The Group of Extensions in rep $Q$

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Throughout, we will assume $Q$ is a given acyclic quiver, and we will work in the category of representations of $Q$ over a given base field $k$. In §1, we define $\mathcal{E}(M, N)$, the extensions of $M$ by $N$ for two representations $M, N$, and define a binary operation on $\mathcal{E}(M, N)$. In §2, we show that this operation makes $\mathcal{E}(M, N)$ an abelian group. In §3, we describe an isomorphism of $\mathcal{E}(M, N)$ with $\text{Ext}^1(M, N)$.

1 Definition of $\mathcal{E}(M, N)$

Much of this discussion of extensions parallels extensions in the category of groups or $R$-modules. For some discussion of the $R$-module version, see Weibel’s Introduction to Homological Algebra [2].

**Definition 1.1.** Let $M, N \in \text{rep } Q$. An **extension** $\zeta$ of $M$ by $N$ is a short exact sequence of the form

\[ 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0 \]

**Definition 1.2.** Two extensions $\zeta, \zeta'$ of $M$ by $N$ are **equivalent** if there is a commutative diagram

\[
\begin{array}{c}
0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0 \\
\downarrow \text{Id}_N \quad \downarrow \phi \quad \downarrow \text{Id}_M \\
0 \longrightarrow N' \longrightarrow E' \longrightarrow M \longrightarrow 0
\end{array}
\]

Note that by the Five Lemma, any such $\phi$ is an isomorphism.

**Definition 1.3.** The **group of extensions** $\mathcal{E}(M, N)$ of $M$ by $N$ is the set of equivalence classes of extensions of $M$ by $N$. (We haven’t yet defined a group structure on this set, but we will.)

Our first objective is to define an abelian group structure on $\mathcal{E}(M, N)$. Our second objective is to show that $\mathcal{E}(M, N) \cong \text{Ext}^1(M, N)$ as abelian groups, after defining $\text{Ext}^1(M, N)$.

First, we define a binary operation on extensions. Then we will show that it is well defined on equivalence classes of extensions.

**Definition 1.4.** Let $M, N \in \text{rep } Q$, and let $\zeta, \zeta'$ be the following extensions of $M$ by $N$.

1
Define
\[
E'' = \{(x, x') \in E \oplus E' : g(x) = g'(x')\}
\]
\[
D'' = \{(f(n), -f'(n)) \in E \oplus E' : n \in N\}
\]

Then define \( F = E''/D'' \). Finally, the extension \( \zeta + \zeta' \) is defined to be
\[
0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0
\]

where \( f''(n) = (f(n), 0) \) and \( g''(x, x') = g(x) \).

This addition is called the **Baer sum**, at least in the context of \( R \)-modules.

**Lemma 1.1.** The definition above makes sense. More specifically,
1. \( E'' \) and \( D'' \) are representations of \( Q \), and \( D'' \) is a subrepresentation of \( E'' \).
2. \( f'' \) and \( g'' \) are well defined and are morphisms in \( \text{rep} \, Q \).
3. The sequence involving \( F \) is exact.

**Proof.** (1) \( E'' \) is a representation of \( Q \) by Exercise 1.8 in [1]. \( D'' \) is a subrepresentation of \( E \oplus E' \) by Exercise 1.9 in [1]. Also, \( D'' \subset E'' \), since
\[
g(f(n)) = 0 = g'(-f'(n)) \quad \forall n \in N
\]

(2) It is clear that \( f'' \) is well defined and is a morphism. We check that \( g'' \) is well defined by showing that it vanishes on \( D'' \).
\[
g''(f(n), -f'(n)) = gf(n) = 0
\]

It is clear that \( g'' \) is a morphism, since \( g \) is a morphism.

(3) We check that the sequence involving \( F \) is exact. First, we show injectivity of \( f'' \). If \( n \in \ker f'' \), then there exists \( n' \in N \) such that
\[
f''(n) = (f(n), 0) = (f(n'), -f'(n')) \implies 0 = -f'(n')
\]

