# Math 991 <br> Homological algebra and sheaves 

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## Contents

1 Presheaves ..... 4
1.1 Defining presheaves ..... 4
1.2 Examples of presheaves ..... 4
1.3 Structure (pre)sheaf on a variety ..... 6
1.3.1 Algebraic sets ..... 6
1.3.2 Irreducibility ..... 7
1.3.3 Rational functions ..... 8
1.3.4 Defining the structure sheaf ..... 9
1.4 Morphisms of presheaves ..... 10
2 Sheaves ..... 12
2.1 Defining sheaves ..... 12
2.2 Examples of sheaves ..... 14
2.3 Structure sheaf on a variety ..... 16
2.4 Morphisms of sheaves ..... 18
2.5 First look at sheafification ..... 19
3 Limits and colimits ..... 22
3.1 Direct limits ..... 22
3.2 Examples of direct limits ..... 24
3.3 Stalks of (pre)sheaves ..... 27
3.4 Inverse limits ..... 30
3.4.1 Inverse limits of groups ..... 34
3.4.2 Inverse limits in sheaf theory ..... 35
3.5 General categorical limits. ..... 37
3.5.1 Realizing common categorical constructions as limits ..... 39
3.5.2 Representable and adjoint functors ..... 44
4 Stalks of sheaves ..... 46
4.1 Exactness properties of limits ..... 48
4.2 Summary of definitions of exactness ..... 55
4.3 Epimorphisms ..... 56
4.4 Image presheaf ..... 58
5 Sheafification ..... 60
5.1 Local homeomorphisms ..... 60
5.2 Etale space of a presheaf ..... 63
5.3 Sheafification - main result ..... 66
6 Cech cohomology ..... 74
6.1 The Mittag-Leffler problem ..... 74
6.2 Defining Cech cohomology ..... 75
6.3 Resolving dependence on the cover ..... 77
6.4 Classification of vector bundles via Cech cohomology ..... 79
6.5 The Cech cochain complex ..... 84
6.6 Resolving dependence on the cover, in more generality ..... 89
6.7 Sheafified Cech complex ..... 92
6.8 Sheafification revisited, using Čech cohomology ..... 96
7 Representable functors and Yoneda's lemma ..... 106
7.1 Application of Yoneda - affine group schemes and Hopf algebras ..... 108
8 Functors between categories of sheaves ..... 111
8.1 Direct image functor ..... 111
8.2 Inverse image functor ..... 116
8.3 Adjunction between $\left(f^{-1}, f_{*}\right)$ ..... 124
8.4 Extension by zero ..... 133
8.5 Exceptional inverse image ..... 136
8.6 Summary of adjunctions and exactness results ..... 138
9 General homological algebra ..... 139
9.1 Injectives ..... 139
9.2 Projectives ..... 142
9.3 Abelian categories ..... 144
9.4 Homology in abelian categories ..... 151
9.5 Sheaves of $\mathcal{O}$-modules ..... 157
$9.6 \delta$-functors ..... 158
9.7 Right derived functors ..... 159
10 Sheaf cohomology ..... 163
10.1 Defining sheaf cohomology ..... 163
10.2 Higher direct images ..... 167
10.3 Acyclic sheaves ..... 168
10.3.1 Flasque sheaves ..... 168
10.3.2 Fine sheaves ..... 171
10.4 Leray's theorem ..... 172
10.5 Unification of de Rham cohomology, singular cohomology, and sheaf cohomology ..... 176
10.5.1 De Rham cohomology ..... 177
10.5.2 Singular cohomology ..... 183

## 1 Presheaves

### 1.1 Defining presheaves

Definition 1.1. A presheaf $\mathcal{F}$ on a topological space $X$ consists of the following data: For each open set $U \subset X$, a set $\mathcal{F}(U)$, and for each inclusion of open sets $V \subset U$ in $X$, a map of sets $\rho_{V}^{U}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ such that

$$
\rho_{U}^{U}=\operatorname{Id}_{U}
$$

for every open $U \subset X$, and if $W \subset V \subset U$ then

$$
\begin{aligned}
& \rho_{W}^{U}=\rho_{W}^{V} \rho_{V}^{U}
\end{aligned}
$$

Remark 1.2. We can also formulate the previous definition more categorically. Let $\mathrm{Op}(X)$ be the category whose objects are open subsets of $X$, and whose morphisms are just inclusion maps. Then a presheaf on $X$ is just a contravariant functor from $\operatorname{Op}(X)$ to the category of sets.

Having written down this definition, we now realize that we can easily replace sets with any category $\mathcal{C}$. A presheaf on $X$ with values in $\mathcal{C}$ is a contravariant functor from $\operatorname{Op}(X)$ to $\mathcal{C}$. Most often, $\mathcal{C}$ will be abelian groups, or rings, or modules over a ring.

Remark 1.3. We adopt the convention that our presheaf target category has a terminal object $T$, and that if $\mathcal{F}$ is a presheaf, then $\mathcal{F}(\emptyset)=T$. Since most of the presheaves we deal with will be presheaves of abelian groups, this is satisfied - the terminal object is the trivial/zero group.
Definition 1.4. Let $\mathcal{F}$ be a presheaf on $X$ with values in a concrete category $\mathcal{C}$. The elements of the set $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over $U$. The maps $\rho_{V}^{U}$ are called restriction maps. The sections over $X$ are called global sections, that is, $\mathcal{F}(X)$ is the set of global sections.

We often think of the sections (elements of $\mathcal{F}(U)$ ) as functions from $U$ to some space like a ground field, and think of $\rho_{V}^{U}$ as literal restriction of functions, since this is the situation which motivates the definition of a presheaf. We will give an example in a minute. Eventually, we will see that in some sense such examples are universal.

### 1.2 Examples of presheaves

Example 1.5 (Presheaf of continuous functions). Let $X, Y$ be topological spaces. We will define the presheaf of $Y$-valued continuous functions. For $U \subset X$ a nonempty open set, define

$$
\mathcal{F}(U)=\operatorname{Hom}_{\text {cts }}(U, Y)=\{\phi: U \rightarrow Y \mid \phi \text { is continuous }\}
$$

On the empty set, we take $\mathcal{F}(\emptyset)=\{*\}$, the set with one element, since that is the terminal object in the category of sets. For $V \subset U$ both nonempty, we define the restriction map to be

$$
\rho_{V}^{U}:\left.\mathcal{F}(U) \rightarrow \mathcal{F}(V) \quad \phi \mapsto \phi\right|_{V}
$$

If $V$ is the empty set, then there is only one possible map from $\mathcal{F}(U)$ to $\mathcal{F}(V)$, so that determines the restriction map in that case. All of this makes $\mathcal{F}$ a presheaf on $X$ with values in the category of sets.

Example 1.6 (Constant presheaf). Let $X$ be a topological space, and let $E$ be a set. For $U \subset X$ open and nonempty, define $\mathcal{F}(U)=E$. Set $\mathcal{F}(\emptyset)=\{*\}$, the set with one element. For $V \subset U$ both nonempty, set $\rho_{V}^{U}=\operatorname{Id}_{E}$, and define $\rho_{\emptyset}^{U}$ to be the unique map $E \rightarrow\{*\}$. This defines a presheaf on $X$ with values in the category of sets. It is called a constant presheaf. Rather than think of sections $\phi \in \mathcal{F}(U)=E$ as elements of $E$, it is often useful to think of them as constant functions $U \rightarrow E$.

Definition 1.7. Let $X, Y$ be topological spaces. A function $\phi: X \rightarrow Y$ is locally constant if for every $x \in X$, there exists an open neighborhood $U$ so that $\left.\phi\right|_{U}$ is constant.

Note that if $X$ is a connected space, a function $X \rightarrow Y$ is locally constant if and only if it is constant. However, if $X$ is not connected, there are many locally constant functions which are not (globally) constant.

Example 1.8 (Locally constant presheaf). Let $X$ be a topological space, and let $E$ be a set. For $U \subset X$ open and nonempty, define $\mathcal{F}(U)$ to be the set of locally constant maps $U \rightarrow E$. Define $\mathcal{F}(\emptyset)=\{*\}$. Define restriction maps to be actual function restriction.

$$
\rho_{V}^{U}:\left.\mathcal{F}(U) \rightarrow \mathcal{F}(V) \quad \phi \mapsto \phi\right|_{V}
$$

This is a presheaf on $X$ with values in the category of sets. It is called a locally constant presheaf.

Example 1.9 (Presheaf of sections). Let $X, Y$ be topological spaces, and $\pi: Y \rightarrow X$ a continuous map. Recall that a section of $\pi$ is a continuous map $\sigma: X \rightarrow Y$ such that $\pi \sigma=\operatorname{Id}_{X}$. For an opent nonempty set $U \subset X$, define $\mathcal{F}(U)$ to be the set of sections of $\pi$ on $U$. That is,

$$
\mathcal{F}(U)=\left\{\sigma: U \rightarrow Y \mid \sigma \text { is continuous, } \pi \sigma=\operatorname{Id}_{U}\right\}
$$

As before, the empty set is sent to the singleton set, and the restriction maps $\rho_{V}^{U}$ are once again given by restriction of functions. This is called the presheaf of sections of $\pi$.

Example 1.10 (Skyscraper presheaf). Let $X$ be a topological space, and fix a point $p \in X$. Let $E$ be a set. For $U \subset X$ open, define

$$
\mathcal{F}(U)= \begin{cases}E & p \in U \\ \{*\} & p \notin U\end{cases}
$$

To define restriction maps, one needs to consider some cases. If $p \in V \subset U$, then we set $\rho_{V}^{U}=\operatorname{Id}_{E}$. If $p$ is not in $V$, then there is only one possible map to $\mathcal{F}(V)$, since it is the singleton set; the map will either be $E \rightarrow\{*\}$ if $p \in U$, or the identity map $\{*\} \rightarrow\{*\}$ if $p \notin U$. This defines a presheaf on $X$, called the skyscraper presheaf. The name comes from the fact that all of the data is "concentrated" at the point $p$, so in some sense the point $p$ "sticks out" like a skyscraper.

After defining stalks, we will be able to formulate this more precisely. In that language, the stalk at $p$ is $E$, and the stalk at every other point of the skyscraper presheaf is trivial.

### 1.3 Structure (pre)sheaf on a variety

We now have a long aside in order to develop a very important example of a presheaf coming from algebraic geometry. The algebraic geometry is not the focus of this class, but this example is foundational for why presheaves and sheaves are important, so it would be odd to skip over it. For those in the know, we will build up to defining the structure sheaf of an affine variety (or affine scheme, if you like).

Fix an algebraically closed field $K$, and let $K\left[x_{1}, \ldots, x_{n}\right]$ be the polynomial ring over $K$ in $n$ variables.

### 1.3.1 Algebraic sets

Definition 1.11. Given an ideal $I \subset K\left[x_{1}, \ldots, x_{n}\right]$, the vanishing set of $I$ is

$$
V(I)=\left\{\left(a_{1}, \ldots, a_{n}\right) \in K^{n} \mid f\left(a_{1}, \ldots, a_{n}\right)=0 \forall f \in I\right\}
$$

Subsets of $K^{n}$ of the form $V(I)$ are called algebraic sets. They may also be called affine varieties, although some people reserve the term "variety" for an irreducible algebraic set, which we will define shortly.

Definition 1.12. One can check that the sets $V(I)$ obey the needed properties to be a collection of closed sets for a topology (someone tedious exercise, with some tricks involved). The resulting topology on $K^{n}$ is called the Zariski topology.

Remark 1.13. Alternatively, one may define the Zariski topology by defining a basis of open sets. For $f \in K\left[x_{1}, \ldots, x_{n}\right]$, set

$$
D(f)=K^{n} \backslash V(f)=\left\{x \in K^{n}: f(x) \neq 0\right\}
$$

The set $D(f)$ is called the principal open subset of $f$. It is clear from the way that we defined the Zariski topology in terms of closed sets that this is open in the Zariski topology. Instead of starting with the closed sets, one may instead define the sets $D(f)$ as open, and show that they satisfy the requirements to be a basis for a topology. The main thing to prove is that

$$
D(f) \cap D(g)=D(f g)
$$

and

$$
K^{n} \backslash V(I)=\bigcup_{f \in I} D(f)
$$

Taking them to be a basis generates the same topology as our definition of closed sets, though this takes some working out.

