Graph Theory
Homework 1

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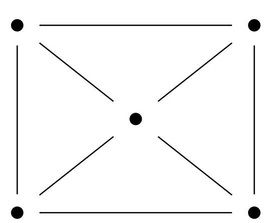
Proposition 0.1 (Exercise 1a). *Let G be the edge graph of the octahedron. G does not contain a subgraph that is a subdivision of either K_5 or $K_{3,3}$. Consequently, G is planar.*

Proof. Suppose G contains a subgraph H that is a subdivision of K_5. Then H must have at least 5 vertices. If H has exactly 5 vertices, it is exactly K_5, so H is a subgraph of G with all vertices of degree 4 containing only 5 vertices. However, any subgraph of G containing only 5 of the 6 vertices must contain vertices of degree 3, so this is impossible, so H must contain all 6 vertices of G.

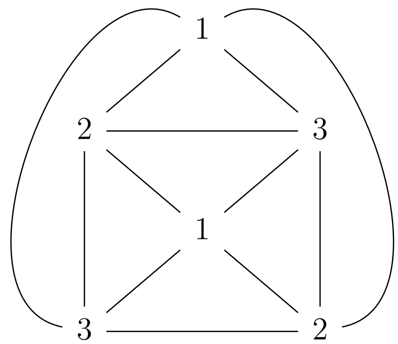
Then H must then have 5 vertices of degree 4 and 1 vertex of degree 2. Since G is 4-regular, to get H we must delete at least 2 edges to get a vertex of degree 2. Since G has 12 edges, H has at most 10 edges. But K_5 has 10 edges, so a subdivision of K_5 with 6 vertices has 11 edges. This is a contradiction, so no such subgraph H exists.

Now suppose G has a subgraph N that is a subdivision of $K_{3,3}$. G has only 6 vertices, so N must contain all 6 vertices of G, so N is isomorphic to $K_{3,3}$. G has 12 edges and $K_{3,3}$ has 9 edges, so N is equal to $G \setminus \{e_1, e_2, e_3\}$. These edges must also be removed so that N is 3-regular. Notice that every vertex of G forms four 3-cycles with its four neighbors.



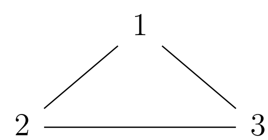
In forming N from G by removing three edges, we cannot remove more than two edges from this subgraph, since removing any three edges from this subgraph results in a vertex of degree 2. We also cannot remove two edges incident to the center vertex, since that would decrease its degree to 2. Any other removal of two edges leaves a 3-cycle. Thus N contains a 3-cycle. But N is isomorphic to $K_{3,3}$, which is bipartite and thus contains no 3-cycles, so we reach a contradiction and conclude that no such subgraph N exists. □

Here is a planar drawing of the octahedron graph, with a 3-coloring of the vertices. (Vertices of color 1 are indicated by a 1.)



Proposition 0.2 (Exercise 1b)**.** *Let G be the edge graph of the octahedron. Then $\chi(G) = 3$.*

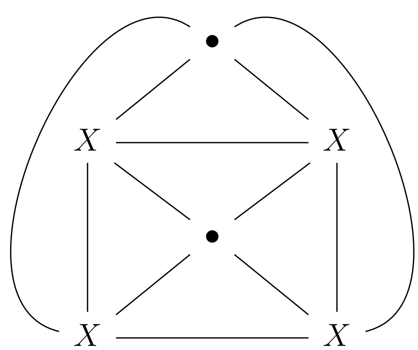
Proof. As an example of a proper 3-coloring, see the picture above. By this example, $\chi(G) \leq 3$. We just need to show that $\chi(3) \geq 3$. Choose one of the triangular subgraphs. This clearly cannot be properly colored with only 2 colors, so $\chi(G) \geq 3$.