which implies \( n' = 0 \) by injectivity of \( f' \). Then \( f(n') = 0 \) so \( f(n) = 0 \) as well, so \( n = 0 \) by injectivity of \( f \). Thus \( f'' \) is injective. Now we show \( g'' \) is surjective. Let \( m \in M \). By surjectivity of \( g, g' \), there exist \( x \in E, x' \in E' \) so that \( g(x) = g'(x') = m \). Then
\[
g''(x, x') = g(x) = m
\]
so \( g'' \) is surjective. Finally, we show that \( \ker g'' = \text{im} f'' \). It is easy to see that \( \text{im} f'' \subset \ker g'' \), since
\[
g'' f''(n) = g''(f(n), 0) = g f(n) = 0
\]
We need to check that \( \ker g'' \subset \text{im} f'' \). Let \((x, x') \in \ker g''\), so \(0 = g(x) = g'(x')\). By exactness of \( \zeta, \zeta', x \in \text{im} f \) and \( x' \in \text{im} f' \), so there exist \( n, n' \in N \) such that \( f(n) = x \) and \( f'(n') = x' \). Then
\[
f''(n + n') = (f(n + n'), 0)
= (f(n) + f(n'), 0)
= (f(n) + f(n'), 0) + (f(-n'), -f'(-n'))
= (f(n), -f'(n'))
= (x, x')
\]
thus \( \ker g'' \subset \text{im} f'' \). \( \square \)

With this lemma in hand, we know that our addition is well defined on exact sequences. Now we need to check that it induces a well defined addition on \( E(M, N) \).

**Definition 1.5.** Let \([\zeta], [\zeta']\) be equivalence classes of extensions in \( E(M, N) \). We define addition in \( E(M, N) \) by
\[
[\zeta] + [\zeta'] = [\zeta + \zeta']
\]

**Lemma 1.2.** This addition on \( E(M, N) \) is well defined.

**Proof.** We need to show that if \([\gamma] = [\zeta]\) and \([\gamma'] = [\zeta']\), then \([\gamma + \gamma'] = [\zeta + \zeta']\). Let \( \zeta, \zeta', \gamma, \gamma', \zeta + \zeta', \gamma + \gamma' \) be the following extensions.

\[
\begin{array}{ccccccccc}
\zeta & 0 & \to & N & \tof & E & \to g & M & \to 0 \\
\zeta' & 0 & \to & N & \tof' & E' & \to g' & M & \to 0 \\
\zeta + \zeta' & 0 & \to & N & \to f' & E' & \to g' & M & \to 0 \\
\gamma & 0 & \to & N & \to h & S & \to j & M & \to 0 \\
\gamma' & 0 & \to & N & \to h' & S' & \to j' & M & \to 0 \\
\gamma + \gamma' & 0 & \to & N & \to h' & S' & \to j' & M & \to 0 \\
\end{array}
\]

where \( F = E''/D'' \) and \( T = S''/R'' \). Because \([\gamma] = [\zeta]\) and \([\gamma'] = [\zeta']\), there is are isomorphisms \( \phi : E \to S \) and \( \phi' : E' \to S' \) making the following diagrams commute.
Then we have an isomorphism $\phi \oplus \phi' : E \oplus E' \to S \oplus S'$ given by $(x, x') \mapsto (\phi(x), \phi'(x'))$. We claim that $\phi \oplus \phi'$ induces an isomorphism $F \to T$ giving an equivalence $[\zeta + \zeta'] = [\gamma + \gamma']$.

First, we claim that $\phi \oplus \phi'|_{D''} : D'' \to S \oplus S''$ has image contained in $S''$. This follows from the right side commutative squares. For $(x, x') \in E''$, we have $g(x) = g'(x')$, so
\[ \phi \oplus \phi'(x, x') = (\phi(x), \phi'(x')) \in S'' \text{ because } j\phi(x) = g(x) = g'(x') = j'\phi'(x') \]

We also claim $S''$ is contained in the image. For $(y, y') \in S''$, we have $j(y) = j'(y')$, so $(\phi^{-1}(y), (\phi')^{-1}(y')) \in E''$ because $g\phi^{-1}(y) = j(y) = j'(y') = g'(\phi')^{-1}(y')$. Thus
\[ \phi \oplus \phi'(\phi^{-1}(y), (\phi')^{-1}(y')) = (y, y') \]

so $S''$ is the image. Now we claim that $\phi \oplus \phi'|_{D''} : D'' \to S \oplus S''$ has image $R''$. Containment and surjection follow from left side commutative squares, as seen below.
\[ \phi \oplus \phi'(f(n), -f'(n)) = (\phi f(n), -\phi' f'(n)) = (h(n), -h'(n)) \in R'' \]

So $\phi \oplus \phi'$ restricts to isomorphisms $E'' \to S''$ and $D'' \to R''$. Thus $\phi \oplus \phi'$ induces an isomorphism $E''/D'' \to S''/R''$, that is, $F \to T$, making the following diagram commute.

\[ \begin{array}{c}
0 \longrightarrow N \overset{f''}{\longrightarrow} F \overset{g''}{\longrightarrow} M \longrightarrow 0 \\
\text{Id} \downarrow \quad \phi \downarrow \quad \text{Id} \\
0 \longrightarrow N \overset{h''}{\longrightarrow} T \overset{j''}{\longrightarrow} M \longrightarrow 0
\end{array} \]

Thus $[\zeta + \zeta'] = [\gamma + \gamma']$. 

\[ \square \]

### 2 Verifying Group Axioms

**Proposition 2.1.** $\mathcal{E}(M, N)$ is an abelian group with this addition.

We break this into several separate propositions, so that the reader can easily find the proof of a particular property.