Definition 1.14. Let $X=V(I) \subset K^{n}$ be a Zariski closed set. The ideal of $X$ is

$$
I(X)=\left\{f \in K\left[x_{1}, \ldots, x_{n}\right]:\left.f\right|_{X}=0\right\}
$$

Note that $I(X)$ is an ideal, and that $I \subset I(X)$ but they are not necessarily equal. $\left.\right|^{1}$ In fact, one can show that $I(X)=\sqrt{I}$, the radical of $I$, see the following statement of Hilbert's Nullstellensatz.

$$
\sqrt{I}=\left\{f \in K\left[x_{1}, \ldots, x_{n}\right]: \exists m \in \mathbb{Z}_{\geq 0} f^{m} \in I\right\}
$$

The coordinate ring of $X$, denoted $K[X]$, is

$$
K[X]=K\left[x_{1}, \ldots, x_{n}\right] / I(X)
$$

$K[X]$ is also called the ring of regular functions on $X$, for reasons which we will shortly explain. One can think of it as functions $X \rightarrow K$ which admit a polynomial representation.

Theorem 1.15 (Hilbert's Nullstellensatz). Let $K$ be an algebraically closed field, and let $J \subset K\left[x_{1}, \ldots, x_{n}\right]$ be an ideal. Then

$$
I(V(J))=\sqrt{J}
$$

### 1.3.2 Irreducibility

Definition 1.16. A closed set $X \subset K^{n}$ is irreducible if $X$ cannot be written as $X=X_{1} \cup X_{2}$ with $X_{1}, X_{2}$ proper closed subsets of $X$.

Example 1.17. In $K\left[x_{1}, x_{2}\right]$, let $I=\left(x_{1} x_{2}\right)$. Then let

$$
X=V(I)=\left\{(a, 0),(0, b) \in K^{2}: a, b \in K\right\}
$$

In terms of a picture, if we draw $K^{2}$ as a Cartesian coordinate plane, $X$ is the union of the vertical and horizontal axes. $X$ is NOT irreducible, since it can be written as the union

$$
X=X_{1} \cup X_{2}=\{(a, 0): a \in K\} \cup\{(0, b): b \in K\}
$$

Also note that $X_{1}, X_{2}$ are closed, since $X_{1}$ is the vanishing of the ideal $\left(x_{1}\right)$, and $X_{2}$ is the vanishing of the ideal $\left(x_{2}\right)$. However, $X_{1}$ and $X_{2}$ are irreducible.

Proposition 1.18. Let $X \subset K^{n}$ be an algebraic set. The following are equivalent.

[^0]1. $X$ is irreducible.
2. $I(X)$ is a prime ideal of $K\left[x_{1}, \ldots, x_{n}\right]$.
3. $K[X]$ is an integral domain.

Proof. The equivalence of (2) and (3) is immediate. The work is just in (1) $\Longleftrightarrow$ (2). We'll leave it to you.

Remark 1.19. The ring $K\left[x_{1}, \ldots, x_{n}\right]$ is Noetherian, so an algebraic set $X \subset K^{n}$ can be written uniquely as a finite union of irreducible algebraic sets, called the irreducible components of $X$. This just relies on some basic point-set topology and commutative algebra, nothing too fancy.

### 1.3.3 Rational functions

Throughout this section, we fix an irreducible algebraic set $X$.
Definition 1.20. Let $X$ be an irreducible algebraic set, with regular functions (coordinate ring) $K[X]$. Since $K[X]$ is an integral domain, we may form its fraction field, which we denote $K(X)$. This it the field of rational functions on $X$.

$$
K(X)=\left\{\frac{g}{h}: g, h \in K[X], h \neq 0\right\}
$$

We can view an element $f=\frac{g}{h} \in K(X)$ as a function with values in $K$, and domain some subset of $X$. The issue is that while $h$ cannot be identically the zero function, it may be zero at some points. Note that while we may write $f=\frac{g}{h}$ in may different ways (by multiplying by $\frac{p}{p}$ for example), the value in $K$ does not depend on the representation, provided the denominator does not vanish.

Definition 1.21. A rational function $f \in K(X)$ is defined at $x \in X$ if there exists a representation $f=\frac{g}{h}$ such that $h(x) \neq 0$.

Definition 1.22. The domain of $f \in K(X)$ is the set of $x \in X$ so that $f$ is defined at $x$. It is denoted $\operatorname{Dom}(f)$. It is a (Zariski) open subset of $X$, because

$$
\operatorname{Dom}(f)=\bigcup_{f=\frac{g}{h}}(D(h) \cap X)
$$

Each principal open subset $D(h)$ is open, so $D(h) \cap X$ is open in $X$, so the union is open in $X$.

### 1.3.4 Defining the structure sheaf

Finally we can define the structure sheaf on an irreducible affine variety, a.k.a irreducible algebraic set.

Definition 1.23. Let $X \subset K^{n}$ be closed and irreducible. For $U \subset X$ nonempty and open, define

$$
\mathcal{O}_{X}(U)=\{f \in K(X): f \text { is defined at all points } x \in U\}
$$

On the empty subset, we take $\mathcal{O}_{X}(\emptyset)=\{*\}$. For $V \subset U$, if $f \in K(X)$ is defined at all points of $U$, then it is also defined at all points of $V$. So we take the restriction map to just be the inclusion map.

$$
\rho_{V}^{U}: \mathcal{O}_{X}(U) \hookrightarrow \mathcal{O}_{X}(V)
$$

This gives a presheaf on $X$, with values in the category of commutative rings. ${ }^{2}$ It is called the structure presheaf on $X$. Once we have discussed sheaves, we will call it the structure sheaf on $X$, since it is in fact a sheaf.

Proposition 1.24. Let $X$ be an irreducible algebraic set. The global sections of the structure sheaf is the coordinate ring of $X$. That is,

$$
\mathcal{O}_{X}(X)=K[X]
$$

Proof. The inclusion $K[X] \subset \mathcal{O}_{X}(X)$ is clear, we just need the reverse inclusion.
Let $f \in \mathcal{O}_{X}(X)$. Then for every $x \in X$, there exist $g_{x}, h_{x} \in K[X]$ with $f=\frac{g_{x}}{h_{x}}$ and $h_{x}(x) \neq 0$. So $f h_{x}=g_{x}$.

Now let $J \subset K[X]$ be the ideal generated by all such $h_{x}$ for $x \in X$. By Hilbert's Nullstellensatz, $I(V(J))=\sqrt{J}$. By definition of $J, V(J)=\emptyset$, so $I(V(J))=K[X]$, thus $K[X]=\sqrt{J}$. In particular, $1 \in \sqrt{J}$, so there exists $j \in J$ such that $j^{n}=1$ for some $n \in \mathbb{Z}_{\geq 0}$. But $J$ is an ideal, so if $j \in J$, then $j^{n}=1 \in J$, hence $1 \in J$. Then we can write 1 as

$$
1=p_{1} h_{x_{1}}+\cdots+p_{r} h_{x_{r}}
$$

with $x_{i} \in X$ and $p_{i} \in K[X]$. Now multiply by $f$.

$$
f=p_{1} h_{x_{1}} f+\cdots+p_{r} h_{x_{r}} f=p_{1} g_{x_{r}}+\cdots+p_{r} g_{x_{r}}
$$

The right hand side is clearly in $K[X]$, so $f \in K[X]$.
This concludes our discussion of the structure sheaf of an affine variety for the moment, though we will return to it briefly later.

[^1]
### 1.4 Morphisms of presheaves

Definition 1.25. Let $X$ be a topological space, and let $\mathcal{F}, \mathcal{G}$ be presheaves on $X$ with values in the same category $\mathcal{C}$. A morphism of presheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of functors. Explicitly, this means that for $U \subset X$ open, there is a morphism (in $\mathcal{C}$ )

$$
\phi_{U}: \mathcal{F}(U) \rightarrow \mathcal{G}(U)
$$

such that the following diagram commutes for every chain of open sets $V \subset U \subset X$.


Remark 1.26. If we fix a topological space $X$, and a category $\mathcal{C}$, then presheaves on $X$ with values in $\mathcal{C}$ form a category. The objects are presheaves, and the morphism are as defined above. This category is denoted $\operatorname{PSh}(X, \mathcal{C})$, or often just written $\operatorname{PSh}(X)$, usually implying that $\mathcal{C}$ is the category of abelian groups.

Definition 1.27. Let $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups on $X$, and let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. For $U \subset V \subset X$, we extend the commutative diagram given by $\phi$ to the kernels.


Then by a simple diagram chase, we observe that

$$
\rho_{V}^{U}(\mathcal{F})\left(\operatorname{ker} \phi_{U}\right) \subset \operatorname{ker} \phi_{V}
$$

So we can complete the diagram.


This allows us to define the kernel presheaf of $\phi$. On open sets $U \subset X$, we define

$$
\mathcal{K}(U)=\operatorname{ker} \phi_{U}
$$

By the previous discussion, the restriction maps $\rho_{V}^{U}$ associated to $\mathcal{F}$ give maps $\mathcal{K}(U) \rightarrow \mathcal{K}(V)$ whenever $V \subset U$. The presheaf $\mathcal{K}$ is called the kernel presheaf of $\phi$.

Definition 1.28. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves (on $X$, values in $\mathcal{C}$ ). In analogy with the above, we define the image presheaf of $\phi$. For $U \subset V \subset X$ open sets, we have the diagram


Then by a diagram chase, we find that

$$
\rho_{V}^{U}(\mathcal{G})\left(\operatorname{im} \phi_{U}\right) \subset \operatorname{im} \phi_{V}
$$

So we can complete our previous diagram.


The image presheaf of $\phi$ is the presheaf $\mathcal{I}$ defined by

$$
\mathcal{I}(U)=\operatorname{im} \phi_{U}
$$

with restriction maps given by $\rho_{V}^{U}(\mathcal{G})$. By the discussion above, $\rho_{V}^{U}(\mathcal{G})$ maps into $\phi_{V}$ when it needs to, so this does define a presheaf.

Remark 1.29. In the category of presheaves, kernels and images are both equally valid constructions, and have no significant issues or subtleties. However, we will see that in the category of sheaves, kernels are much better behaved than images, for whatever reason. To be more precise, the kernel presheaf is a sheaf, while the image presheaf fails the sheaf axioms. To remedy this, we will utilize sheafification. This is all to be discussed in more detail much later.

Definition 1.30. A subpresheaf of a presheaf $\mathcal{F}$ is a subobject in the category $\operatorname{PSh}(X)$. Alternatively, it is a presheaf $\mathcal{K}$ with a morphism of presheaves $\phi: \mathcal{K} \rightarrow \mathcal{F}$ so that for every $U \subset X$ open, the map $\phi_{U}: \mathcal{K}(U) \rightarrow \mathcal{F}(U)$ is injective.

It actually takes some proving to show that these two notions agree, and it involves some subtle category theory, but we'll skim over all of that. The Stacks project has a thorough proof of this https://stacks.math.columbia.edu/tag/00V5.

Example 1.31. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. The kernel presheaf is a subpresheaf of $\mathcal{F}$, and the image presheaf is a subpresheaf of $\mathcal{G}$.

## 2 Sheaves

### 2.1 Defining sheaves

Before defining sheaves, we give an important motivating example.
Example 2.1. Let $X$ be a topological space, and consider the presheaf of continuous functions from $X$ to another topological space, say $\mathbb{R}$ for simplicity.

$$
\mathcal{F}(U)=\operatorname{Hom}_{\mathrm{cts}}(U, \mathbb{R})
$$

If $U_{1}, U_{2} \subset X$ are open, and $f_{1}: U_{1} \rightarrow \mathbb{R}$ and $f_{2}: U_{2} \rightarrow \mathbb{R}$ are sections (continuous maps), and the maps $f_{1}, f_{2}$ agree on the overlap, that is,

$$
\left.f_{1}\right|_{U_{1} \cap U_{2}}=\left.f_{2}\right|_{U_{1} \cap U_{2}}
$$

then there is a unique continuous function $f: U_{1} \cup U_{2} \rightarrow \mathbb{R}$ formed by "gluing" $f_{1}$ and $f_{2}$. That is,

$$
\left.f\right|_{U_{1}}=\left.f_{1} \quad f\right|_{U_{2}}=f_{2}
$$

As a memorable slogan, we summarize all of this by saying that "continuous functions can be glued together."

The previous example shows that a presheaf of continuous functions on $X$ has a lot more structure than just being a presheaf. The restriction maps interact very well with overlaps and unions in a way that allows forming new sections out of other sections by "gluing."

