Definitions
Algebraic topology qualifying course
MSU, Spring 2017

Joshua Ruiter

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This document was made as a way to study the material from the spring semester algebraic topology qualifying course at Michigan State University, in spring of 2017. It serves as a companion document to the "Theorems" review sheet for the same class. The main textbook for the course was *Algebraic Topology*, by Hatcher.

Contents

1 Category Theory

Definition 1.1. Let $f : X \to Y$ and $g : A \to B$ be set maps. The **product map** is the map $f \times g : X \times A \to Y \times B$ defined by $f \times g(x, a) = (f(x), g(a))$. (This is probably a product in the category of set maps, but I don't really care about that right now.)

Definition 1.2. Let C, D be categories. A **covariant functor** $F : C \to D$ assigns each $X \in \mathrm{Ob}(C)$ to an object $F(X) \in \mathrm{Ob}(D)$ and assigns each morphism $f : X \to Y$ to a morphism $F(f) : F(X) \to F(Y)$ such that $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ for morphisms $f : X \to Y$ and $g : Y \to Z$. That is, the diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[d, "F(f)"'] \arrow[dr, "F(g\circ f)"] & \\
F(Y) \arrow[r, "F(g)"'] & F(Z)
\end{tikzcd}$$

Definition 1.3. Let C, D be categories. A **contravariant functor** $F : C \to D$ is just like a covariant functor except that if $f : X \to Y$ then $F(f) : F(Y) \to F(X)$, and $F(g \circ f) = F(f) \circ F(g)$. That is, the diagram commutes:

$$\begin{tikzcd}
F(Z) \arrow[d, "F(g)"'] \arrow[dr, "F(g\circ f)"] & \\
F(Y) \arrow[r, "F(f)"'] & F(X)
\end{tikzcd}$$

Definition 1.4. Let C, D be categories, and let $F, G : C \to D$ be covariant functors. A **natural transformation** $\eta : F \to G$ assigns each object $X \in \mathrm{Ob}(C)$ to a morphism $\eta_X : F(X) \to G(X)$ such that for every morphism $f : X \to Y$, we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. That is, the diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"'] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}$$

1.1 Chain Complexes

Definition 1.5. A **chain complex** is a sequence of abelian groups (or more generally R-modules)

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

such that the composition of two successive homomorphisms is always zero. That is, $\partial_n \circ \partial_{n+1} = 0$ for every n. The ***n*th homology** group of the chain complex is

$$H_n(C) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

Elements of $\ker \partial_n$ are **cycles** and elements of $\operatorname{im} \partial_{n+1}$ are **boundaries**. Elements of H_n are cosets of $\operatorname{im} \partial_{n+1}$, and these cosets are **homology classes**. Two cycles in the same homology class are **homologous** (so their difference is a boundary).

Definition 1.6. Let C_n and D_n be chain complexes. A **chain map** or **morphism of chain complexes** is a sequence of maps $f_n : C_n \to D_n$ making the following diagram commute.

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Note that a chain map induces homomorphisms on the homology groups of the chain complexes.

Definition 1.7. A **short exact sequence of chain complexes** is a short exact sequence of chain maps. For example, if A, B, C are chain complexes, and $f : A \to B$ and $g : B \to C$ are chain maps, we can depict this in the following diagram, which has exact columns and chain complexes for rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & B_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By applying the Snake Lemma, out of a short exact sequence of chain complexes, we always get a long exact sequence relating the homology groups of all three chain complexes:

$$\dots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow \dots$$

Definition 1.8. Let C_n, D_n be chain complexes and let $f, g : C \to D$ be chain maps. A **chain homotopy** between f and g is a sequence of maps $h_n : C_n \to D_{n+1}$ such that

$$f_n - g_n = \partial_D h_n + h_{n+1} \partial_C$$

This fits into the following diagram. (This diagram is not intended to be commutative.)

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] \arrow[dl, "h_n"] & C_{n-1} \arrow[r] \arrow[d] \arrow[dl, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial"'] & D_n \arrow[r, "\partial"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Note that chain homotopic maps always induce the same homomorphism on homology.

2 Homotopy and Cell Complexes

Throughout, X refers to a topological space and $I = [0, 1]$.