□

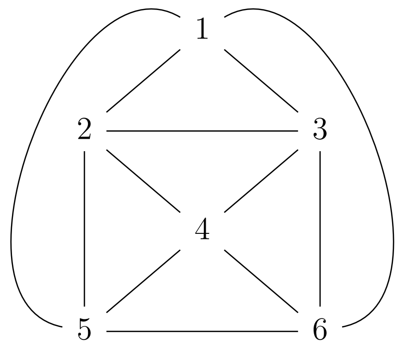
Proposition 0.3 (Exercise 1c)**.** *Let G be the edge graph of the octahedron. Then $\kappa(G) = 4$.*

Proof. First, we exhibit a 4-element cutset S of G. The vertices marked X are in S. The remaining graph is just two vertices with no edges, which is disconnected.



Now we claim that no 3 vertices of G form a cutset. First, it is clear that no 3 element set of S forms a cutset, since including any X vertex back forms a path graph with 3 vertices. If we try to form a 3 element cutset using both of the remaining nodes, again we have a path graph, which is connected. If we use 2 vertices of S and one of the other vertices, we also get a path graph. Thus $\kappa(G) > 3$, so $\kappa(G) = 4$. □

(Exercise 1e)
Let G be the octahedron graph as above. We label the vertices so that we can talk about the adjacency matrix.



The adjacency matrix for G is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Using a CAS, the characteristic polynomial is $\lambda^6 - 12\lambda^4 - 16\lambda^3$, and the roots with multiplicity are $4, -2, -2, 0, 0, 0$. The eigenvector corresponding to 4 is $(1, 1, 1, 1, 1, 1)$, the eigenvectors corresponding to -2 are $(-1, 1, 0, -1, 0, 1)$ and $(-1, 0, 1, -1, 1, 0)$, and the eigenvectors corresponding to zero are $(0, -1, 0, 0, 0, 1), (0, 0, -1, 0, 1, 0), (-1, 0, 0, 1, 0, 0)$.

When we order the eigenvalues as $4 \geq 0 \geq 00 \geq -2 \geq -2$, we have $\lambda_2 = -2$, so we verify that the result $d \geq \kappa(G) \geq d - \lambda_2$ holds, since

$$4 = d \geq \kappa(G) = 4 \geq d - \lambda_2 = 4 - 0$$

Lemma 0.4 (for Exercise 2)**.** *Let n be a positive integer. There is a 1-regular connected graph on n vertices if and only if $n = 2$.*

Proof. Any 1-regular graph is a disjoint union of copies of the complete graph on 2 vertices. Such a graph is only connected if it has exactly two vertices. □

Proposition 0.5 (Exercise 2)**.** *For (k, n) positive integers with $1 < k < n$, there exists a k-regular connected graph on n vertices if and only if kn is even. (For the case $k = 1$, see the previous lemma.)*

Proof. First we show that it is necessary for kn to be even. Let $G = (V, E)$ be a k-regular graph with n vertices. Then

$$\sum_{v \in V} d(v) = \sum_{i=1}^{n} k = kn$$

Since this is also equal to $2|E|$, kn is even. Now we construct such graphs when they exist do demonstrate that this condition is also sufficient. First consider the case where k is even. Let S be the following set of elements from the group $\mathbb{Z}/n\mathbb{Z}$:

$$S = \{1, -1, 2, -2, \ldots, k/2, -k/2\} \subset \mathbb{Z}/n\mathbb{Z}$$

None of these elements are the same, since $k < n$, so S has k elements. Then let G be the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generating set S. This graph is then k-regular, since there were k distinct generators.

The only remaining case is where n is even and k is odd. Now let T be the the following set of elements from $\mathbb{Z}/n\mathbb{Z}$.

$$T = \{n/2, 1, -1, 2, -2, \ldots, (k-1)/2, -(k-1)/2\}$$

None of these elements are the same since $k < n$. S has k elements, since it has $k-1$ pairs of elements $(1, -1), \ldots, (k-1)/2, -(k-1)/2$ and the element $n/2$. Then let G be the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generating set T. This graph is k-regular, since there were k distinct generators. □

Proposition 0.6 (Exercise 3)**.** *Let G be a graph. Then at least one of G and $\overline{G}$ is connected.*

Proof. It is sufficient to show that if G is disconnected, then $\overline{G}$ is connected. Let G be disconnected with components $G_1 = (V_1, E_2), \ldots, G_k = (V_k, E_k)$.