**Proposition 2.2.** The split extension is an additive identity in $\mathcal{E}(M, N)$.

**Proof.** First, we claim that the equivalence class of the sequence $[\alpha]$, depicted below,
acts as an additive identity in \( E(M, N) \). Let \([\zeta] \in E(M, N)\) with representative
\[
0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0
\]
Then we set
\[
E'' = \{(e, (n, m)) \in E \oplus (N \oplus M) : g(e) = \pi(n, m)\}
\]
\[
D'' = \{(f(n), -\iota(n)) \in E \oplus (N \oplus M) : n \in N\}
\]
\[
F = E'' / D''
\]
Then \([\zeta + \alpha]\) is represented by
\[
0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0
\]
where \(f''(n) = (f(n), (0, 0)) = (f(n), (0, 0))\) and \(g''(e, (n, m)) = g(e)\). We claim that \([\zeta + \alpha] = [\zeta]\). To show this equivalence, we exhibit an explicit equivalence of extensions. Define \(\phi : E \to F\) by \(e \mapsto (e, (0, g(e)))\) and \(\psi : F \to E\) by \((e, (n, m)) \mapsto e + f(n)\). It is straightforward to see that \(\phi\) is well defined, maps into \(F\), and is a morphism. We check that \(\psi\) is well defined by checking that it vanishes on \(D''\).
\[
\psi(f(n), (-n, 0)) = f(n) + f(-n) = 0
\]
It is clear that \(\psi\) maps into \(E\) and is a morphism. Now we show that \(\phi, \psi\) are inverse.
\[
\psi(\phi(e)) = \phi((0, g(e))) = e + f(0) = e
\]
\[
\phi(\psi(e, (n, m))) = \phi(e + f(n)) = (e + f(n), (0, g(e + f(n))))
\]
Finally, we check that the following diagram commutes.
\[
\begin{array}{ccc}
0 & \longrightarrow & N \\
\downarrow{\text{Id}} & & \downarrow{\phi} \\
0 & \longrightarrow & F \\
\downarrow{\text{Id}} & & \downarrow{\text{Id}} \\
0 & \longrightarrow & M \\
\end{array}
\]
\[
\phi f(n) = (f(n), (0, 0)) = f''(n)
\]
\[
g'' \phi(e) = g''(e, (0, g(e))) = g(e)
\]
Thus \([\zeta + \alpha] = [\zeta]\), so \([\alpha]\) is an identity in \(E(M, N)\).

**Proposition 2.3.** Addition in \(E(M, N)\) is associative.

**Proof.** Let \(\zeta_i\) for \(i = 1, 2, 3\) be extensions of \(M\) by \(N\).
\[
\zeta_i \quad 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0
\]
Let
\[
E_{ij} = \{(x_i, x_j) \in E_i \oplus E_j : g_i(x_i) = g_j(x_j)\}
\]
\[
D_{ij} = \{(f_i(n), -f_j(n)) : n \in N\}
\]
\[
F_{ij} = E_{ij}/D_{ij}
\]
and define \( f_{ij} : N \to F_{ij}\) by \( f_{ij}(n) = (f_i(n), 0)\) and \( g_{ij} : F_{ij} \to M\) by \( g_{ij}(x_i, x_j) = g_i(x_i)\).
That is, \( \zeta_i + \zeta_j\) is represented by
\[
0 \longrightarrow N \xrightarrow{f_{ij}} F_{ij} \xrightarrow{g_{ij}} M \longrightarrow 0
\]
Then set
\[
E_{(ij)k} = \left\{ (x_i, x_j, x_k) \in F_{ij} \oplus E_k : g_{ij}(x_i, x_j) = g_k(x_k) \right\}
\]
\[
D_{(ij)k} = \{(f_{ij}(n), f_k(n)) : n \in N\}
\]
\[
F_{(ij)k} = E_{(ij)k}/D_{(ij)k}
\]
\[
E_{(jk)} = \left\{ (x_i, x_j, x_k) \in E_i \oplus F_{jk} : g_i(x_i) = g_{jk}(x_j, x_k) \right\}
\]
\[
D_{(jk)} = \{(f_i(n), f_{jk}(n)) : n \in N\}
\]
\[
F_{(jk)} = E_{(jk)}/D_{(jk)}
\]
and let \( f_{(ij)k}, g_{(ij)k}\) and \( f_{(jk)}, g_{(jk)}\) so that \( (\zeta_i + \zeta_j) + \zeta_k\) and \( \zeta_i + (\zeta_j + \zeta_k)\) are respectively represented by
\[
0 \longrightarrow N \xrightarrow{f_{(ij)k}} F_{(ij)k} \xrightarrow{g_{(ij)k}} M \longrightarrow 0
\]
\[
0 \longrightarrow N \xrightarrow{f_{(jk)}} F_{(jk)} \xrightarrow{g_{(jk)}} M \longrightarrow 0
\]
We care about the case \( i = 1, j = 2, k = 3.\) We define \( \Psi : E_{(12)3} \to E_{1(23)}\) by
\[
\Psi \left( (x_1, x_2), x_3 \right) = (x_1, (x_2, x_3))
\]
First, we need to check that this is well defined; for this it is sufficient to check that \( \Psi\) vanishes on the zero element of \( E_{(12)3}\). We can represent the zero element of \( E_{(12)3}\) by \((0, 0, 0), 0\), which clearly goes to the zero element of \( E_{1(23)}\) under \( \Psi\), so it is well defined.