Definition 2.1. Let X be a space and $\sim$ an equivalence relation on X. The **quotient space** $X/\sim$ is the set of equivalence classes,

$$(X/\sim) = \{[x] : x \in X\}$$

Note that $\pi : X \to X/\sim$ given by $x \mapsto [x]$ is surjective. We define a set $U \subset X/\sim$ to be open if $\pi^{-1}(U)$ is open in X. This gives rise to a topology on $X/\sim$, called the **quotient topology**.

Definition 2.2. Let X be a space and A a subspace. Then $\sim$ given by $a \sim a'$ for $a, a' \in A$ is an equivalence relation. The **quotient space X/A** is then $X/\sim$. In this case, the quotient map $\pi : X \to X/A$ is given by

$$\pi(x) = \begin{cases} A & x \in A \\ x & x \notin A \end{cases}$$

Definition 2.3. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let X be the cartesian set product $\prod_{i \in I} X_i$. We define the **product topology** on X by defining a basis to be sets of the form $\prod_{i \in I} U_i$ where $U_i \subset X_i$ is open and $U_i \neq X_i$ for only finitely many i.

Definition 2.4. A **homotopy** is a family of maps $f_t : X \to Y$ for $t \in I$ such that the map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous. Two maps f_0, f_1 are **homotopic** if there is a homotopy f_t between them. This is denoted $f_0 \simeq f_1$.

Definition 2.5. A **deformation retraction** of a space X onto a space A is a family of maps $f_t : X \to X$ for $t \in I$ such that $f_0 = \mathrm{Id}_X$ and $f_1(X) = A$ and $f_t|_A = \mathrm{Id}_A$ for $t \in I$, and such that the map $X \times I \to X$ given by $(x, t) \mapsto f_t(x)$ is continuous.

Definition 2.6. A **deformation retraction in the weak sense** of a space X onto a space A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t.

Definition 2.7. Let X, Y be spaces and $f : X \to Y$ be a continuous map. Consider the space $(X \times I) \sqcup Y$, and define an equivalence relation $(x, 1) \sim f(x)$. Then we define the **mapping cylinder** of f, denoted $\boldsymbol{M_f}$, to be $\big((X \times I) \sqcup Y\big)/\sim$.

Note: The mapping cylinder M_f always deformation retracts to the subspace Y.

Definition 2.8. Let $f : X \to Y$ be a map, with mapping cylinder M_f. The **mapping cone**, denoted $\boldsymbol{C_f}$, is the quotient $M_f/(X \times \{0\})$. Using the language of attaching maps, $C_f = Y \sqcup_f CX$, where CX is the cone $(X \times I)/(X \times \{0\}$ and we attach CX to Y along $X \times \{1\}$ via $(x, 1) \sim (f(x)$.

Definition 2.9. Let X be a space and $A \subset X$. A **retraction** of X onto A is a map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathrm{Id}_A$. This is equivalent to saying that $r^2 = r$.

Alternative way to define deformation retraction: A deformation retraction is a homotopy from Id_X to a retraction of X onto A.

Definition 2.10. Let $f_t : X \to Y$ be a homotopy and $A \subset X$. We say that f_t is a **homotopy relative to $\boldsymbol{A}$** or a **homotopy rel $\boldsymbol{A}$** if $f_t|_A = \mathrm{Id}_A$ for all $t \in I$.

Definition 2.11. Let X, Y be spaces. A **homotopy equivalence** is a map $f : X \to Y$ such that there exists a map $g : Y \to X$ such that $fg \simeq \mathrm{Id}_Y$ and $gf \simeq \mathrm{Id}_X$. If there is a homotoopy between X and Y, then they are **homotopic** spaces, denoted $X \simeq Y$. Homotopic spaces are said to have the same **homotopy type**.

Definition 2.12. Let X, Y be spaces and let $A = X \cap Y$. A **homotopy equivalence rel $\boldsymbol{A}$** is a map $f : X \to Y$ such that there exists $g : Y \to X$ so that $f|_A = g|_A = \mathrm{Id}_A$ and $gf \simeq \mathrm{Id}_X$ and $fg \simeq \mathrm{Id}_Y$ via homotopies rel A. We say that X, Y are **homotopic rel $\boldsymbol{A}$** if there is a homotopy equivalence rel A.