Not only can new sections be formed by gluing, but the resulting section is unique, and this uniqueness aspect is nearly as important as the existence of such a section. The definition of a sheaf captures these properties in a bit more abstraction.

Definition 2.2. Let $\mathcal{F}$ be a presheaf on $X . \mathcal{F}$ is a sheaf if for every open $U \subset X$ and every open covering

$$
U=\bigcup_{\alpha \in I} U_{\alpha}
$$

the following two conditions hold.

1. (Uniqueness) If there are sections $s, t \in \mathcal{F}(U)$ such that $\rho_{U_{\alpha}}^{U}(s)=\rho_{U_{\alpha}}^{U}(t)$ for all $\alpha \in I$, then $s=t$.
2. (Gluing) Given a collection of $s_{\alpha} \in \mathcal{F}\left(U_{\alpha}\right)$ for each $\alpha \in I$ such that they "agree on the overlaps," that is,

$$
\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\alpha}}\left(s_{\alpha}\right)=\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\beta}}\left(s_{\beta}\right)
$$

for all $\alpha, \beta \in I$, then there exists $s \in \mathcal{F}(U)$ such that $\rho_{U_{\alpha}}^{U}(s)=s_{\alpha}$ for all $\alpha \in I$. (By condition 1 , such $s$ is unique.)

Definition 2.3. A presheaf satisfying only condition (1) above is called a separated presheaf.

Remark 2.4. For the moment, think of sections as functions. The uniqueness condition says that if two functions $s, t$ restrict to the same function "everywhere locally," then they are the same function "globally" on $U$. This matches our intuition of functions.

The gluing condition captures the gluing phenomenon we saw in the example, where two functions that have the same restriction to every overlap can be glued together to give a "global" function on $U$. When sections are not exactly functions, the intuition goes away, but this is where the definition comes from.

Remark 2.5. If $\mathcal{F}$ is a presheaf of abelian groups, we can restate the uniqueness condition as the following: If $\rho_{U_{\alpha}}^{U}(s)=0$ for all $\alpha \in I$, then $s=0$.

Remark 2.6. We can extend the previous remark further. If $\mathcal{F}$ is a presheaf of abelian groups, we can express both the uniqueness and gluing properties together as exactness of a certain sequence, which we now describe. So let $U \subset X$ be an open set, and $\bigcup_{\alpha \in I} U_{\alpha}$ be an open cover of $U$. First, we have the product of all the restriction maps:

$$
\phi: \mathcal{F}(U) \rightarrow \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right) \quad s \mapsto\left(\rho_{U_{\alpha}}^{U}(s)\right)
$$

The uniqueness condition is equivalent to $\phi$ being injective, which is equivalent to saying that the following sequence is exact.

$$
0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi} \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right)
$$

Now consider the maps

$$
\begin{gathered}
\psi_{1}: \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right) \rightarrow \prod_{(a, b) \in I \times I} \mathcal{F}\left(U_{a} \cap U_{b}\right) \\
\psi_{2}: \prod_{\beta \in I} \mathcal{F}\left(U_{\beta}\right) \rightarrow \prod_{(a, b) \in I \times I} \mathcal{F}\left(U_{a} \cap U_{b}\right) \\
\overbrace{\beta}) \mapsto\left(s_{\alpha}\right) \\
\widetilde{s}_{\alpha}= \begin{cases}\rho_{U_{a} \cap U_{b}}^{U_{\alpha}}\left(s_{\alpha}\right) & a=\alpha \\
0 & a \neq \alpha\end{cases} \\
\widehat{s}_{\beta}= \begin{cases}\rho_{U_{a} \cap U_{b}}^{U_{\beta}}\left(s_{\beta}\right) & b=\beta \\
0 & b \neq \beta\end{cases} \\
\widetilde{s}_{\alpha} \in \mathcal{F}\left(U_{\alpha} \cap U_{b}\right)
\end{gathered}
$$

Now define $\psi=\psi_{1}-\psi_{2}$. By construction,

$$
\begin{aligned}
\operatorname{ker} \psi & =\left\{\left(s_{\alpha}\right) \in \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right): \psi_{1}\left(\left(s_{\alpha}\right)\right)=\psi_{2}\left(\left(s_{\beta}\right)\right)\right\} \\
& =\left\{\left(s_{\alpha}\right) \in \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right):\left(\widetilde{s}_{\alpha}\right)=\left(\widehat{s}_{\beta}\right)\right\} \\
& =\left\{\left(s_{\alpha}\right) \in \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right): \widetilde{s}_{\alpha}=\widehat{s}_{\beta} \text { whenever they are in the same set }\right\}
\end{aligned}
$$

The sections $\widetilde{s}_{\alpha}, \widehat{s}_{\beta}$ lie in the same set $\mathcal{F}\left(U_{a} \cap U_{b}\right)$ when $a=\alpha$ and $b=\beta$, in which case they are equal when $\rho_{U_{a} \cap U_{b}}^{U_{\alpha}}\left(s_{\alpha}\right)=\rho_{U_{a} \cap U_{b}}^{U_{\beta}}\left(s_{\beta}\right)$. Thus

$$
\operatorname{ker} \psi=\left\{\left(s_{\alpha}\right) \in \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right): \rho_{U_{\alpha} \cap U_{\beta}}^{U_{\alpha}}\left(s_{\alpha}\right)=\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\beta}}\left(s_{\beta}\right)\right\}
$$

Thus $\operatorname{ker} \psi=\operatorname{im} \phi$ if and only if $\mathcal{F}$ satisfies the gluing condition. Hence a presheaf $\mathcal{F}$ is a sheaf if and only if the following sequence is exact.

$$
0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi} \prod_{\alpha \in I} \mathcal{F}\left(U_{\alpha}\right) \xrightarrow{\psi} \prod_{(\alpha, \beta) \in I \times I} \mathcal{F}\left(U_{\alpha} \cap U_{\beta}\right)
$$

### 2.2 Examples of sheaves

Example 2.7. Let $X, Y$ be topological spaces, and let $\mathcal{F}$ be the presehaf of $Y$-valued continuous functions on $X$.

$$
\mathcal{F}(U)=\operatorname{Hom}_{\mathrm{cts}}(U, Y)
$$

Then $\mathcal{F}$ is a sheaf, for exactly the reasons discussed in the motivating example 2.1. Continuous functions have the gluing property, and if two functions restrict to the same function everywhere locally, they are the same globally.

Remark 2.8. A subpresheaf of a sheaf is always separated (satisfies the uniqueness condition), but need not satisfy gluing. Essentially, the glued section which always exists as a section for the larger sheaf may fail to be a section for the subpresheaf. The next example gives a case where gluing fails for a subpresheaf of a sheaf.

Example 2.9. Let $\mathcal{F}$ be the sheaf of $\mathbb{R}$-valued continuous functions on $X=\mathbb{R}$, and let $\mathcal{G}$ be the subpresheaf of $\mathcal{F}$ consisting of bounded continuous functions.

$$
\begin{aligned}
\mathcal{F}(U) & =\{f: U \rightarrow \mathbb{R} \mid f \text { is continuous }\} \\
\mathcal{G}(U) & =\{f: U \rightarrow \mathbb{R} \mid f \text { is continuous and bounded }\}
\end{aligned}
$$

Then $\mathcal{G}$ is separated, but it does not satisfy the gluing axiom. For example, take $U=X=\mathbb{R}$ with the open cover

$$
\mathbb{R}=\bigcup_{n \in \mathbb{Z}} U_{n} \quad U_{n}=(n-1, n+1)
$$

On each $U_{n}, \mathcal{G}$ has the section $f_{n}: U_{n} \rightarrow \mathbb{R}, x \mapsto x$. This is bounded since $U_{n}$ is bounded. As $\mathcal{F}$ is a sheaf, there is a global section $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x$, but this global section is not a global section for $\mathcal{G}$, since it is not bounded.

Example 2.10. The constant presheaf is usually not a sheaf. For example, let $X$ be a topological space containing at least two disjoint open sets $U_{1}, U_{2}$, so $U_{1} \cap U_{2}=\emptyset$. Let $E$ be a set with at least two elements, and let $\mathcal{F}$ be the constant presheaf on $X$ with values in $E$.

$$
\mathcal{F}(U)= \begin{cases}E=\text { constant functions } U \rightarrow E & U \neq \emptyset \\ \{*\} & U=\emptyset\end{cases}
$$

We will show that gluing fails for this presheaf. Take $U=U_{1} \cup U_{2}$, with open cover given by $U_{1}, U_{2}$. As $E$ has at least distinct elements $e_{1}, e_{2}$, view $e_{1} \in \mathcal{F}\left(U_{1}\right)$ and $e_{2} \in \mathcal{F}\left(U_{2}\right)$. These trivially agree on the overlap, because $U_{1} \cap U_{2}=\emptyset$ and $\mathcal{F}(\emptyset)=\{*\}$. However, for any "global section" $e \in \mathcal{F}(U)=E$, we have

$$
\rho_{U_{1}}^{U}(e)=e_{1} \quad \rho_{U_{2}}^{U}(e)=e_{2}
$$

so there is no hope of finding a section on $U$ which glues $e_{1}, e_{2}$.
More conceptually, if we view elements of $\mathcal{F}(U)$ as constant functions $U \rightarrow E$, what is going on in this example is that we are trying to take constant functions on $U_{1}, U_{2}$ with different values and glue them to obtain a constant function on $U_{1} \cup U_{2}$, which is impossible. That is to say, a locally constant function need not be globally constant.

Example 2.11. In contrast with the previous example, if $\mathcal{F}$ is a presheaf of locally constant functions (with values in some set $E$ ), then $\mathcal{F}$ is a sheaf.

Example 2.12. In at least one case, the presheaf of constant functions is a sheaf. Let $X$ be a space, and let $\mathcal{F}$ be the presheaf of abelian groups on $X$ which takes the trivial group for every open subset of $X$, including the empty set.

$$
\mathcal{F}(U)=\{0\}
$$

There is only one possible way to define the restriction maps, since there is a unique map from the trivial group to itself. So for any $V \subset U \subset X, \rho_{V}^{U}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is the zero map/identity map from the trivial group to itself. This is a sheaf; all the axioms are basically vacuous. It is called the zero sheaf on $X$.

Example 2.13. The skyscraper presheaf is a sheaf, as we now show. Let $X$ be a topological space, and fix $p \in X$, and let $E$ be a set. Recall that the skyscraper sheaf is defined by

$$
\mathcal{F}(U)= \begin{cases}E & p \in U \\ \{*\} & p \notin U\end{cases}
$$

with restriction maps given either by the identity $E \rightarrow E$ or by the unique map $E \rightarrow\{*\}$ as necessary. Uniqueness is fairly trivial. If $s, t \in \mathcal{F}(U)$ have restrictions which agree everywhere, then either $s, t \in\{*\}$, in which case they have to be the same point, or $s, t \in E$ with $\operatorname{Id}_{E}(s)=\operatorname{Id}_{E}(t)$, so either way $s=t$.

Gluing is also fairly trivial. If we have an open covering $U=\bigcup_{\alpha} U_{\alpha}$, and $s_{\alpha} \in U_{\alpha}$ with agreeing restrictions, there are two possibilities. If $p \notin U$, then all $s_{\alpha}$ are the same point in $\{*\}$, so they trivially glue to the unique section on $U$. If $p \in U$, then some $s_{\alpha}$ may be the unique point of $\{*\}$, but some may be points in $E$. Since they all have the same restriction, any $s_{\alpha} \in E$ must be the same value, so that common value is the resulting glued section on $U$.

### 2.3 Structure sheaf on a variety

We now return to considering the structure presheaf on an algebraic variety, which we can now prove is a sheaf. Let $K$ be an algebraically closed field, and $X \subset K^{n}$ an irreducible (Zariski) closed subset. Recall that the structure presheaf was defined by

$$
\mathcal{O}_{X}(U)=\text { subring of } K(X) \text { of rational functions defined at all points of } U
$$

and restriction maps were just given by inclusion maps. Before we show it is a sheaf, note that the following are equivalent.

1. $X$ is irreducible.
2. Every nonempty open subset of $X$ is dense.
3. Any two nonempty open subsets of $X$ have nonempty intersection.