We also need to check that the image is contained in \( E_{1(23)}\). For \( (x_1, x_2), x_3 \in E_{(12)3}\) we have \( g_{12}(x_1, x_2) = g_3(x_3),\) so \( g_1(x_1) = g_2(x_2) = g_3(x_3)\) (because \( (x_1, x_2) \in E_{12}\)). Thus \( g_1(x_1) = g_{23}(x_2, x_3),\) so the image is contained in \( E_{1(23)}\) as desired.

Now we claim that \( \Psi\) maps \( D_{(12)3}\) to \( D_{1(23)}\). For \( n \in N,\)
\[
\Psi \left( f_{12}(n), f_3(n) \right) = \Psi \left( (f_1(n), 0), f_3(n) \right) = \Psi \left( (0, f_2(n)), f_3(n) \right) = (0, (f_2(n), f_3(n)) \in D_{1(23)}
\]
Thus \( \Psi\) induces a morphism \( F_{(12)3} \to F_{1(23)}\). Finally, we need to check that the following diagram commutes.
\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{f_{(ij)k}} F_{(ij)k} \xrightarrow{g_{(ij)k}} M \rightarrow 0 \\
\downarrow \text{Id} \quad \downarrow \Phi \quad \downarrow \text{Id} \\
0 \rightarrow N \xrightarrow{f_{(jk)}'} F_{(jk)}' \xrightarrow{g_{(jk)k}} M \rightarrow 0
\end{array}
\]

Note that \(f_{(12)3}(n) = (f_{12}(n), 0)\) and \(f_{1(23)}(n) = (f_1(n), (0, 0))\) and \(g_{(12)3}(x_1, x_2, x_3) = g_{1(23)}(x_1, (x_2, x_3)) = g_1(x_1)\).

\[
\Psi f_{(12)3}(n) = \Psi((f_{12}(n), 0)) = \Psi((f_1(n), 0, 0)) = (f_1(n), (0, 0)) = f_{1(23)}(n)
\]

Thus the diagram commutes and \(\Psi\) is an equivalence of extensions. (Note that by the Five Lemma, we any morphism making this commute is an isomorphism.)

\textbf{Proposition 2.4.} \(\text{If } [\zeta] \in \mathcal{E}(M, N), \text{ there is an extension } -\zeta \text{ so that } [\zeta] + [-\zeta] = [0].\)

\textbf{Proof.} Let \(\zeta\) be the extension

\[
0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0
\]

Then we have another extension, which we call \(-\zeta\),

\[
0 \rightarrow N \xrightarrow{-f} E \xrightarrow{g} M \rightarrow 0
\]

