L$ that satisfies $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in L$.

As a consequence of this definition, the bracket must be antisymmetric.
Brief excursion: connection to Lie groups

**Definition**

\[ \text{SL}(3, \mathbb{C}) = \{ A \in M(3, \mathbb{C}) : \det A = 1 \} \]

\[
\begin{align*}
\alpha(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 + 1 & t^3 + 2t \\ 0 & t & t^2 + 1 \end{pmatrix} \\
\alpha(0) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\det(\alpha(t)) &= 1 \implies \alpha(t) \in \text{SL}(3, \mathbb{C}) \\
\alpha'(t) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2t & 3t^2 + 2 \\ 0 & 1 & 2t \end{pmatrix} \\
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Definition

Let $V, W$ be vector spaces. A **linear map** is a function $\phi : V \to W$ such that $\phi(av_1 + v_2) = a\phi(v_1) + \phi(v_2)$.

For each $x \in \mathfrak{sl}(3, \mathbb{C})$, we can associate to $x$ a map from $L \to \mathbb{C}$ by $y \to [x, y]$. This map associated to $x$ is called $\text{ad } x$.

Proposition

*For* $x \in \mathfrak{sl}(3, \mathbb{C})$, $\text{ad } x$ *is linear.*

Proof.

Follows from linearity of the bracket in the 2nd entry.
**Linear maps and ad \( x \)**

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The subalgebra of diagonal matrices

\[ H = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} : d_1 + d_2 + d_3 = 0 \right\} \subset \mathfrak{sl}(3, \mathbb{C}) \]

Proposition

Let \( A, B \in H \). Then \([A, B] = 0\), hence \([A, B] \in H\).

Proof.

\( A, B \) are diagonal, so \( AB = BA \), so \([A, B] = AB - BA = 0\).

Definition

A subalgebra of a Lie algebra \( L \) is a vector subspace \( H \) that is closed under the bracket \((x, y \in H \implies [x, y] \in H)\).
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Ideals: $sl(3, \mathbb{C})$ is simple

**Definition**

An **ideal** of a Lie algebra $L$ is a vector subspace $I$ such that for $x \in L$, $a \in I$, $[x, a] \in I$.

Note that every ideal is a subalgebra, but not every subalgebra is an ideal.

**Proposition**

$sl(3, \mathbb{C})$ is simple, that is, it has no nonzero proper ideals.

**Definition**

A Lie algebra $L$ is **semisimple** if it can be written as a direct sum of simple Lie algebras.
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Sketch of proof that $\mathfrak{sl}(n, \mathbb{C})$ is simple

Suppose $I$ is a nonzero proper ideal with $\nu \neq 0$, $\nu \in I$. Let

$$\nu = \sum_{i \neq j} c_{ij} e_{ij} + \sum_{i=1}^{n} d_i e_{ii}$$

where $c_{ij}, d_i \in \mathbb{C}$. Then

$$\begin{align*}
[e_{kl}, [e_{kl}, \nu]] &= -2c_{lk}^l e_{kl} \\
[e_{kl}, \nu] &= (d^l - d^k) e_{kl}
\end{align*}$$

From this it follows that $e_{ij} \in I$ for some $i \neq j$. One can then show that if an ideal of $\mathfrak{sl}(n, \mathbb{C})$ contains some $e_{ij}$, then $I = \mathfrak{sl}(n, \mathbb{C})$. 

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From this it follows that \( e_{ij} \in I \) for some \( i \neq j \). One can then show that if an ideal of \( \mathfrak{sl}(n, \mathbb{C}) \) contains some \( e_{ij} \), then \( I = \mathfrak{sl}(n, \mathbb{C}) \).
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where $c_{ij}, d_{i} \in \mathbb{C}$. Then

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where $c_{ij}, d_i \in \mathbb{C}$. Then

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From this it follows that $e_{ij} \in I$ for some $i \neq j$. One can then show that if an ideal of $\mathfrak{sl}(n, \mathbb{C})$ contains some $e_{ij}$, then $I = \mathfrak{sl}(n, \mathbb{C})$. 
Let $h$ be the diagonal matrix with entries $d_1, d_2, d_3$. Then

$$\text{ad } h(e_{ij}) = [h, e_{ij}] = he_{ij} - e_{ij}h = d_i e_{ij} - d_j e_{ij} = (d_i - d_j)e_{ij}$$

**Definition**

A linear map $\phi : V \to V$ is **diagonalizable** if there is a basis of $V$ consisting of eigenvectors for $\phi$.