Proposition 2.14. Let $X$ be an irreducible variety. Then $\mathcal{O}_{X}$ is a sheaf.
Proof. Separatedness is clear. For gluing, let $U \subset X$ be open and let $U=\bigcup_{\alpha} U_{\alpha}$ be an open cover, and suppose we have sections $f_{\alpha} \in \mathcal{O}_{X}\left(U_{\alpha}\right)$ for which restrictions agree.

$$
\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\alpha}}\left(f_{\alpha}\right)=\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\beta}}\left(f_{\beta}\right) \quad \forall \alpha, \beta
$$

By the equivalences above, any two $U_{\alpha}, U_{\beta}$ have nonempty intersection, and the restriction maps $\rho$ are just inclusions, so the equation above says that $f_{\alpha}=f_{\beta}$ inside $\mathcal{O}_{X}\left(U_{\alpha} \cap U_{\beta}\right)$ for every pair $\alpha, \beta$. That is, all of the $f_{\alpha}$ are equal, with common value $f$. This rational function $f$ is defined on each $U_{\alpha}$, so it is defined on all of $U$, so $f \in \mathcal{O}_{X}(U)$ is the needed global section.

Proposition 2.15. Let $X$ be an irreducible variety, and $\mathcal{O}_{X}$ the structure sheaf. If $U=$ $\bigcup_{\alpha} U_{\alpha}$ is an open subset of $X$, then

$$
\mathcal{O}_{X}(U)=\bigcap_{\alpha} \mathcal{O}_{X}\left(U_{\alpha}\right)
$$

Proof. The inclusion $\supset$ is obvious, since a rational function defined on all $U_{\alpha}$ is defined on the union $U$. The reverse inclusion $(\subset)$ is also easy, since if $f \in \mathcal{O}_{X}(U)$ is defined on $U$, it is defined on each $U_{\alpha}$, so it lies in the intersection $\bigcap_{\alpha} \mathcal{O}_{X}\left(U_{\alpha}\right)$.

Remark 2.16. The previous proposition says that it suffices to determine the structure sheaf $\mathcal{O}_{X}$ on a basis for the Zariski topology on $X$. In fact, this is true in general - for any sheaf, it is determined by the values on a basis.

Following the previous remark, we now determine $\mathcal{O}_{X}$ on the principal open subsets. Recall the notation

$$
D(p)=K^{n} \backslash V(p)=\left\{x \in K^{n}: p(x) \neq 0\right\}
$$

where $p \in K\left[x_{1}, \ldots, x_{n}\right]$ is some polynomial. We use the notation $D_{X}(p)=X \cap D(p)$ for the principal open subset of $X$ determined by $p$.

Proposition 2.17. Let $X$ be an irreducible algebraic variety and let $p \in K\left[x_{1}, \ldots, x_{n}\right]$. Then

$$
\mathcal{O}_{X}\left(D_{X}(p)\right)=K[X]_{p}
$$

where $K[X]_{p}$ denotes the localization of the ring $K[X]$ at the set $\left\{1, p, p^{2}, \ldots\right\}$.
Proof. The inclusion $K[X]_{p} \subset \mathcal{O}_{X}\left(D_{X}(p)\right)$ is obvious. The proof of the opposite inclusion is basically a repeat of the proof of Proposition 1.24 , but we include it anyway.

Let $f \in \mathcal{O}_{X}\left(D_{X}(p)\right)$. Then for every $x \in D_{X}(p)$, there exists a representation of $f$, $f=\frac{g_{x}}{h_{x}}$ with $g_{x}, h_{x} \in K[X]$ with $h_{x}(x) \neq 0$. Let $I \subset K[X]$ be the ideal generated by all $h_{x}$, and let $Y=V_{X}(I)$. Then $Y \cap D_{X}(p)=\emptyset$, so $Y \subset V_{X}(p)=X \backslash D_{x}(p)$. That is to say, $p$ vanishes on all of $Y$, and $p$ vanishes on all zeros of $I$. Hence $p \in \sqrt{I}$ by Hilbert's Nullstellensatz, so there exists $d \in \mathbb{Z}_{\geq 0}$ with $p^{d} \in I$. So we may write $p^{d}$ as

$$
p^{d}=r_{1} h_{x_{1}}+\cdots r_{t} h_{x_{t}}
$$

with $x_{i} \in D_{X}(p)$ and $r_{i} \in K[X]$. Multiplying this by $f$, we obtain

$$
f p^{d}=r_{1} h_{x_{1}} f+\cdots+r_{t} h_{x_{t}} f=r_{x} g_{x_{1}}+\cdots+r_{t} g_{x_{t}}
$$

The right hand side is in $K[X]$, so we can divide by $p^{d}$ and see that $f=\frac{g}{p^{d}}$ for some $d \geq 0$, that is, $f \in K[X]_{p}$.

Example 2.18. Let $X=K^{2}$, with coordinate ring $K[X]=K[x, y]$. Let $U_{x}, U_{y}, U_{x y}$ be the principal open subsets

$$
\begin{aligned}
U_{x} & =D_{X}(x)=\left\{(a, b) \in K^{2}: a \neq 0\right\} \\
U_{y} & =D_{X}(y)=\left\{(a, b) \in K^{2}: b \neq 0\right\} \\
U_{x y} & =D_{X}(x y)=\left\{(a, b) \in K^{2}: a b \neq 0\right\}
\end{aligned}
$$

Geometrically, $U_{x}$ and $U_{y}$ are each a disjoint union of two open half planes, cut in half by a missing axis, and $U_{x y}$ is a disjoint union of four open quarter planes. This is compatible with what Remark 1.13 tells us, that

$$
U_{x} \cap U_{y}=D_{X}(x) \cap D_{X}(y)=D_{X}(x y)=U_{x y}
$$

By Proposition 2.17,

$$
\begin{aligned}
\mathcal{O}_{X}\left(U_{x}\right) & =K[X]_{x}=K\left[x, x^{-1}, y\right] \\
\mathcal{O}_{X}\left(U_{y}\right) & =K[X]_{y}=K\left[x, y, y^{-1}\right] \\
\mathcal{O}_{X}\left(U_{x y}\right) & =K[x, y]_{x y}=K\left[x, y,(x y)^{-1}\right]
\end{aligned}
$$

We can also consider the open subset $U_{0}=U_{x} \cap U_{y}=K^{2} \backslash\{(0,0)\}$. Using Proposition 2.15.

$$
\mathcal{O}_{X}\left(U_{0}\right)=\mathcal{O}_{X}\left(U_{x}\right) \cap \mathcal{O}_{X}\left(U_{y}\right)=K\left[x, y, x^{-1}\right] \cap K\left[x, y y^{-1}\right]=K[x, y]=K[X]
$$

This gives an interesting example of how the sections over a proper subset may be the same as the global sections.

### 2.4 Morphisms of sheaves

Definition 2.19. A morphism of sheaves is just a morphism of the underlying presheaves.
Remark 2.20. Fix a topological space $X$ and a category $\mathcal{C}$. Sheaves on $X$ with values in $\mathcal{C}$ form a category with the morphisms described above, denoted $\operatorname{Sh}(X, \mathcal{C})$. Usually the category $\mathcal{C}$ is understood (usually sets, or abelian groups, or rings) and we just write $\operatorname{Sh}(X)$. $\operatorname{Sh}(X)$ is a full subcategory of $\operatorname{PSh}(X)$.

This will have to wait until much later in the course, but the big advantage of studying $\operatorname{Sh}(X)$ instead of $\operatorname{PSh}(X)$ is that $\operatorname{Sh}(X)$ is an abelian category, while $\operatorname{PSh}(X)$ is not. Eventually we will describe what this means, and give a proof.

Recall that earlier we defined the kernel and image presheaves of a morphism of presheaves in Definition 1.27, in the case where the target category was abelian groups.
Lemma 2.21. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves of abelian groups, and let $\mathcal{K}$ be the kernel presheaf. Then $\mathcal{K}$ is a sheaf.
Proof. We know that a subpresheaf of a sheaf is always separated, so $\mathcal{K}$ is separated. We just need to show that gluing holds. Let $U \subset X$ be an open set, with open cover $U=\bigcup_{\alpha} U_{\alpha}$, and suppose we have $s_{\alpha} \in U_{\alpha}$ so that they agree on the overlaps.

$$
\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\alpha}}\left(s_{\alpha}\right)=\rho_{U_{\alpha} \cap U_{\beta}}^{U_{\beta}}\left(s_{\beta}\right) \quad \forall \alpha, \beta
$$

(The restriction maps above are the restriction maps for $\mathcal{K}$. These are effectively the same as the restriction maps for $\mathcal{F}$ as well, restricted to the kernel of associated maps for $\phi$.) By the gluing property for $\mathcal{F}$, there is a section $s \in \mathcal{F}(U)$ such that $\rho_{U_{\alpha}}^{U}(s)=s_{\alpha}$ for each $\alpha$. We just need to show that $s \in \mathcal{K}(U)=\operatorname{ker} \phi_{U}$. This comes down to a diagram chase. For each $\alpha$, we have the following commutative diagram.

$$
\begin{aligned}
& \mathcal{K}\left(U_{\alpha}\right)=\operatorname{ker} \phi_{U_{\alpha}} \longleftrightarrow \mathcal{F}\left(U_{\alpha}\right) \xrightarrow[\phi_{U_{\alpha}}]{ } \mathcal{G}\left(U_{\alpha}\right)
\end{aligned}
$$

To show that $s \in \mathcal{K}(U)$, we need to show that $\phi_{U}(s)=0$. Equivalently, we need to show that $\phi_{U}(s)$ is zero "everywhere locally," meaning $\rho_{U_{\alpha}}^{U}(\mathcal{G}) \circ \phi_{U}(s)=0$ for each $\alpha$. But this follows immediately from the diagram above, since

$$
\rho_{U_{\alpha}}^{U}(\mathcal{G}) \circ \phi_{U}(s)=\phi_{U_{\alpha}} \circ \rho_{U_{\alpha}}^{U}(\mathcal{F})(s)=\phi_{U_{\alpha}}\left(s_{\alpha}\right)=0
$$

since $s_{\alpha} \in \mathcal{K}\left(U_{\alpha}\right)=\operatorname{ker} \phi_{U_{\alpha}}$. Thus $s \in \mathcal{K}(U)$, so $\mathcal{K}$ satisfies gluing and is hence a sheaf.
Remark 2.22. The previous lemma raises the question - do other constructions work this well? Are the image, cokernel, etc. presheaves also sheaves? It seems like things ought to work out. Unfortunately, this is really the only one of those where it works out. In order to remedy this and be sure that $\operatorname{Sh}(X)$ is a "good" category (has kernels, cokernels, images, etc.) we need a tool called sheafification. (Here "good" just means abelian, more on that much later.

### 2.5 First look at sheafification

In the following theorem, the word "morphism" at first appears ambiguous - is a morphism in $\operatorname{Sh}(X)$ or in $\operatorname{PSh}(X)$ ? However, we omit specifying, because it doesn't actually matter. A morphism of sheaves has no additional structure on top of being a morphism of presheaves.

Theorem 2.23. Let $\mathcal{F}$ be a presheaf on a space $X$. There exists a sheaf $\mathcal{F}^{+}$and a morphism $\theta: \mathcal{F} \rightarrow \mathcal{F}^{+}$with the following universal property. If $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism, then there exists a unique morphism $\psi: \mathcal{F} \rightarrow \mathcal{G}$ making the following diagram commute.


Proof. Much later in the course.
Definition 2.24. The sheaf $\mathcal{F}^{+}$of the previous theorem is called the sheafification of $\mathcal{F}$.
Remark 2.25. A fuller statement of the previous theorem includes information about stalks, but that will have to wait until after we define stalks.

Remark 2.26. As usual for universal properties, it follows immediately that $\mathcal{F}^{+}$is unique up to unique isomorphism.

Remark 2.27. Let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves, and let $\theta_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}^{+}$and $\theta_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{G}^{+}$be the sheafifications with associated morphisms. By the universal property, there is a unique morphism $\alpha^{+}: \mathcal{F}^{+} \rightarrow \mathcal{G}^{+}$making the following diagram commute.


This diagram makes a lot more sense written as a square.