We claim that \([\zeta] + [-\zeta] = [0],\) that is, \(\zeta + (-\zeta)\) is equivalent to the split extension. Let’s describe \(\zeta + (-\zeta)\). It is

\[
0 \rightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \rightarrow 0
\]

where

\[
E'' = \{(x, x') \in E \oplus E' : g(x) = g(x')\}
\]

\[
D'' = \{(f(n), f(n)) : n \in N\}
\]

\[
F = E''/D''
\]

and \(f''(n) = (f(n), 0)\) and \(g''(x, x') = g(x) = g(x')\). We define a morphism \(\phi : N \oplus M \rightarrow F\) as follows. For \(m \in M\), there exists \(x \in E\) so that \(g(x) = m\) by surjectivity of \(g\). We define \(\phi(n, m) = (f(n) + x, x)\). We need to check that this is well defined. Suppose \(x, x'\) are two different lifts of \(m\). Then \(x - x' \in \ker g\), so there exists \(n \in N\) with \(f(n') = x - x'\), so for \(n \in N\), we have

\[
(f(n) + x, x') - (f(n) + x', x) = (x - x', x - x') = (f(n'), f(n')) \in D''
\]

which implies that \(f(n) + x, x') = (f(n) + x', x')\). Thus \(\phi\) is well defined. We verify that the diagram below commutes, and thus \(\phi\) is an isomorphism, and we have \(\zeta + (-\zeta) = [0].\)
Proposition 2.5. Addition in $E(M, N)$ is commutative.

Proof. Let $\zeta_i$ for $i = 1, 2$ be extensions of $M$ by $N$.

$$\zeta_i \quad 0 \to N \overset{f_i}{\to} E_i \overset{g_i}{\to} M \to 0$$

Let

$$E_{ij} = \{(x_i, x_j) \in E_i \oplus E_j : g_i(x_i) = g_j(x_j)\}$$
$$D_{ij} = \{(f_i(n), -f_j(n)) : n \in N\}$$
$$F_{ij} = E_{ij}/D_{ij}$$

and define $f_{ij} : N \to F_{ij}$ by $f_{ij}(n) = (f_i(n), 0)$ and $g_{ij} : F_{ij} \to M$ by $g_{ij}(x_i, x_j) = g_i(x_i)$. That is, $\zeta_i + \zeta_j$ is represented by

$$0 \to N \overset{f_{ij}}{\to} F_{ij} \overset{g_{ij}}{\to} M \to 0$$

We have the obvious isomorphism $\Psi : E_{12} \to E_{21}$ given by $(x_1, x_2) \mapsto (x_2, x_1)$. $\Psi$ restricts to an isomorphism $D_{12} \to D_{21}$, because

$$\Psi(f_1(n), -f_2(n)) = (-f_2(n), f_1(n)) = (f_2(-n), -f_1(-n))$$

Thus $\Psi$ induces an isomorphism $F_{12} \to F_{21}$, and we verify that the following diagram commutes.

$$0 \to N \overset{f_{12}}{\to} F_{12} \overset{g_{12}}{\to} M \to 0$$

$$\Psi f_{12}(n) = \Psi(f_1(n), 0) = (0, f_1(n)) = (0, f_1(n)) + (f_2(n), -f_1(n)) = (f_2(n), 0) = f_{21}(n)$$
$$g_{21}(x_1, x_2) = g_{21}(x_2, x_1) = g_2(x_2) = g_1(x_1) = g_{12}(x_1, x_2)$$

This completes the proof that $E(M, N)$ is an abelian group.
3 Isomorphism $\mathcal{E}(M, N) \cong \text{Ext}^1(M, N)$

Now that we know that $\mathcal{E}(M, N)$ is an abelian group, we can describe it’s relationship with the functor $\text{Ext}^1$. First we recall the definition of $\text{Ext}^1$. Remember that every representation of $Q$ has a two-term projective resolution.

**Definition 3.1.** Let $M \in \text{rep } Q$. Let

$$
0 \longrightarrow P_1 \overset{f}{\longrightarrow} P_2 \overset{g}{\longrightarrow} M \longrightarrow 0
$$

be a projective resolution of $M$. Then for $N \in \text{rep } Q$, we define $\text{Ext}^1(M, N)$ as the cokernel of $f^*$ in the following sequence.

$$
0 \longrightarrow \text{Hom}(M, N) \overset{g^*}{\longrightarrow} \text{Hom}(P_0, N) \overset{f^*}{\longrightarrow} \text{Hom}(P_1, N)
$$

That is, $\text{Ext}^1(M, N) := \text{Hom}(P_1, N)/\text{im } f^*$. In particular, the following sequence is exact.

$$
0 \longrightarrow \text{Hom}(M, N) \overset{g^*}{\longrightarrow} \text{Hom}(P_0, N) \overset{f^*}{\longrightarrow} \text{Hom}(P_1, N) \longrightarrow \text{Ext}^1(M, N) \longrightarrow 0
$$

**Note:** It is not clear from this definition why $\text{Ext}^1(M, N)$ does not depend on the choice of projective resolution. However, there are “standard” results in homological algebra that it does not. That is, $\text{Ext}^1(M, N)$ depends on only $M$ and $N$.