Definition 2.13. A map is **nullhomotopic** if it is homotopic to a constant map.

Definition 2.14. A space is **contractible** if it is homotopic to a point. This is equivalent to the identity map being nullhomotopic.

Definition 2.15. We use the notation $\boldsymbol{D^n}$ for the closed n-ball.

Definition 2.16. A **$\boldsymbol{n}$-cell**, denoted $\boldsymbol{e^n}$, is an n-fold cartesian product of $(0, 1)$. It is homeomorphic to the open ball $B(0, 1)$ in $\mathbb{R}^n$.

Definition 2.17. A **CW complex** or **cell complex** is a space built up from attaching n-cells to $n - 1$ cells. More precisely, begin with a set X^0 of points (0-cells). Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. That is, X^n is the space

$$X^n = \left(X^{n-1} \bigsqcup_\alpha D^n_\alpha \right) / \sim$$

where $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha \cong S^{n-1}$. That is,

$$X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$$

If this process terminates for some n, then $X = X^n$ has the expected quotient topology, and n is the **dimension** of X. If the process does not terminate, then $A \subset X$ is open iff $A \cap X^n$ is open in X^n for every n. (This is called the **weak topology**.)

Definition 2.18. Let X be a cell complex. For each cell e^n_α, we define the **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map ϕ_α, by defining Φ_α to be the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \textstyle\bigsqcup_\alpha D^n_\alpha \longrightarrow X^n \hookrightarrow X$$

where the left map and right maps are inclusions, and the middle map is the quotient map defining X^n.

Definition 2.19. A **graph** is a one dimensional cell complex.

Definition 2.20. Let X be a CW complex. The **Euler characteristic** of X is the number of even-dimensional cells minus the number of odd dimensional cells.

Definition 2.21. Let X be a topological space and $A \subset X$. The pair (X, A) has the **homotopy extension property** if for every homotopy $f_t : A \to Y$ and every map $F_0 : X \to Y$ such that $F_0|_A = f_0$, there exists a homotopy $F_t : X \to Y$ such that $F_t|_A = f_t$ for all t.

Definition 2.22. Let X be a cell complex. A **subcomplex** is a closed subspace $A \subset X$ that is a union of cells in X. (A is a cell complex.)

Definition 2.23. A **CW pair** is a pair (X, A) where X is a cell complex and A is a subcomplex.

Definition 2.24. Let X, Y be cell complexes where the cells of X are e^m_α and the cells of Y are e^n_β. Then **product cell complex** is the set cartesian product $X \times Y$, which has cells $e^m_\alpha \times e^n_\beta$.

Definition 2.25. Let (X, A) be a CW pair. The **quotient cell complex** is the quotient space X/A with a cell complex structure inherited from X. The cells of X/A are the cells of $X \setminus A$ with the addition of a new 0-cell, which is the image of A in X/A (under the map $X \mapsto X/A$ given by $x \mapsto [x]$). If e^n_α is a cell of $X \setminus A$ with attaching map $\phi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition

$$S^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{x \mapsto [x]} X^{n-1}/A^{n-1}$$

Definition 2.26. Let X be a space. The **cone** CX is the space $(X \times I)/(X \times \{0\})$. (Note: The cone of a space is always contractible.)

Definition 2.27. Let X be a space. The **suspension** SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. (This may be viewed as the union of two copies of the cone of X.)

Definition 2.28. Let X be a CW complex and $x_0 \in X$ a 0-cell. The **reduced suspension**, denoted $\boldsymbol{\Sigma X}$ is formed by collapsing the line segment $\{x_0\} \times I$ inside the supsension SX. Note that $\Sigma X = X \wedge S^1$. (See definition of smash product below.)

Definition 2.29. Let X, Y be spaces. The **join** of X and Y, denoted $X * Y$ is the quotient space of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. A convenient way to think about/parametrize this space is as the set of formal linear combinations

$$\{t_1 x + t_2 y : t_1, t_2 \in [0, 1], t_1 + t_2 = 1, x \in X, y \in Y\}$$

where $0x + 1y = y$ and $1x + 0y = x$.

Note: The join generalizes the cone and suspension. (Explain more how this works?)