That is to say, sheafification is not just an association on objects, but also induces morphisms. So sheafification is a covariant functor $S: \operatorname{PSh}(X) \rightarrow \operatorname{Sh}(X)$.
Remark 2.28. Let $\mathcal{F}$ be a presheaf with sheafification $\theta_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}^{+}$and let $\mathcal{G}$ be a sheaf. The universal property gives a bijection

$$
\operatorname{Hom}_{\operatorname{Sh}(X)}\left(\mathcal{F}^{+}, \mathcal{G}\right) \rightarrow \operatorname{Hom}_{\operatorname{PSh}(X)}(\mathcal{F}, \mathcal{G}) \quad \psi \mapsto \theta_{\mathcal{F}} \psi
$$

This isomorphism is "natural," meaning that it really comes from a natural isomorphism of bifunctors. More concretely, it means that the following diagrams commute. Let $\mathcal{H}$ be another presheaf on $X$, with sheafification $\theta_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{H}^{+}$, and let $\alpha: \mathcal{F} \rightarrow \mathcal{H}$ be a morphism, and let $\alpha^{+}: \mathcal{F}^{+} \rightarrow \mathcal{H}^{+}$be the induced morphism on the sheafifications. Then the following diagram commutes.


Similarly, if $\mathcal{K}$ is another sheaf, and $\beta: \mathcal{G} \rightarrow \mathcal{K}$ is a morphism, then the following diagram commutes.


Together, these commutative diagrams say that there is a natural isomorphism of bifunctors

$$
\operatorname{Hom}_{\operatorname{Sh}(X)}\left((-)^{+}, *\right) \cong \operatorname{Hom}_{\operatorname{PSh}(X)}(-, *)
$$

The first diagram captures naturality in the first argument, and the second commutative diagram captures naturality in the second argument. Another way to say this is using the language of "adjoint" functors, which we describe next.

Definition 2.29. Let $\mathcal{A}, \mathcal{B}$ be categories and $S: \mathcal{A} \rightarrow \mathcal{B}, T: \mathcal{B} \rightarrow \mathcal{A}$ be covariant functors. $S, T$ are adjoint if there is a natural isomorphism of bifunctors

$$
\operatorname{Hom}_{\mathcal{B}}(S(-), *) \cong \operatorname{Hom}_{\mathcal{A}}(-, T(*))
$$

"Natural isomorphism of bifunctors" means that for every $A \in \operatorname{Ob}(\mathcal{A}), B \in \operatorname{Ob}(\mathcal{B})$, there is an isomorphism

$$
\tau_{A B}: \operatorname{Hom}_{\mathcal{B}}(S A, B) \rightarrow \operatorname{Hom}_{\mathcal{A}}(A, T B)
$$

with commutative diagram properties analogous to the previous remark. Specifically, if $f: A \rightarrow A^{\prime}$ is a morphism in $\mathcal{A}$ and $g: B \rightarrow B^{\prime}$ is a morphism in $\mathcal{B}$, then the following diagrams commute.


Remark 2.30. Using the language above of adjoint functors, we can say more about the isomorphism given by sheafification on hom sets. Let $S: \operatorname{PSh}(X) \rightarrow \operatorname{Sh}(X)$ be the sheafification functor, and let $T: \operatorname{Sh}(X) \rightarrow \operatorname{PSh}(X)$ be the forgetful functor. Then $S, T$ are adjoint. This is the content of remark 2.28 ,

## 3 Limits and colimits

At this point in the course, we take a break from discussing sheaves and presheaves to do some general category theory. In particular, we need to develop definitions and basic properties of limits and colimits, since these ideas are necessary to define and study stalks of sheaves.

### 3.1 Direct limits

Definition 3.1. A filtered set is a set $I$ with a relation $\leq$ which is reflexive and transitive, with the additional property that for every $i, j \in I$, there exists $k \in I$ with $i \leq k$ and $j \leq k$.

Remark 3.2. Often one encounters the previous definition in the form where $\leq$ is taken to be a partial ordering on $I$, but this includes the additional property that $i \leq j$ and $j \leq i$ implies $i=j$. Since we want to use filtered sets in some situations where this property does not hold, we leave it out of our definition. But it doesn't really hurt anything either way.

Definition 3.3. A morphism of filtered sets is a set map $f: I \rightarrow J$ which preserves the ordering, i.e. $i \leq j \Longrightarrow f(i) \leq f(j)$.

Definition 3.4. Let $\mathcal{C}$ be a category and $I$ a filtered set. A direct system in $\mathcal{C}$, also called an inductive system, is a family of objects $\left\{A_{i}: i \in I\right\}$ along with morphisms $\tau_{i}^{j}: A_{i} \rightarrow A_{j}$ whenever $i \leq j$, such that $\tau_{i}^{i}=\operatorname{Id}_{A_{i}}$ and whenever $i \leq j \leq k$, the following diagram commutes.


Definition 3.5. Let $\left\{A_{i}, \tau_{j}^{i}: i \in I\right\}$ and $\left\{B_{j}, \sigma_{j}^{i}: j \in J\right\}$ be directed systems (over possibly different filtered sets) with values in the same category $\mathcal{C}$. A morphism of directed systems is a morphism $\psi: I \rightarrow J$ of filtered sets, along with a family of maps $\psi_{i}: A_{i} \rightarrow B_{\psi(i)}$ such that for every $i \leq j$ in $I$, the following diagram commutes.


Definition 3.6. Let $\left\{A_{i}, \tau_{i}^{j}\right\}$ be a direct system in $\mathcal{C}$. A direct limit of the system is an object $A=\underline{\lim } A_{i}$ with morphisms $\sigma_{i}: A_{i} \rightarrow A$ making the following diagrams commute whenever $i \leq \vec{j}$,


Additionally, $A$ satisfies the following universal property. If $B$ is any other object in $\mathcal{C}$ with morphisms $\phi_{i}: A_{i} \rightarrow B$ making the analogous triangles commute, i.e.

then there exists a unique map $h: A \rightarrow B$ making the following diagram commute (for every $i \leq j$ ).


Remark 3.7. A morphism of directed systems induces a morphism (in the target category) between the direct limit objects, in the following way. Suppose $\left\{A_{i}, \tau_{i}^{j}\right\}$ and $\left\{B_{i}, \phi_{i}^{j}\right\}$ are directed systems over the respective filtered sets $I, J$, with values in a category $\mathcal{C}$ which has direct limits. Let $\psi: I \rightarrow J, \psi_{i}: A_{i} \rightarrow B_{\psi(i)}$ be a morphism of directed systems. Let

$$
A=\underset{\longrightarrow}{\lim } A_{i} \quad B=\underset{\longrightarrow}{\lim } B_{i}
$$

be the direct limits, with associated maps $\sigma_{i}: A_{i} \rightarrow A$ and $\theta_{i}: B_{i} \rightarrow B$. Then for $i \leq j$ in $I$, we have the following commutative diagram.


Thus by the universal property, there exists a unique map $h: A \rightarrow B$ making the diagram commute.


This map $h$ is the map induced on the direct limits by the morphism $\psi$ of directed systems. We write this as $h=\underset{\longrightarrow}{\lim } \psi: \xrightarrow{\lim } A_{i} \rightarrow \underset{\longrightarrow}{\lim } B_{i}$.

Remark 3.8. The previous construction of induced map on directed systems makes $\underset{\longrightarrow}{\lim }$ into a covariant functor from the category of directed systems (with values in a fixed category $\mathcal{C}$ ) to the category $\mathcal{C}$.

### 3.2 Examples of direct limits

Proposition 3.9. Direct limits exist in the category of sets. More specifically, if $\left\{A_{i}, \tau_{i}^{j}\right\}_{i, j \in I}$ is a directed system of sets, the direct limit is

$$
\underset{\longrightarrow}{\lim } A_{i}=\left(\bigsqcup_{i \in I} A_{i}\right) / \sim
$$

where $\sim$ is an equivalence relation determined by $a \sim \tau_{i}^{j}(a)$ for any $a \in A_{i}$.
Proof. Let $\left\{A_{i}, \tau_{i}^{j}\right\}$ be a directed system of sets, over the filtered set $I$. Define

$$
\widetilde{A}=\bigsqcup_{i \in I} A_{i}
$$

Then define an equivalence relation on $\widetilde{A}$ as follows. For $a_{i} \in A_{i}$ and $a_{j} \in A_{j}$, we say $a_{i} \sim a_{j}$ if there exists $k \in I$ with $i, j \leq k$ such that

$$
\tau_{i}^{k}\left(a_{i}\right)=\tau_{j}^{k}\left(a_{j}\right)
$$

More succinctly, for every $a_{i} \in A_{i}$ and every $j$ such that $i \leq j, a_{i} \sim \tau_{i}^{j}\left(a_{i}\right)$ and $\tau_{i}^{j}\left(a_{i}\right) \sim a_{i}$. It is clear that $\sim$ is reflexive and symmetric. Transitivity can be worked out in tedious detail if necessary, but we omit it here. Hence $\sim$ is an equivalence relation. Now define

$$
A=\widetilde{A} / \sim
$$

There are obvious choices of maps $\sigma_{i}: A_{i} \rightarrow A$, given by $a_{i} \mapsto\left[a_{i}\right]$, where $\left[a_{i}\right]$ is the equivalence class of $a_{i}$. We claim that $A$ with these maps $\sigma_{i}$ is a (the) direct limit of the system $\left\{A_{i}, \tau_{i}^{j}\right\}$. It suffices to verify the universal property. Suppose we have a set $B$ with maps $\phi_{i}: A \rightarrow B$ such that the following diagram commutes for every $i \leq j$ in $I$.


Then define $h: A \rightarrow B$ by $\left[a_{i}\right] \mapsto \phi_{i}\left(a_{i}\right)$. This is well defined, because of $\left[a_{i}\right]=\left[a_{j}\right]$, then there exists $k$ with $i, j \leq k$ and $\tau_{i}^{k}\left(a_{i}\right)=\tau_{j}^{k}\left(a_{j}\right)$. Applying $\phi_{k}: A_{k} \rightarrow B$ to both sides of this, we get

$$
\phi_{i}\left(a_{i}\right)=\phi_{k} \tau_{i}^{k}\left(a_{i}\right)=\phi_{k} \tau_{j}^{k}\left(a_{j}\right)=\phi_{j}\left(a_{j}\right)
$$

Thus $h$ is well defined. From the definition of $h$, it is immediate that the following diagram commutes.


It is also relatively immediate that $h$ is unique. Any map $t: A \rightarrow B$ making the left triangle commute satisfies

$$
\phi_{i}\left(a_{i}\right)=t \sigma_{i}\left(a_{i}\right)=t\left[a_{i}\right]
$$

so it must be the same as our $h$.
Remark 3.10. Roughly the same construction as above works to show that direct limits exist in the following categories: groups, abelian groups, modules over a ring $R$. In each case, the part that changes the most is that disjoint union is replaced by the coproduct in the appropriate category. So for example in abelian groups or modules, the coproduct is direct sum, so the direct limit is

$$
\lim _{\longrightarrow} A_{i}=\left(\bigoplus_{i} A_{i}\right) / \sim
$$

where the $\sim$ is roughly the same equivalence relation. In groups, the coproduct is the free product, which is somewhat more complicated to work out usually.

In fact, the construction above generalizes to any abelian category with arbitrary coproducts (recall that an abelian category is required to have all finite coproducts, but may fail to have infinite coproducts, which may be necessary to have some direct limits).

Example 3.11. We give an example which demonstrates that the ambient category has a large impact on the direct limit, even using "the same" directed system. Consider the directed system


The number on the arrow indicates that the morphism is multiplication by that number, so for example $3: \mathbb{Z} / 2 \mathbb{Z} \rightarrow \mathbb{Z} / 6 \mathbb{Z}$ sends 1 to 3 , and 2 to $6=0$. The above is a directed system, and we may consider it as a directed system in the category of abelian groups, or in the category of groups. We will show that the direct limits are diffent depending on this choice of perspective.

In the category of abelian groups, as we said the direct limit is a quotient of the direct sum, so the direct limit is a quotient of $\mathbb{Z} / 2 \mathbb{Z} \oplus \mathbb{Z} / 4 \mathbb{Z} \oplus \mathbb{Z} / 6 \mathbb{Z}$. It is enough for this example to note that it has finite order, $\leq 48$.

In the category of groups, the direct limit is a quotient of the free product $\mathbb{Z} / 2 \mathbb{Z} * \mathbb{Z} / 4 \mathbb{Z} *$ $\mathbb{Z} / 6 \mathbb{Z}$, and one can work out that the free product in question is infinite, and quotiet subgroup has infinite index, so that the resulting direct limit has infinite order. It is also relatively easy to show that this direct limit is nonabelian. So in multiple ways, it cannot possibly be the same as the direct limit in the category of abelian groups.