Note that $\text{Ext}^1(M, N)$ is a $k$-vector space, so it is also an abelian group. Now we will show that it is isomorphic to $\mathcal{E}(M, N)$ as an abelian group.

**Definition 3.2.** Fix a projective resolution $\mathcal{P}$ of $M$.

$$
0 \longrightarrow P_1 \overset{f}{\longrightarrow} P_0 \overset{g}{\longrightarrow} M \longrightarrow 0
$$

Let $[\zeta] \in \mathcal{E}(M, N)$ with representative short exact sequence $\zeta$.

$$
0 \longrightarrow N \overset{s}{\longrightarrow} E \overset{t}{\longrightarrow} M \longrightarrow 0
$$

Since $P_0$ is projective and $t$ is surjective, there exists $a : P_0 \rightarrow E$ making the following diagram commute (by the universal property of projectives).

$$
\begin{array}{ccc}
0 & \longrightarrow & P_1 \\
\downarrow a & & \downarrow \text{id} \\
0 & \longrightarrow & N
\end{array}
\quad
\begin{array}{ccc}
& & \\
& & \\
\quad & 0 & \longrightarrow M
\end{array}
$$

By commutativity of this diagram, $taf = gaf = 0$, that is, $af : P_0 \rightarrow \ker t = \text{im } s$. Since $s : N \rightarrow \text{im } s$ is surjective and $P_1$ is projective, again using the universal property of projectives, there is $b : P_1 \rightarrow N$ making the following diagram commute.

$$
\begin{array}{ccc}
0 & \longrightarrow & P_1 \\
\downarrow b & & \downarrow a \\
0 & \longrightarrow & N
\end{array}
\quad
\begin{array}{ccc}
& & \\
& & \\
\quad & 0 & \longrightarrow E \overset{t}{\longrightarrow} M
\end{array}
\quad
\begin{array}{ccc}
& & \\
& & \\
\quad & 0 & \longrightarrow M
\end{array}
$$
Recall that $\text{Ext}^1(M, N) = \text{Hom}(P_1, N)/\text{im } f^*$, so $b$ is a representative of some class $\overline{b} \in \text{Ext}^1(M, N)$. We define $\Phi_P : \mathcal{E}(M, N) \to \text{Ext}^1(M, N)$ by $\Phi_P[\zeta] = \overline{b}$.

To save space, we'll just denote $\Phi_P$ by $\Phi$. There is some homological algebra behind the scenes which says that the choice of $P$ doesn't really matter, but we won't concern ourselves with that.

**Proposition 3.1.** $\Phi$ is an isomorphism $\mathcal{E}(M, N) \to \text{Ext}^1(M, N)$.

We prove the following four statements, in this order.

1. $\Phi$ does not depend on the choice of $a$ and $b$.
2. If $[\zeta] = [\zeta']$, then $\Phi[\zeta] = \Phi[\zeta']$.
3. $\Phi$ is a group homomorphism.
4. $\Phi$ is bijective.

**Proposition 3.2.** $\Phi$ does not depend on the choice of $a$ and $b$.

**Proof.** Suppose that when computing $\Phi[\zeta]$, we choose $a_1 : P_0 \to E$ and $b_1 : P_1 \to E$. Then we recompute, and choose different morphisms $a_2 : P_0 \to E$ and $b_2 : P_1 \to E$. We need to verify that $\overline{b_1} = \overline{b_2}$ in $\text{Ext}^1(M, N)$. That is, we need to show that $b_2 - b_1 \in \text{im } f^*$.

Since $ta_1 = ta_2 = g$, we have $t(a_2 - a_1) = 0$. Thus $a_2 - a_1 : P_0 \to E$ has image contained in $\ker t = \text{im } s$. Then by projectivity of $P_0$, there exists $q : P_0 \to N$ making the following diagram commute.

Then $sqf = (a_2 - a_1)f = a_2f - a_1f = sb_2 - sb_1 = s(b_2 - b_1)$. By injectivity of $s$, this implies $qf = b_2 - b_1$, that is, $f^*q = b_2 - b_1$. \hfill $\square$

**Proposition 3.3.** If $[\zeta] = [\zeta']$, then $\Phi[\zeta] = \Phi[\zeta']$.

**Proof.** Let $\zeta, \zeta'$ be two equivalent extensions of $M$ by $N$ (i.e. $[\zeta] = [\zeta']$).

$\zeta$ 0 $\longrightarrow$ N $\longrightarrow$ E $\longrightarrow$ M $\longrightarrow$ 0