Definition 2.30. A **simplex** is the join of n points. If we take the n standard basis vectors for $\mathbb{R}^n$, their join is

$$\Delta^{n-1} = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_i \in [0, 1] \text{ and } t_1 + \ldots + t_n = 1\}$$

Definition 2.31. Let (X, x_0) and (Y, y_0) be pointed spaces. The **wedge sum** $X \vee Y$ is the quotient of $X \sqcup Y$ by identifying x_0, y_0 to a single point.

Definition 2.32. Let X, Y be spaces. The **smash product** is $X \wedge Y = X \times Y / X \vee Y$.

Definition 2.33. Let X_0, X_1 be spaces, and $A \subset X_1$. Given a map $f : A \to X_0$, we define an equivalence relation on $X_0 \sqcup X_1$ by $a \sim f(a)$ for $a \in A$. Then the quotient space $(X_0 \sqcup X_1)/\sim$ is called **$\boldsymbol{X_0}$ with $\boldsymbol{X_1}$ attached along $\boldsymbol{A}$ via $\boldsymbol{f}$**. This space is denoted $\boldsymbol{X_0 \sqcup_f X_1}$. The map f is called the **attaching map**.

Note: The most common form of attaching map is when $(X_1, A) = (D^n, S^{n-1})$ and $f : S^{n-1} \to X_0$. This attaches an n-cell to X_0 via f.

Definition 2.34. Let X be a space and $A \subset X$. The pair (X, A) has the **homotopy extension property** if for every map $f_0 : X \to Y$ and a homotopy $f_t : A \to Y$ of $f_0|_A$, there exists a homotopy $f_t : X \to Y$ of f_0. Equivalently, (X, A) has the homotopy extension property if every pair of maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$ can be extended to a map $X \times I \to Y$.

3 The Fundamental Group

Definition 3.1. A **path** in a space X is a continuous map $f : I \to X$.

Definition 3.2. A **homotopy of paths** is a homotopy $f_t : I \to X$ such that $f_t(0) = x_0$ and $f_t(1) = x_1$ for all t.

Definition 3.3. Two paths $f, g : I \to X$ are **homotopic** if there is a homotopy of paths h_t so that $h_0 = f$ and $h_1 = g$. This is denoted $\boldsymbol{f \simeq g}$.

Definition 3.4. Let $f : I \to X$ be a path. The **homotopy class** of f, denoted $[\boldsymbol{f}]$, is the equivalence class of f under the equivalence relation of homotopy of paths.

Definition 3.5. Let $f, g : I \to X$ be paths such that $f(1) = g(0)$. The **composition path** or **product path** $f \cdot g$ is defined by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

This traverses f in the first half of the interval, then traverses g in the second half of the interval.

Definition 3.6. A **loop** is a path $f : I \to X$ satisfying $f(0) = f(1)$. The **basepoint** of the loop is $f(0)$.

Definition 3.7. Let X be a space and $x_0 \in X$. The **fundamental group** of X based at x_0, denoted $\pi_1(X, x_0)$, is the set of homotopy classes $[f]$ of loops f based at x_0.

Definition 3.8. A **reparametrization** of a path $f : I \to X$ is a composition $f\phi$ where $\phi : I \to I$ is a continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$.

Definition 3.9. Let $f : I \to X$ be a path. The **inverse path** is the path $\overline{f} : I \to X$ given by $\overline{f}(t) = f(1-t)$. (Note that $f \cdot \overline{f}$ is homotopic to a constant path.)

Definition 3.10. Let X be a space and $x_0, x_1 \in X$. Let $h : I \to X$ be a path from x_0 to x_1. A **change-of-basepoint** map is a map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $\beta_h[f] = [h \cdot f \cdot \overline{h}]$. (Note: β_h is an isomorphism.)

Definition 3.11. A space is **simply connected** if it is path connected and has trivial fundamental group.

Definition 3.12. Let $p : \widetilde{X} \to X$ be a map. An open neighborhood U in X is **evenly covered** if $p^{-1}(U)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically to U via p. The preimages of U are called **sheets** of $\widetilde{X}$ over U.

Definition 3.13. A **covering map** is a map $p : \widetilde{X} \to X$ such that every $x \in X$ has a neighborhood U that is evenly covered.