Example 3.12 (Union). In this example we realize the usual union of sets as a direct limit. Suppose $I$ is a totally ordered set and $\left\{A_{i}\right\}$ is a directed system of sets, with all maps $\tau_{i}^{j}: A_{i} \rightarrow A_{j}$ just being inclusion maps. Then the direct limit is just the union.

$$
\xrightarrow[\longrightarrow]{\lim } A_{i}=\bigcup_{i \in I} A_{i}
$$

Example 3.13 (Localization). In this example we realize the construction of localization of a ring as a direct limit $3^{3}$. Fix a commutative ring $R$, and for the sake of simplicity assume $R$ is an integral domain (this is not necessary, but without it we would have to deal with many technicalities involving zero divisors).

Let $S \subset R$ be a multiplicative subset, and for $s \in S$, let $R_{s}$ be the localization of $R$ at the multiplicative set $\left\{1, s, s^{2}, \ldots\right\}$. The set $S$ is filtered with respect to $s \leq t \Longleftrightarrow \exists u \in$
 a map

$$
\tau_{s}^{t}: R_{s} \rightarrow R_{t} \quad \frac{a}{s^{n}} \mapsto \frac{a u^{n}}{t^{n}}=\frac{a}{s^{n}}
$$

We also have maps $R \rightarrow R_{s}, r \mapsto \frac{r}{1}$, which are compatible with the $\tau$ maps in the sense that the following diagrams commute for $s \leq t$.

[^2]

Thus the rings $R_{s}$ with maps $\tau_{s}^{t}$ form a directed system. We claim that $R_{S}=\underset{\longrightarrow}{\lim } R_{s}$, but omit the verification. (Recall that $R_{S}$ is the localization of $R$ at all of $S$.) The maps associated with the direct limit are just inclusions $\theta_{S}: R_{s} \hookrightarrow R_{S}$, and the universal property is relatively straightforward to verify, using the universal property of localizations.

### 3.3 Stalks of (pre)sheaves

Definition 3.14. Let $X$ be a topological space, and let $\mathcal{F}$ be a presheaf on $X$ with values in a category $\mathcal{C}$ which has direct limits (such as sets, or abelian groups). Fix a point $x$. Define

$$
\mathcal{U}=\{U \subset X: U \text { is open, and } x \in U\}
$$

Then partially order $\mathcal{U}$ by reverse inclusion, that is, $U \leq V$ if and only if $U \supset V$. Then $\mathcal{U}$ is a filtered/directed set, and because $\mathcal{F}$ is a presheaf, we have maps

$$
\rho_{V}^{U}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)
$$

whenever $U \leq V$ (equivalently $V \subset U$ ). Thus

$$
\{\mathcal{F}(U): U \in \mathcal{U}\}
$$

is a directed system with values in $\mathcal{C}$. The stalk of $\mathcal{F}$ at the point $x$ is the directed limit of this system. It is denoted $\mathcal{F}_{x}$.

$$
\mathcal{F}_{x}=\lim _{x \in U} \mathcal{F}(U)
$$

Remark 3.15. Suppose $\mathcal{F}$ is a presheaf of sets. How can we describe elements of the stalk $\mathcal{F}_{x}$ ? Recall that the direct limit is constructed as a disjoint union of the sets $\mathcal{F}(U)$, modulo some equivalence relation. So an element $\phi \in \mathcal{F}_{x}$ has a representative as a section $f \in \mathcal{F}(U)$, and moreover two such sections $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$ represent the same element $\phi$ in the stalk $\mathcal{F}_{x}$ if and only if there is some smaller open set $W$ to which the sections $f, g$ have the same restriction, that is, if

$$
\rho_{W}^{U}(f)=\rho_{W}^{V}(g)
$$

So intuitively speaking, an element of the stalk $\mathcal{F}_{x}$ is a "section" on a neighborhood of $x$, except it's identified with "similar" sections which agree on small neighborhoods of $x$. So an element of $\mathcal{F}_{x}$ is sort of like a germ of a function at $x$. In the case where $\mathcal{F}$ is a sheaf of continuous functions, then the stalk $\mathcal{F}_{x}$ is literally germs of functions at $x$, as we will see in examples soon.

Example 3.16 (Constant presheaf). Let $X$ be a space and $E$ a set, and $\mathcal{F}$ be the constant presheaf with value $E$ (or think of $\mathcal{F}$ as the presheaf of constant-valued functions in $E$.)

$$
\mathcal{F}(U)= \begin{cases}E & U \neq \emptyset \\ \{*\} & U=\emptyset\end{cases}
$$

with the obvious restriction maps. Then the stalk any any point $x \in X$ is $E$, since the directed system of $\mathcal{F}(U)$ for $x \in U$ is just a directed system where every $\mathcal{F}(U)=E$ and every map is the identity map. (The set $\{*\}$ is not involved, since $x \notin \emptyset$.)

Example 3.17. As a particular example of a constant presheaf, let $\mathcal{F}$ be the zero sheaf. Then the stalks $\mathcal{F}_{x}$ are all zero (that is, the trivial group).

The next example pairs well as a contrast with the constant presheaf, and will we see shortly that they are very importantly related.

Example 3.18 (Locally constant (pre)sheaf). Let $X$ be space and $E$ a set, and $\mathcal{F}$ be the locally constant presheaf with values in $E$.

$$
\mathcal{F}(U)=\{\text { locally constant functions } U \rightarrow E\}
$$

Then we claim that for $x \in X$, the stalk $\mathcal{F}_{x}$ is isomorphic to $E$. Why is this? Intuitively speaking, germs of locally constant functions are determined by the value at $x$.

More precisely, $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are both sections with $x \in U \cap V$, and they represent the same element of the stalk $\mathcal{F}_{x}$, then $f, g$ coincide (as functions) on some open neighborhood of $x$, so in particular, $f(x)=g(x)$. Conversely, if $f(x)=g(x)$, then because $f, g$ are locally constant there exists a neighborhood $W$ of $x$ so that $\left.f\right|_{W}=\left.g\right|_{W}$, which is to say,

$$
\rho_{W}^{U}(f)=\rho_{W}^{V}(g)
$$

meaning that $f, g$ represent the same element of the stalk $\mathcal{F}_{x}$. All this to say, $f \sim g$ in $\mathcal{F}_{x}$ if and only if $f(x)=g(x)$. Hence there is a map

$$
\mathcal{F}_{x} \rightarrow E \quad[f] \mapsto f(x)
$$

which is well defined and injective, by the preceding discussion. It is also clearly surjective, since any $e \in E$ is the image of the class of a constant function with value $e$. Thus $\mathcal{F}_{x} \cong E$.

Remark 3.19. We will see later that the locally constant sheaf with values in $E$ is the sheafification of the constant sheaf with values in $E$. The fact that in the preceding two examples, these two sheaves have isomorphic stalks at every point (meaning for $x \in X$, the stalk of the locally constant sheaf is the same as the stalk of the constant sheaf, not that they are the same for different points), is a general phenomenon. That is to say, sheafification does not change stalks (up to isomorphism).

Example 3.20 (Skyscraper sheaf). Let $X$ be a space, $E$ be a set, and fix $p \in X$. Let $\mathcal{F}$ be the skyscraper sheaf at $p$.

$$
\mathcal{F}(U)= \begin{cases}E & p \in U \\ \{*\} & p \notin U\end{cases}
$$

with the obvious restriction maps. Then the stalk at $p$ is $\mathcal{F}_{p} \cong E$, basically by the same argument as with the constant sheaf.

However, for $x \neq p$, the stalk $\mathcal{F}_{x}$ depends on the topological properties of $X$. If $x \in \overline{\{p\}}$ ( $x$ is in the closure of the point $p$ ), then every open neighborhood of $p$ contains $x$, then $\mathcal{F}_{x} \cong E$, since every neighborhood of $x$ has $\mathcal{F}(U)=E$.

On the other hand, if $x \notin \overline{\{p\}}$, then there exists an open neighborhood of $x$ which does not contain $p$. In this case, the directed system which determines $\mathcal{F}_{x}$ includes the set $\{*\}$ as one of the sets (probably many), so $\mathcal{F}_{x} \cong\{*\}$.

Remark 3.21. From the previous example, the lesson is that stalks often capture information about "separation of points," in a topological sense. If two points are so "close" together that they share all the same neighborhoods, then they will have the same stalk. This doesn't come up too often in useful topological spaces, but an example is something like the real line with two origins (see https://ncatlab.org/nlab/show/line+with+two+origins or other sources).

In particular, such a phenomenon means the space is not Hausdorff, but even most nonHausdorff spaces don't have such strange "non-separated points." For example, the Zariski topology on an affine variety is far from being Hausdorff, but at least such spaces do not have pairs of "non-separated points."

Example 3.22 (Structure sheaf of a variety). Let $K$ be an algebraically closed field, and $X \subset K^{n}$ an Zariski-closed subset, a.k.a. $X$ is an affine variety. We defined the structure sheaf $\mathcal{O}_{X}$ previously, and computed various aspects of it. Now we compute the stalk $\mathcal{O}_{X, x}$ for a point $x \in X$.

If $U \subset X$ is an open neighborhood of $x$, and $f \in \mathcal{O}_{X}(U)$, then $f$ is a rational function on $X$ which is defined on $U$. In particular, $f$ is defined at $x$. Conversely, if $f$ is defined at $x$, we can write $f$ as $f=\frac{g}{h}$ with $g, h \in K[X]$ and $h(x) \neq 0$. So $f$ is defined on the principal open subset

$$
D_{X}(h)=\{p \in X: h(p) \neq 0\}
$$

Thus

$$
\mathcal{O}_{X, x}=\underset{x \in U}{\lim } \mathcal{O}_{X}(U)=\{\text { rational functions on } X, \text { defined at } x\}
$$

We can describe this more algebraically, using localization. Let $m_{x} \subset K[X]$ be the ideal of functions that vanish at $x$. Since $K[X] / m_{x} \cong K, m_{x}$ is a maximal ideal. By definition, $f \in K[X]$ is defined at $x$ is we can write $f=\frac{g}{h}$ with $h(x) \neq 0$, which is to say, $h \notin m_{x}$. Thus $\mathcal{O}_{X, x}$ is all rational functions with denominator not in $m_{x}$. That is,

$$
\mathcal{O}_{X, x}=K[X]_{m_{x}}
$$

(Recall that the localization at a prime ideal means localizing at the complement of the prime ideal, so things outside of $m_{x}$ get inverted.) Note that we can also take the direct limit over just principal open sets containing $x$,

$$
\mathcal{O}_{X, x}=\underset{x \in U}{\lim _{x}} \mathcal{O}_{X}(U)=\underset{p \in X, \overrightarrow{x \in D_{X}(p)}}{\lim _{X}} \mathcal{O}_{X}\left(D_{X}(p)\right)
$$

Using this description and our previous knowledge that $\mathcal{O}_{X}\left(D_{X}(p)\right)=K[X]_{p}$, we can fit all of this together as

$$
\mathcal{O}_{X, x}=K[X]_{m_{x}}=\underset{x \in \xrightarrow[D_{X}(p)]{ }}{\lim _{X}} \mathcal{O}_{X}\left(D_{X}(p)\right)=\underset{x \in \overrightarrow{D_{X}}(p)}{\lim } K[X]_{p}
$$

This isn't saying anything new, just confirming things in different ways.
Remark 3.23. A morphism of presheaves induces a morphism on the stalks in a formal categorical way, which we now describe. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on $X$, for simplicity assume it is a presheaf of sets. Then for every $V \subset U \subset X$ open sets, we have a commutative diagram


By considering such diagrams as $U$ ranges over open neighborhoods of a fixed point $x \in X$, we get a morphism of directed systems

$$
\{\mathcal{F}(U): x \in \mathcal{U}\} \rightarrow\{\mathcal{G}(U): x \in U\}
$$

This morphism of direct systems induces a morphism on the direct limits, which are the respective stalks.

$$
\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}
$$

Remark 3.24. The previous remark has signification consequences for the sheafification functor, as we will see later. If $\mathcal{F}$ is a presheaf and $\theta: \mathcal{F} \rightarrow \mathcal{F}^{+}$is the sheafification, then $\theta$ induces a morphism on stalks

$$
\theta_{x}: \mathcal{F}_{x} \rightarrow \mathcal{F}_{x}^{+}
$$

Later we will prove that $\theta_{x}$ is always an isomorphism for every $x \in X$. That is to say, a presheaf and its sheafification have naturally isomorphic stalks.