$\zeta'$ 0 $\longrightarrow$ N $\longrightarrow$ E $\longrightarrow$ M $\longrightarrow$ 0
Let \( a, a' : P_0 \to E \) and \( b, b' : P_1 \to N \) be the morphisms constructed for \( \Phi[\zeta] \) and \( \Phi[\zeta'] \) respectively.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \overset{f}{\longrightarrow} & P_0 & \overset{g}{\longrightarrow} & M & \longrightarrow & 0 \\
\downarrow{b} & & \downarrow{a} & & \downarrow{\text{Id}} & & & & \\
\zeta & 0 & \longrightarrow & N & \overset{s}{\longrightarrow} & E & \overset{t}{\longrightarrow} & M & \longrightarrow & 0
\end{array}
\]

By part (1), we can choose \( a' \) to be any morphism making the following diagram commute.

\[
\begin{array}{ccc}
P_0 & \overset{a'}{\longrightarrow} & E' \\
\downarrow{g} & & \downarrow{\text{Id}} \\
M & \longrightarrow & 0
\end{array}
\]

In particular, we can choose \( a' = \theta a \), since then the diagram commutes, as demonstrated by the following calculation.

\[ t'a' = t\theta^{-1}\theta a = ta = g \]

(\( ta = g \) by the original diagram for \( \zeta \).) We can also choose \( b' \) to be any morphism making the following diagram commute.

\[
\begin{array}{ccc}
P_1 & \overset{b'}{\longrightarrow} & N \\
\downarrow{\text{Id}} & \overset{a'f = \theta af}{\longrightarrow} & \text{im } s' \\
M & \longrightarrow & 0
\end{array}
\]

In particular, we can choose \( b' = b \), since then the diagram commutes, as demonstrated by the following calculation.

\[ s'b' = \theta sb = \theta af \]

(\( sb = af \) by the original diagram for \( \zeta \).) Thus \( \Phi[\zeta] = \overline{b} \) and \( \Phi[\zeta'] = \overline{b} \).

\[ \square \]

**Proposition 3.4.** \( \Phi \) is a group homomorphism.

**Proof.** Let \([\zeta], [\zeta'] \in \mathcal{E}(M, N)\). We need to show that \( \Phi[\zeta + \zeta'] = \Phi[\zeta] + \Phi[\zeta'] \). Choose representatives \( \zeta, \zeta' \).

\[
\begin{array}{ccccccccc}
\zeta & 0 & \longrightarrow & N & \overset{s}{\longrightarrow} & E & \overset{t}{\longrightarrow} & M & \longrightarrow & 0 \\
\zeta' & 0 & \longrightarrow & N & \overset{s'}{\longrightarrow} & E' & \overset{t'}{\longrightarrow} & M & \longrightarrow & 0
\end{array}
\]

Then we let

\[
\begin{align*}
E'' &= \{(x, x') \in E \oplus E' : t(x) = t'(x')\} \\
D'' &= \{(s(n), -s'(n)) : n \in N\} \\
F &= E'' / D''
\end{align*}
\]

and we have a representative of \( \zeta + \zeta' \). 

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where \( s''(n) = (s(n), 0) \) and \( t''(x, x') = t(x) \). Let \( a, a' : P_0 \to E \) and \( b, b' : P_1 \to N \) be morphisms constructed to compute \( \Phi[\zeta], \Phi[\zeta'] \) respectively.

Then we define \( a'' : P_0 \to F \) by \( a''(p) = (a(p), a'(p)) \). Notice that this lies in \( F \) because \( t'a'(p) = ta(p) = g(p) \) by the commutative triangles above. Then we have \( t''a''(p) = ta(p) = g(p) \), so the following diagram also commutes.

By construction of \( b, b' \), we also have commutative diagrams

Then we define \( b'' = b + b' \), and we calculate \( s''b''(p) = s''(b(p) + b'(p)) = (sb(p) + sb'(p), 0) = (sb(p), s'b'(p)) = (af(p), a'f(p)) = a''f(p) \) so the following diagram commutes.

Thus \( \Phi[\zeta + \zeta'] = \overline{b'} \), by our proposition about the freedom to choose our \( a'', b'' \). Thus

\[
\Phi[\zeta + \zeta'] = \overline{b'} = \overline{b + b'} = \overline{b} + \overline{b'} = \Phi[\zeta] + \Phi[\zeta']
\]

\[ \square \]

**Proposition 3.5.** \( \Phi \) is bijective.

**Proof.** We define an inverse mapping. Given \( \overline{b} \in \text{Ext}^1(M, N) \), choose any representative \( b \), which is a morphism \( P_1 \to N \). Then let \( E \) be the pushout of \( b \) and \( f \) (see Exercise 1.9 of Schiffer). Namely,

\[ E = (P_0 \oplus N)/\{(f(x), -b(x)) : x \in P_1\} \]

By Exercise 1.9, we then have a commutative diagram with exact rows.
where \( s(n) = (0, n) \) and \( a(p) = (p, 0) \) and \( t(p, n) = g(p) \). Then take \([\zeta] \in \mathcal{E}(M, N)\) represented by the exact sequence on the bottom. This gives us an assignment \( \Psi : \text{Ext}^1(M, N) \to \mathcal{E}(M, N) \).