Definition 3.14. A **covering space** is a space $\widetilde{X}$ along with a covering map $p : \widetilde{X} \to X$.

Definition 3.15. Let $p : \widetilde{X} \to X$ be a covering map. Let $f : I \to X$ be a path in X. A **lift** of f is a path $\widetilde{f} : I \to \widetilde{X}$ such that $f = p\widetilde{f}$. That is, there exists $\widetilde{f}$ making the diagram commute.

$$\begin{tikzcd}
& \widetilde{X} \arrow[d, "p"] \\
I \arrow[r, "f"'] \arrow[ur, dotted, "\widetilde{f}"] & X
\end{tikzcd}$$

Definition 3.16. Let $p : \widetilde{X} \to X$ be a covering map. Let $f_t : I \to X$ be a homotopy of paths in X. A **lifted homotopy** of f_t is a homotopy of paths $\widetilde{f}_t : I \to \widetilde{X}$ such that for all t, we have $f_t = p\widetilde{f}_t$. That is, for each t, we get a lift $\widetilde{f}_t$ of f_t,

$$\begin{tikzcd}
& \widetilde{X} \arrow[d, "p"] \\
I \arrow[ru, dotted, "\widetilde{f}_t"] \arrow[r, "f_t"'] & X
\end{tikzcd}$$

and this lift is continuous, in the sense that the map $F : I \times I \to X$ given by $F(s,t) = f_t(s)$ lifts to a continous $\widetilde{F} : I \times I \to X$.

$$\begin{tikzcd}
& \widetilde{X} \arrow[d, "p"] \\
I \times I \arrow[ru, dotted, "\widetilde{F}"] \arrow[r, "F"'] & X
\end{tikzcd}$$

Definition 3.17. Let $\phi : X \to Y$ be a continuous map such that $\phi(x_0) = \phi(y_0)$. The **induced homomorphism** $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is defined by $[f] \mapsto [\phi f]$. (Note that one can check that this is well defined, and that it is a group homomorphism.)

3.1 Van Kampen's Theorem

Definition 3.18. Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of groups. The **free product** $*_\alpha G_\alpha$ is the group of reduced words of finite length, using letters that are elements from some G_α, and adjacent letters belonging to different G_α. The group operation is concatenation, followed by reducing.

Definition 3.19. Let m, n be relatively prime positive integers. The **torus knot** $K_{m,n} \subset \mathbb{R}^3$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subset \mathbb{R}^3$ defined by $f(z) = (z^m, z^n)$.

3.2 Covering Spaces

For the definition of a covering space, see the section on the fundamental group.

Definition 3.20. Let $p : \widetilde{X} \to X$ be a covering space, and let $f : Y \to X$. A **lift** of f is a map $\widetilde{f} : Y \to \widetilde{X}$ so that $p\widetilde{f} = f$, that is, the following diagram commutes.

$$\begin{tikzcd}
& \widetilde{X} \arrow[d, "p"] \\
Y \arrow[ru, dotted, "\widetilde{f}"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Definition 3.21. Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be coverings. A **morphism of covers** is a map $f : \widetilde{X}_1 \to \widetilde{X}_2$ making the following diagram commute.

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[r, "f"] \arrow[rd, "p_1"'] & \widetilde{X}_2 \arrow[d, "p_2"] \\
& X
\end{tikzcd}$$

An **isomorphism of covers** is a morphism of covers with an inverse that is also a morphism of covers. Equivalently, an isomorphism of covers is a homeomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ making the above diagram commute.

Definition 3.22. A space X is **semilocally simply connected** if every point $x \in X$ has a neighborhood U so that the inclusion map $U \hookrightarrow X$ induces the trivial map $\pi_1(U) \to \pi_1(X)$. (Note: This is not as strong as requiring $\pi_1(U)$ to be the trivial group.)

Definition 3.23. Let X be path connected, locally path connected, and semilocally simply connected. Then the unique simply connected covering space $p : \widetilde{X} \to X$ is the **universal cover**.