Let $x, y \in V$. If they lie in different components of G, then the edge xy is not in G, so it is in $\overline{G}$, so we have a path connecting them in $\overline{G}$ in this case. If they lie in the same componenet of G, choose some other vertex z from a different component of G (such a component exists because G has at least two components). Then $\overline{G}$ contains the edges xz and yz, so xzy is a path from x to y in $\overline{G}$. Thus any two vertices in $\overline{G}$ have a path between them, so it is connected. □

Proposition 0.7 (Exercise 4)**.** *Let G be a graph with $n \geq 3$ vertices. The following are equivalent.*

1. *$\kappa(G) \geq 2$, that is, G is connected with no cutvertex*
2. *Any two vertices lie on a cycle.*
3. *Any two edges lie on a cycle, and there are no isolated vertices.*
4. *For any vertices x, y, z, there is a path from x to y to z.*

Proof. We prove only (4) $\implies$ (1), (3) $\implies$ (1), and (3) $\implies$ (2) $\implies$ (1), which does not suffice for the full set of equivalences, so this proof is incomplete.

(2) $\implies$ (1). We prove the contrapositive. Clearly if $\kappa(G) = 0$, there are vertices that do not lie on a cycle, so we just consider the case $\kappa(G) = 1$. Let v be a cutvertex. Since $n \geq 3$, there are vertices x and y in different components of $G \setminus v$. We claim that there is no cycle including both x and y. Since v is a cutvertex, any path from x to y goes through v, and similarly any path from y to x goes through v. A cycle including both x and y must include a path xPy and a path yQx. Since these paths both contain v, the resulting cycle is not in fact a cycle, since the vertex v is repeated. Thus x and y do not lie on a cycle, so the contrapositive is proved.

(3) $\implies$ (1). Suppose (3) holds. Let $x, y \in V(G)$. They are not isolated, so there are edges $xw, yz \in E(G)$. By (3), there is a cycle containing xw and yz, which includes a path xPy. Thus G is connected.

We prove G has no cutvertex by contradiction. Suppose G has a cutvertex v. Choose two neighbors x and y of v in different connected components of $G \setminus v$. Any cycle containing the edge vy must lie entirely in $\{v\}$ union with the connected component of $G \setminus v$ containing y. A similar statement holds for x. Thus no cycle contains both xv and yv, contradicting (3). Thus G has no cutvertex, so (1) is proved.

(3) $\implies$ (2). Suppose (3) holds, and let x, w be any two vertices. By (3), they are not isolated, so there are edges xy and wz for some vertices y, z. Then by (3), the two edges xy and wz lie on a cycle, so x and w both lie on this cycle. This proves (2).

(4) $\implies$ (1). Suppose (4) holds. Then G is connected, since there is a path from any x to any z. If $n = 3$, then because G is connected, it must be K_3, since the path graph P_3 doesn't satisfy (4). Since K_3 satisfies (1), we may assume $n \geq 4$.

We prove G has no cutvertex by contradiction. Suppose G has a cutvertex v. Let U_1, U_2 be the vertex sets of two connected components of $G \setminus v$. Since $n \geq 4$, at least one of U_1, U_2 has two vertices, or there is a 3rd connected component U_3. If there are three components, choose $x_i \in U_i$. Then there is no path from x_1 to x_2 to x_3, since any path $x_i P x_j$ passes through v. If there are only two components, choose $x_0, x_1 \in U_1$ and $x_2 \in U_2$. Then there is no path from x_0 to x_2 to x_1, since any paths $x_0 P x_2$ and $x_2 Q x_1$ pass through v. Thus we see that G having a cutvertex implies the existence of three points x, y, z such that there is no path from x to y to z, which contradicts (4). Thus G has no cutvertex, so (1) is proved. $\square$