### 3.4 Inverse limits

Basically, to get inverse limits, take all the definitions for direct limits and reverse all of the arrows.

Definition 3.25. Let $\mathcal{C}$ be a category and $I$ a filtered set. An inverse system in $\mathcal{C}$ indexed by $I$ is a family of objects $\left\{S_{i} \mid i \in I\right\}$ in $\mathcal{C}$ and morphisms

$$
\pi_{i}^{j}: S_{j} \rightarrow S_{i} \quad i \leq j
$$

such that $\pi_{i}^{i}=\operatorname{Id}_{S_{i}}$ and the following diagram commutes whenever $i \leq j \leq k$.


Remark 3.26. We leave it to the reader to formulate the definition of a morphism of inverse systems. It is identical to the definition of morphism of directed systems, with arrows reversed.

Definition 3.27. Let $\left\{S_{i}, \pi_{i}^{j}\right\}$ be an inverse system in $\mathcal{C}$, indexed by a filtered set $I$. An inverse limit of the system is an object $S$ with morphissm $\gamma_{i}: S \rightarrow S_{i}$ such that the following diagrams commute for every $i \leq j$ in $I$,

and such that $S$ is universal in this diagram. Explicitly, that means that if $T$ is any object with morphisms $\psi_{i}: T \rightarrow S_{i}$ making the analogous triangle as above commute, then there exists a unique morphism $h: T \rightarrow S$ making the following diagram commute.


When $S$ exists, we write $S=\lim _{\leftrightarrows} S_{i}$.
Remark 3.28. As with directed systems, a morphism of inverse systems induces a morphism on the inverse limits, provided the limits exist. This is essentially a consequence of the universal property.

Remark 3.29. If an inverse limit exists, it is unique up to isomorphism. This is an immediate consequence of the universal property.

Proposition 3.30. Inverse limits exist in the category of sets.
Proof. Let $\left\{S_{i}, \pi_{i}^{j}\right\}$ be an inverse system of sets. Consider

$$
\widetilde{S}=\prod_{i \in I} S_{i}
$$

with projection maps

$$
\widetilde{\gamma}_{i}: \widetilde{S} \rightarrow S_{i}
$$

Define

$$
S=\left\{\left(s_{i}\right) \in \widetilde{S} \mid \pi_{i}^{j}\left(s_{j}\right)=s_{i}, \forall i \leq j\right\}
$$

with projection maps $\gamma_{i}=\left.\widetilde{\gamma}_{i}\right|_{S}: S \rightarrow S_{i}$. We claim that $S$ with the maps $\gamma_{i}$ is an inverse limit of the system. The verification of various commutative diagrams and the universal property are all straightforward.

Remark 3.31. Even when it exists, the inverse limit may display somewhat "pathological" behavior, in the sense that the constructed set $S$ may be empty. See the following example.

Example 3.32. Let $I=\mathbb{N}=\{1,2,3 \ldots\}$ with the usual ordering. For $i \in I$, let $S_{i}=\mathbb{N}$, and for $i \leq j$ set

$$
\pi_{i}^{j}: S_{i} \rightarrow S_{j} \quad n \mapsto n+(j-i)
$$

Since $i \leq j, j-i \in \mathbb{Z}_{\geq 0}$, so the map $\pi_{i}^{j}$ does land in $\mathbb{N}$. This is an inverse system, but we claim that the inverse limit is the empty set. Let $S=\varliminf_{\longleftarrow} S_{i}$, and suppose $\left(s_{i}\right) \in \varliminf_{\longleftarrow}{ }_{i} S_{i}$. Set $n=s_{1} \in S_{1}$. By definition of $\pi_{i}^{j}$,

$$
\pi_{1}^{n+1}\left(s_{n+1}\right)=s_{n+1}+n+1-1=s_{n+1}+n
$$

On the other hand, because $\left(s_{i}\right) \in \lim S_{i}$, it has the property that

$$
\pi_{1}^{n+1}\left(s_{n+1}\right)=s_{1}=n
$$

Hence $s_{n+1}+n=n$ so $s_{n+1}=0$. But $0 \notin \mathbb{N}$, so this is impossible. Thus ${\underset{L i m}{~}}_{\rightleftarrows} S_{i}$ contains no sequences, it is empty.

Remark 3.33. The previous example shows that while inverse limits of sets always exist, they don't always behave that well. The next proposition shows that under certain conditions, the inverse limit is not empty, although the proof is more complicated than you would think.

Proposition 3.34. Let I be a filtered set and $\left\{S_{i}, \pi_{i}^{j}\right\}$ an inverse system of nonempty finite sets over I. Then $\varliminf_{\leftrightarrows} S_{i}$ is not empty.

Proof. Give each $S_{i}$ the discrete topology, so it is compact. By Tychonoff's theorem, $\widetilde{S}=$ $\prod_{i \in I} S_{i}$ is compact in the product topology. For each $j \in I$, set

$$
T_{j}=\left\{\left(s_{i}\right) \in \widetilde{S} \mid \pi_{i}^{j}\left(s_{j}\right)=s_{i}, \forall i \leq j\right\}
$$

Note that $T_{j}$ is not empty, because we can take $s_{j} \in S_{j}$ and set $s_{i}:=\pi_{i}^{j}\left(s_{i}\right)$ for all $i \leq j$ and take $s_{k} \in S_{k}$ arbitrary for the other sets.

We claim $T_{j}$ is closed in $\widetilde{S}$ (in the product topology). We will show the complement is open by taking an arbitary element of $\widetilde{S} \backslash T_{j}$ and finding an open neighborhood for it. Suppose $s=\left(s_{i}\right) \in \widetilde{S} \backslash T_{j}$. Then by definition of $T_{j}$, there exists $i \in I$ with $i \leq j$ such that $\pi_{i}^{j}\left(s_{j}\right) \neq s_{i}$. Then define for $k \in I$,

$$
V_{k}= \begin{cases}\left\{s_{i}\right\} & k=i \\ \left\{s_{j}\right\} & k=j \\ S_{k} & k \neq i \text { and } k \neq j\end{cases}
$$

Then the following set $W$ is an open subset of $\widetilde{S}$, by definition of the product topology.

$$
W=\prod_{k \in I} V_{k}
$$

Also, $W$ is an open neighborhood of $s=\left(s_{i}\right)$, and $W \cap T_{j}=\emptyset$. Thus the complement of $T_{j}$ is open, so $T_{j}$ is closed.

Note that if $i \leq j$, then $T_{j} \supset T_{i}$. So since $I$ is filtered, any finite intersection of the $T_{j}$ is nonempty. Since $\widetilde{S}$ is compact, and any finite intersection of the $T_{j}$ is nonempty, by a standard result in point-set topology $5^{5}$

$$
\bigcap_{j \in I} T_{j} \neq \emptyset
$$

Also, it is clear that $S=\lim _{\rightleftharpoons} S_{i}=\bigcap_{j} T_{j}$, hence the inverse limit is not empty.
Remark 3.35. The previous proposition generalizes to the following: an inverse limit of nonempty compact Hausdorff spaces is nonempty.
Definition 3.36. Let $I$ be a filtered set and $\left\{A_{i}, \pi_{i}^{j}\right\}$ an inverse system indexed by $I$. A subset $J \subset I$ is cofinal if for all $i \in J$, there exists $j \in J$ with $i \leq j$.
Remark 3.37. If $\left\{A_{i}, \pi_{i}^{j}\right\}_{i, j \in I}$ is an inverse system indexed by $I$ and $J \subset I$ is cofinal, then $\left\{A_{i}, \pi_{i}^{j}\right\}_{i, j \in J}$ is also an inverse system, and there is a natural isomorphism

$$
{\underset{\dddot{i m}}{\overleftarrow{\prime}}} A_{i} \cong \lim _{\overleftarrow{i \in J}} A_{i}
$$

given by projection. Intuitively speaking, the inverse limit is sometimes determined by what happens on a subset $J$ of the indexing set $I$, so one can "throw away" the unimportant indices and still get the same limit.

[^3]Remark 3.38. A filtered set $I$ can be thought of as a small category, whose objects are the elements of $I$, and whose morphisms are given by the $\leq$ relation. That is, if $i \leq j$, there is unique morphism $i \rightarrow j$, along with the necessary identity arrows. In this way, an inverse system is a functor from $I$ to $\mathcal{C}$.

### 3.4.1 Inverse limits of groups

Remark 3.39. Inverse limits exist in the category of groups. The construction is basically the same as in the category of sets, except that all the maps involved are group homomorphisms instead of arbitrary set maps.

Also, it is clear that the inverse limit of groups is always nonempty, since the inverse limit group must have an identity element. However, it may still be the trivial group.

Definition 3.40. An inverse limit of finite groups is called a profinite group.
If $G=\lim G_{i}$ is profinite with each $G_{i}$ finite, then giving each $G_{i}$ the discrete topology gives a topology to $G$ as a subspace of the product space. This topology on $G$ is in general not discrete, but it is compact, Hausdorff, and totally disconnected. These are all pretty immediate from basic point-set topological facts.

What is more interesting is the converse: a topological group $G$ is profinite if and only if it is compact, Hausdorff, and totally disconnected. The converse is harder to prove, and not useful for this course, so we skip over the details.

Remark 3.41. An important application of inverse limits of groups is the generalization of Galois theory for finite field extensions to Galois theory for infinite field extensions. Let $L / K$ be a finite field extension. In finite Galois theory, the main theorem is an inclusion-reversing correspondence

$$
\begin{aligned}
\text { \{subgroups of } \operatorname{Gal}(L / K)\} & \longleftrightarrow\{\text { intermediate subfields } K \subset F \subset L\} \\
H & \longleftrightarrow L^{H} \\
\operatorname{Gal}(L / F) & \longleftrightarrow F
\end{aligned}
$$

As stated, this does not generalize to the case where $L / K$ is a field extension of infinite degree. However, the Galois group $\operatorname{Gal}(L / K)$ is a profinite group:

$$
\operatorname{Gal}(L / K) \cong \lim _{\leftrightarrows} \operatorname{Gal}(F / K)
$$

where in the inverse limit $F$ ranges over all finite Galois extensions of $K$. This induces a topology on $\operatorname{Gal}(L / K)$, and using this topology, the main theorem of Galois theory generalizes to an inclusion-reversing correspondence

$$
\begin{aligned}
\{\text { closed subgroups of } \operatorname{Gal}(L / K)\} & \longleftrightarrow\{\text { intermediate subfields } K \subset F \subset L\} \\
H & \longleftrightarrow L^{H} \\
\operatorname{Gal}(L / F) & \longleftrightarrow F
\end{aligned}
$$

This theorem is not even possible to state without using the topology on $\operatorname{Gal}(L / K)$, which really comes from the process of forming the inverse limit.

### 3.4.2 Inverse limits in sheaf theory

The next goal is to discuss extending a sheaf from a basis to arbitrary open subsets. The main tool for this will be inverse limits.

To be a bit more precise, consider space $X$ with a sheaf of sets $\mathcal{F}$. Let $\mathcal{B}$ be a basis for the topology on $X$, meaning $\mathcal{B}$ is a collection of open sets such that any open subset $U \subset X$ is a union of elements of $\mathcal{B}$, and $\mathcal{B}$ is stable under finite intersections. The goal is to show that if $U \subset X$ is any open subset, then $\mathcal{F}(U)$ is determined entirely by values of $\mathcal{F}$ on elements of the basis $\mathcal{B}$, and not only $\mathcal{F}(U)$ is determined, but restriction maps are also determined.

We do this as follows. Consider an arbitrary open subset $U \subset X$, and define

$$
\mathcal{U}=\{V \in \mathcal{B}: V \subset U\}
$$

We order $\mathcal{U}$ by inclusion, meaning $V_{2} \leq V_{1} \Longleftrightarrow V_{2} \subset V_{1}$. Then $\mathcal{U}$ is a partially ordered set. Note that $\mathcal{U}$ is not necessarily filtered, meaning given $V_{1}, V_{2}$ there may not exist $V_{3}$ with $V_{1}, V_{2} \leq V_{3}$. However, this does not cause any problems for whether our limit will exist. The collection

$$
\{\mathcal{F}(V): V \in \mathcal{U}\}
$$

is an inverse system with respect to the restriction maps of $\mathcal{F}$, meaning for $V_{2} \leq V_{1}$, we have

$$
\rho_{V_{2}}^{V_{1}}: \mathcal{F}\left(V_{1}\right) \rightarrow \mathcal{F}\left(V_{2}\right)
$$

As noted, the fact that $\mathcal{U}$ lacks one part of the filtered set property does not actually matter, the inverse limit still exists.

$$
\lim _{\grave{V \in \mathcal{U}}} \mathcal{F}(V)
$$

Proposition 3.42. Let $X, \mathcal{F}, U, \mathcal{U}$ be as above. Then there is a natural map

$$
\mathcal{F}(U) \rightarrow{\underset{V}{V \in \mathcal{U}}}^{\mathcal{F}}(V)
$$

which is an isomorphism.
Proof. For $V_{2} \subset V_{1} \subset U$ with $V_{1}, V_{2} \in \mathcal{U}$, the restriction maps $\rho$ fit into the following commutative diagram.