Definition 3.24. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space. Then the **action via lifting** of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is the group action

$$\pi_1(X, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0) \qquad ([\gamma], \widetilde{x}) \mapsto [\gamma]_L \widetilde{x} = \widetilde{\gamma}(1)$$

where $\widetilde{\gamma}$ is the unique lift of γ satisfying $\widetilde{\gamma}(0) = \widetilde{x}$.

Definition 3.25. Let $p : \widetilde{X} \to X$ be a covering space. A **deck transformation** is an isomorphism of covers $\widetilde{X} \to \widetilde{X}$. The set of deck transformations forms a group under function composition.

Definition 3.26. A covering space $p : \widetilde{X} \to X$ is **normal** if the deck transformation group acts transitively every fiber $p^{-1}(x)$ (for any $x \in X$). That is, for any $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x)$, there is a deck transformation mapping $\widetilde{x}_1$ to $\widetilde{x}_2$.

Definition 3.27. Let X be a space. The set of homeomorphisms $X \to X$ is a group under composition, and is denoted $\mathrm{Homeo}(X)$.

Definition 3.28. Let G be a group and X be a space. An **action** of G on X is a group homomorphism $G \to \mathrm{Homeo}(X)$. Equivalently, it is a map $\phi : G \times X \to X$ where for each $g \in G$, the map $x \mapsto \phi(g, x)$ is a homeomorphism.

Definition 3.29. Let G be a group and X be a space. A **covering space action** of G on X is an action of G on X such that for each $x \in X$, there is a neighborhood U of x so that for any $g_1, g_2 \in G$, we have $g_1(U) \cap g_2(U) = \emptyset$. (Note: If $p : \widetilde{X} \to X$ is a covering space, then the group of deck transformations acts in this way on $\widetilde{X}$.)

Definition 3.30. Let a group G act on a space X. The **orbit** of $x \in X$ is the set

$$Gx = \{g(x) | g \in G\}$$

The set of orbits partition X. The quotient space under this equivalence relation is denoted X/G, and called the **orbit space** of the action. (For a normal covering space $p : \widetilde{X} \to X$, he orbit space $\widetilde{X}/G(\widetilde{X})$ is X.)

4 Simplicial and Singular Homology

4.1 Δ-Complexes

Definition 4.1. An **n-simplex** is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \ldots, v_n$ that do not lie in a hyperplane of dimension n. Equivalently, the vectors $v_0 - v_1, \ldots, v_n - v_0$ are linearly independent. The points v_i are the **vertices** of the simplex, and the simplex is denoted $[v_0, \ldots, v_n]$. It is important to remember that the vertices are ordered $v_0 < v_1 < \ldots < v_n$ and this ordering induces an orientation on each edge $[v_i, v_j]$ (in the direction of increasing subscripts).

Definition 4.2. The **standard n-simplex** is

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \ \forall i \right\}$$

Definition 4.3. Let $[v_0, \ldots, v_n]$ be an n-simplex. Each point can be written uniquely as $\sum_i t_i v_i$ where $\sum_i t_i = 1$ and $0 \leq t_i$. The coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i$.

Definition 4.4. A **face** of an n-simplex $[v_0, \ldots, v_n]$ is the span of an $(n-1)$-simplex formed by deleting one vertex, $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$.

Definition 4.5. The **boundary** of an n-simplex is the union of the faces.

Definition 4.6. The **open n-simplex**, denoted $\overset{\circ}{\Delta}{}^n$, is the interior of Δ^n.

Definition 4.7. A **Δ-complex** structure on a space X is a collection of (continuous) maps $\sigma_\alpha : \Delta^n \to X$ (where n depends on α) such that

1. The restriction $\sigma_\alpha | \overset{\circ}{\Delta}{}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha | \overset{\circ}{\Delta}{}^n$.
2. Each restriction of σ_α to a face Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
3. A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

As a consequence of (3), X can be built as a quotient space of a collection of disjoint simplices Δ^n_α, with one for each $\sigma_\alpha : \Delta^n \to X$, where we form the quotient space by identifying each face of σ^n_α with the σ^{n-1}_β corresponding to the restriction of σ_α to the face in question.

One can think of a Δ-complex built inductively, similar to constructing a CW-complex, by starting with a discrete set of vertices, then attaching edges to produce a graph, then attaching 2-simplices, etcetera.