Specifically, $\text{ad } h$ is diagonalizable.
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Proposition (Lemma 16.7)

Let $x_1, \ldots, x_k : V \rightarrow V$ be diagonalizable linear transformations. There is a basis of $V$ the simultaneously diagonalizes all $x_i$ if and only if each pair $x_i, x_j$ commutes.

Application: If $h_1, h_2, \ldots, h_k$ are diagonal elements of $\mathfrak{sl}(3, \mathbb{C})$, then $\text{ad } h_1, \text{ad } h_2, \ldots, \text{ad } h_k$ are all simultaneously diagonalized in the usual basis, so they all pairwise commute.
Let $h = \text{diag}(d_1, d_2, d_3)$, and let $L_{ij} = \text{span}\{e_{ij}\}$. Based on our computations so far, we have

$$H \subset \{x \in \text{sl}(3, \mathbb{C}) : \text{ad} h(x) = 0, \text{ for all } h \in H\}$$

$$L_{ij} \subset \{x \in \text{sl}(3, \mathbb{C}) : \text{ad} h(x) = (d_i - d_j)x, \text{ for all } h \in H\}$$

Actually, we can replace $\subset$ with $=$, this takes a bit more work. Define $\epsilon_i : H \rightarrow \mathbb{C}$ by $\epsilon_i(\text{diag}(d_1, d_2, d_3)) = d_i$. Then we can rewrite this as

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This makes $L_{ij}$ a **root space** with associated **root** $(\epsilon_i - \epsilon_j)$. 
Definition

A **root** is a linear map $\alpha : H \to \mathbb{C}$ such that

$$\{ x \in \mathfrak{sl}(3, \mathbb{C}) : \text{ad } h(x) = \alpha(h)x \text{ for all } h \in H \}$$

is a nonzero subspace of $\mathfrak{sl}(3, \mathbb{C})$.

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Root space decomposition

Proposition

$\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$ is the entire set of roots for $\mathfrak{sl}(3, \mathbb{C})$.

Recall:

$\mathfrak{sl}(3, \mathbb{C}) = \text{span}\{e_{11} - e_{22}, e_{22} - e_{33}\} \oplus \bigoplus_{i \neq j} \text{span}\{e_{ij}\}$

Now that we have the language of roots, we can write this as

$\mathfrak{sl}(3, \mathbb{C}) = H \oplus \bigoplus_{i \neq j} L_{ij} = H \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha}$

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This is called the root space decomposition.
A **root system** is a subset $R$ of a real inner-product space $E$ satisfying:

(R1) $R$ is finite, it spans $E$, and it does not contain 0.
(R2) If $\alpha \in R$, then the only scalar multiples of $\alpha$ in $R$ are $\pm \alpha$.
(R3) If $\alpha \in R$, then the reflection $s_\alpha$ permutes $R$.
(R4) If $\alpha, \beta \in R$, then $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$.

**Proposition**

Let $L$ be a complex semisimple Lie algebra, and let $\Phi$ be a set of roots of $L$. Then $\Phi$ is a root system.
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Root diagram for $\mathfrak{sl}(3, \mathbb{C})$

$\alpha = \epsilon_1 - \epsilon_2 \quad \beta = \epsilon_2 - \epsilon_3$
Theorem (Cartan)

*Up to isomorphism, there is just one complex semisimple Lie algebra for each root system.*

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*With five exceptions, every finite-dimensional simple Lie algebra over \( \mathbb{C} \) is isomorphic to one of the classical Lie algebras \( \mathfrak{sl}(n, \mathbb{C}) \), \( \mathfrak{so}(n, \mathbb{C}) \), and \( \mathfrak{sp}(2n, \mathbb{C}) \).

The five exceptional Lie algebras are known as \( \mathfrak{e}_6 \), \( \mathfrak{e}_7 \), \( \mathfrak{e}_8 \), \( \mathfrak{f}_4 \), and \( \mathfrak{g}_2 \).
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