We need to check that this doesn’t depend on the choice of \( b \). Suppose we have \( b_1, b_2 : P_1 \to N \) with \( b_1 = b_2 \). Let \( E_i \) be the pushout of \( b_i, f \), with associated morphisms \( s_i, t_i \).

\[
E_i = (P_0 \oplus N)/\{f(p), -b_i(n)\} \quad s_i(n) = (0, n) \quad t_i(p, n) = g(p)
\]

By definition, \( \Psi(b_1) \) and \( \Psi(b_2) \) are represented by the following exact sequences.

\[
\Psi(b_1) \quad 0 \longrightarrow N \overset{s_1}{\longrightarrow} E_1 \overset{t_1}{\longrightarrow} M \longrightarrow 0
\]

\[
\Psi(b_2) \quad 0 \longrightarrow N \overset{s_2}{\longrightarrow} E_2 \overset{t_2}{\longrightarrow} M \longrightarrow 0
\]

We need a morphism \( \gamma : E_1 \to E_2 \) making the diagram above commute, so that \([\Psi(b_1)] = [\Psi(b_2)]\). Because \( b_1 = b_2 \), we have \( b_1 - b_2 \in \text{im} f^* \), so there exists \( \beta : P_0 \to N \) with \( f^* \beta = \beta f = b_1 - b_2 \). Define \( \gamma : E_1 \to E_2 \) by \( \gamma(p, n) = (p, n + \beta(p)) \). Note that \( \gamma \) is well defined because it vanishes on \( \{f(p), -b_1(n)\} \), by the following calculation.

\[
\gamma(f(x), -b_1(x)) = (f(x), -b_1(x) + \beta f(x)) = (f(x), -b_2(x)) = 0
\]

And by the following calculation, the required diagram commutes.

\[
\gamma s_1(n) = \gamma(0, n) = (0, n + \beta(0)) = (0, n) = s_2(n)
\]

\[
t_2 \gamma(p, n) = t_2(p, n + \beta(p)) = g(p) = t_1(p, n)
\]

The result of all of this is that we have a well defined function \( \Psi : \text{Ext}^1(M, N) \to \mathcal{E}(M, N) \).

Finally, we claim that \( \Psi \) is an inverse to \( \Phi \). It is immediate from the definition of \( \Psi \) that \( \Phi \Psi(b) = b \). It remains to show that \( \Phi \Psi[\zeta] = [\zeta] \). Let \([\zeta]\) have representative extension

\[
0 \longrightarrow N \overset{s}{\longrightarrow} E \overset{t}{\longrightarrow} M \longrightarrow 0
\]

then \( \Phi[\zeta] = b \) fits into the following commutative diagram.

\[
0 \longrightarrow P_1 \overset{f}{\longrightarrow} P_0 \overset{g}{\longrightarrow} M \longrightarrow 0
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & P_1 \\
\downarrow b & & \downarrow a \\
0 & \longrightarrow & N \\
\end{array}
\]

\[
0 \longrightarrow N \overset{s}{\longrightarrow} E \overset{t}{\longrightarrow} M \longrightarrow 0
\]

Then \( \Phi \Psi[\zeta] = \Psi(b) \) is the pushout of \( b \) and \( f \).
Where $E' = (P_0 \oplus N)/\sim$ and $s'(n) = (0, n)$ and $a'(p) = [p, 0]$ and $t'(p, n) = g(p)$. We define $\gamma : P_0 \oplus N \to E$ by $\gamma(p, n) = a(p) + s(n)$. Then

$$\gamma((f(x), -b(x))) = af(x) - sb(x) = 0$$

so $\gamma$ induces a morphism $E' \to E$ by $\gamma(p, n) = a(p) + s(n)$. Furthermore, we check that the following diagram commutes, which makes $\gamma$ an equivalence between $[\zeta]$ and $\Psi\Phi[\zeta]$.

Thus $\Phi\Psi$ and $\Psi\Phi$ are the respective identities, so $\Phi$ is a bijection.

This concludes the proof that $\Phi$ is an isomorphism of abelian groups.

**References**