4.2 Simplicial Homology

Definition 4.8. Let X be a Δ-complex. Let $\Delta_n(X)$ be the free abelian group with basis as the open n-simplices e_α^n of X. Elements of $\Delta_n(X)$ are **n-chains**. They can be written as formal sums $\sum_\alpha k_\alpha e_\alpha^n$ with $k_\alpha \in \mathbb{Z}$.

Definition 4.9. Let X be a Δ-complex, and let $\Delta_n(X)$ be as above. The **boundary homomorphism** is the map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ defined on basis elements by

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha | [v_0, \dots, \widetilde{v}_i, \dots, v_n]$$

Example calculations of the boundary homomorphism:

$$\begin{aligned}
\partial[v_0, v_1] &= [v_1] - [v_0] \\
\partial[v_0, v_1, v_2] &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \\
\partial[v_0, v_1, v_2, v_3] &= [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]
\end{aligned}$$

Definition 4.10. Let X be a Δ-complex, and let (Δ_n, ∂_n) be the chain complex constructed above. The nth homology of this chain complex, denoted $H_n^\Delta(X)$ is the nth **simplicial homology group** of X.

4.3 Singular Homology

Definition 4.11. Let X be a space. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$.

Definition 4.12. Let X be a space. Then $\boldsymbol{C}_n(X)$ is the free abelian group generated by the set of singular n-simplices in X. Elements of $C_n(X)$ are **singular n-chains**, and can be written as finite formal sums $\sum_i n_i \sigma_i$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$.

Definition 4.13. Let X be a space. The **boundary map** $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma | [v_0, \dots, \widetilde{v}_i, \dots, v_n]$$

(We are implicity identifying $[v_0, \dots, \widetilde{v}_i, \dots, v_n]$ with the standard simplex Δ^{n-1} so that we can regard $\sigma | [v_0, \dots, \widetilde{v}_i, \dots, v_n]$ as a map $\Delta^{n-1} \to X$.)

Definition 4.14. Let X be a space, with singular chain complex

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \dots$$

The nth homology of this complex, denoted $H_n(X)$, is the nth **singular homology group** of X.

Some geometric interpretation: $H_0(X)$ counts the connected components of X. Elements of $H_1(X)$ are represented by maps of oriented loops in X. Elements of $H_2(X)$ are represented by maps of closed oriented surfaces into X.

Definition 4.15. Let X be a nonempty space. The **augmented chain complex** for X is

$$\dots \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is defined by

$$\epsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$$

The **reduced homology groups** of X are the homology groups of this chain complex. They are denoted $\widetilde{H}_n(X)$. Note that for $n \neq 0$, we have $\widetilde{H}_n(X) = H_n(X)$, and in dimension zero, $\widetilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)$.

4.4 Homotopy Invariance of Singular Homology

Definition 4.16. Let $f : X \to Y$. The **induced map** $f_\# : C_n(X) \to C_n(Y)$ is defined on basis elements by $f_\#(\sigma) = f \circ \sigma$. Then we extend f linearly, i.e.

$$f_\#\left(\sum_i n_i\sigma_i\right) = \sum_i n_i f_\#(\sigma_i)$$

Note that $f_\#\partial = \partial f_\#$.

Definition 4.17. A **good pair** is a pair (X, A) where X is any space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X. (For example, if X is a CW complex and $A \subset X$ is a subcomples, then (X, A) is always a good pair.)

4.5 Relative Homology

Definition 4.18. Let X be a space and $A \subset X$. The group of **relative n-chains**, denoted $\boldsymbol{C}_n(X, A)$ is the quotient group $C_n(X)/C_n(A)$. Note that the boundary map $\partial C_n(X) \to C_{n-1}(X)$ induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$, giving rise to the following chain complex.

$$\dots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} C_{n-2}(X, A) \xrightarrow{\partial} \dots$$

The homology groups of this complex are the **relative homology groups** $H_n(X, A)$.

Definition 4.19. A **map of pairs** $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$.

Definition 4.20. Let $f : (X, A) \to (Y, B)$ be a map of pairs. The **induced map** on singular n-chains is $f_\# : C_n(X, A) \to C_n(Y, B)$. The **induced map** on homology is $f_* : H_n(X, A) \to H_n(Y, B)$.