Thus by the universal property of the inverse limit, there exists a unique map $\tau: \mathcal{F}(U) \rightarrow$ $\varliminf_{\rightleftarrows} \mathcal{F}(V)$ fitting into the following commutative diagram.


The arrows coming out of $\lim \mathcal{F}(V)$ are the canonical maps associated with the inverse limit. Now we use the sheaf axioms to show that $\tau$ is an isomorphism. We know that $U$ is the union over $\mathcal{U}$,

$$
U=\bigcup_{V \in \mathcal{U}} V
$$

First we prove injectivity of $\tau$. Suppose we have $s, t \in \mathcal{F}(U)$ such that $\tau(s)=\tau(t)$. Since the previous diagram commutes, $\rho_{V}^{U}(s)=\rho_{V}^{U}(t)$ for all $V \in \mathcal{U}$. Hence by the separation axiom for $\mathcal{F}, s=t$. Hence $\tau$ is injective.

Now we prove surjectivity of $\tau$. Let $\left(a_{V}\right) \in \lim \mathcal{F}(V)$, and recall our description of the inverse limit as a subset of the direct product,

$$
\lim _{\rightleftharpoons} \mathcal{F}(V)=\left\{\left(a_{V}\right) \in \prod_{V \in \mathcal{U}} \mathcal{F}(V) \mid \rho_{V_{2}}^{V_{1}}\left(a_{V_{1}}\right)=a_{V_{2}}, \quad \text { whenever } V_{2} \subset V_{1}\right\}
$$

Let $V_{1}, V_{2} \in \mathcal{U}$. Since the basis is closed under finite intersections, $V_{1} \cap V_{2} \in \mathcal{U}$ also, hence

$$
\rho_{V_{1} \cap V_{2}}^{V_{1}}\left(a_{V_{1}}\right)=a_{V_{1} \cap V_{2}}=\rho_{V_{1} \cap V_{2}}^{V_{2}}\left(a_{V_{2}}\right)
$$

Thus by the gluing axiom, there exists $a \in \mathcal{F}(U)$ such that $\rho_{V_{i}}^{U}(a)=a_{V_{i}}$, hence $\tau(a)=\left(a_{V_{i}}\right)$. Thus $\tau$ is surjective.

Remark 3.43. The previous proposition shows that the sections of a sheaf $\mathcal{F}$ are determined entirely by its sections and restriction maps on a basis of open sets, and inverse limits were the key tool to make the connection. One useful aspect of this is that often one may define a sheaf by prescribing only the values on a basis, and then extending uniquely using this fact.

Remark 3.44. If $\mathcal{F}$ is just a presheaf but not necessarily a sheaf, the same use of the universal property above will give a $\operatorname{map} \mathcal{F}(U) \rightarrow \lim \mathcal{F}(V)$, it will just not necessarily be an isomorphism.

Remark 3.45. The previous result showed that a sheaf $\mathcal{F}$ has sections on an arbitrary open set $U$ determined (as an inverse limit) by the sections $\mathcal{F}(V)$ for $V$ ranging over a basis for the topology on $X$, along with restriction map data for the basis. It is also true that the restriction maps for $\mathcal{F}$ (outside the basis) are determined by defining $\mathcal{F}$ and its restriction maps on a basis, as follows. Given $U_{2} \subset U_{1} \subset X$ two arbitrary open sets with one contained in the other, consider

$$
\mathcal{U}_{i}=\left\{V \in \mathcal{B} \mid V \subset U_{i}\right\}
$$

for $i=1,2$. There is an obvious embedding $\mathcal{U}_{2} \hookrightarrow \mathcal{U}_{1}$, which gives a morphism of inverse systems

$$
\left\{\mathcal{F}(V): V \in \mathcal{U}_{1}\right\} \rightarrow\left\{\mathcal{F}(V): V \in \mathcal{U}_{2}\right\}
$$

which induces a map on the inverse limits

$$
\lim _{V \in \mathcal{U}_{1}} \mathcal{F}(V) \rightarrow \lim _{V \in \mathcal{U}_{2}} \mathcal{F}(V)
$$

This map is concretely describable in terms of the direct product, but we leave that to the reader to work out. The important fact, which we also omit verification for, is that the following diagram commutes.


The horizontal map on the bottom is the map just described, and the vertical maps are the isomorphisms of Proposition 3.42. In this way, defining $\mathcal{F}$ on a basis determines $\mathcal{F}$ on all open subsets, along with restriction morphisms everywhere.

### 3.5 General categorical limits

Definition 3.46. Let $\mathcal{C}, I$ be categories (usually $I$ is a small category). A diagram of shape $\mathbf{I}$ in $\mathcal{C}$ is a functor $\mathcal{F}: I \rightarrow \mathcal{C}$. For objects $i$ of $I$, the objects $\mathcal{F}(i)$ are called vertices of $\mathcal{F}$, and for a morphism $\phi: i \rightarrow j$ in $I$, the morphisms $\mathcal{F}(\phi)$ are called edges of $\mathcal{F}$.

Example 3.47. Let $\mathcal{C}, I$ be any categories and $A$ an object of $\mathcal{C}$. The constant diagram $c_{A}: I \rightarrow \mathcal{C}$ is defined by $c_{A}(i)=A$ for every object $i$ of $I$, and $c_{A}(\phi)=\operatorname{Id}_{A}$ for every morphism $\phi$ of $I$.

Remark 3.48. Fix categories $I, \mathcal{C}$, and let $A, B$ be objects in $\mathcal{C}$. Given a morphism $\theta: A \rightarrow$ $B$ in $\mathcal{C}$, there is an obvious choice for induced natural transformation $\widetilde{\theta}: c_{A} \rightarrow c_{B}$, described concretely as follows. For each object $i \in I, \widetilde{\theta}: c_{A}(i) \rightarrow c_{B}(i)$ is just $\theta: A \rightarrow B$, which clearly makes the diagram below commute for any morphism $\phi: i \rightarrow j$ in $I$.

$$
\begin{gathered}
c_{A}(i)=A \xrightarrow{c_{A}(\phi)=\operatorname{Id}_{A}} c_{A}(j)=A \\
\underset{\downarrow}{ }(\phi)=\theta \quad \mid \widetilde{\theta}(\phi)=\theta \\
c_{B}(i)=B \xrightarrow{c_{B}(\phi)=\text { Id }_{B}} c_{B}(j)=B
\end{gathered}
$$

Example 3.49. Let $I$ be the finite category depicted below.

with the identity arrows not depicted. Then a diagram of shape $I$ in $\mathcal{C}$ is a commutative square in $\mathcal{C}$.


It must commute because in $I$, going around the square on the top or bottom must be equal to the unique arrow along the diagonal.
Definition 3.50. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $I$, and let $A$ be an object of $\mathcal{C}$. A natural transformation $\gamma: c_{A} \rightarrow \mathcal{F}$ is called a cone over $\mathcal{F}$ with tip $A$.

We explain the imagery behind the word "cone." A natural transformation $\gamma: c_{A} \rightarrow \mathcal{F}$ means that for every morphism $\phi: i \rightarrow j$ in $I$, there is a commutative square


Collapsing the redundant identity arrow here, we write it as


So visually speaking, a natural transformation $c_{A} \rightarrow \mathcal{F}$ just means to look at the whole image of $\mathcal{F}$, and then for every object an arrow $\gamma_{i}: A \rightarrow \mathcal{F}(i)$, such that all such triangles commute. Thinking of $\mathcal{F}(I)$ as some sort of "base space," this can be visualized as a cone, with the object $A$ sitting at the point. Hence the terminology "cone with tip $A$."

Definition 3.51. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $A$, and let $A$ be an object of $\mathcal{C}$. A natural transformation $\mathcal{F} \rightarrow c_{A}$ is called a cocone under $\mathcal{F}$ with tip $A$.
Definition 3.52. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $I$. An object $X$ of $\mathcal{C}$ is a limit of $\mathcal{F}$, denoted $X=\lim _{I} \mathcal{F}$ or $X=\lim _{I} \mathcal{F}$ if there exists a natural transformation $\gamma: c_{X} \rightarrow \mathcal{F}$ such that given any object $Y$ of $\mathcal{C}$ and a natural transformation $\gamma^{\prime}: c_{Y} \rightarrow \mathcal{F}$, there exists a unique morphism $\tau: Y \rightarrow X$ such that $\gamma^{\prime}=\gamma \circ \widetilde{\tau}$, where $\widetilde{\tau}: c_{Y} \rightarrow c_{X}$ is the natural transformation induced by $\tau$.

Alternatively, we may phrase this in terms of cones over $\mathcal{F} . X$ is a limit of $\mathcal{F}$ if there exists a cone $\sigma$ over $\mathcal{F}$ with tip $X$ such that for any cone $\widetilde{\sigma}^{\prime}$ over $\mathcal{F}$ with tip $Y$, then there exists a unique morphism $\tau: Y \rightarrow X$ such that the induced morphism $\widetilde{\tau}: c_{Y} \rightarrow c_{X}$ satisfies $\gamma^{\prime}=\gamma \circ \widetilde{\tau}$.

Definition 3.53. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $I$. An object $A$ of $\mathcal{C}$ is a colimit or direct limit of $\mathcal{F}$, denoted $A=\lim _{I} \mathcal{F}$ or $A=\operatorname{colim}_{I} \mathcal{F}$ if there exists a natural transformation $\sigma: \mathcal{F} \rightarrow c_{A}$ such that given any object $B$ of $\mathcal{C}$ and a natural transformation $\sigma^{\prime}: \mathcal{F} \rightarrow c_{B}$, there exists a unique morphism $\theta: A \rightarrow B$ such that for the corresponding natural transformation $\widetilde{\theta}: c_{A} \rightarrow c_{B}$ we have $\sigma^{\prime}=\widetilde{\theta} \circ \sigma$.

Alternatively, we can phrase this in terms of cocones. $A$ is a colimit of $\mathcal{F}$ if there exists a cocone $\sigma$ under $\mathcal{F}$ with tip $A$ such that for any cocone $\sigma^{\prime}$ under $\mathcal{F}$ with tip $B$, there exists a unique morphism $\theta: A \rightarrow B$ such that the induced morphism $\widetilde{\theta}: c_{A} \rightarrow c_{B}$ satisfies $\sigma^{\prime}=\widetilde{\theta} \circ \sigma$.

Remark 3.54. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $I$. There is a category whose objects are cones over $\mathcal{F}$. Given two cones $\gamma: c_{A} \rightarrow \mathcal{F}, \eta: c_{B} \rightarrow \mathcal{F}$, a morphism between them in this category is a morphism $\theta: A \rightarrow B$ whose induced morphism $\widetilde{\theta}: c_{A} \rightarrow c_{B}$ makes the following diagram commute.


Similarly, there is a category whose objects are cocones under $\mathcal{F}$. If $\gamma^{\prime}: \mathcal{F} \rightarrow c_{a}, \eta^{\prime}: \mathcal{F} \rightarrow c_{B}$ are cocones, a morphism between them is given by a morphism $\theta: A \rightarrow B$ whose induced morphism $\tilde{\theta}: c_{A} \rightarrow c_{B}$ makes the following diagram commute.


Using the language of these categories, we can give our last formulation of the definition of limits and colimits. A limit of a diagram $\mathcal{F}: I \rightarrow \mathcal{C}$ (if it exists) is the terminal object in the category of cones over $\mathcal{F}$. A colimit of $\mathcal{F}$ (if it exists) is the initial object in the category of cocones under $\mathcal{F}$.

### 3.5.1 Realizing common categorical constructions as limits

In this section, we try to demonstrate that limits and colimits are not merely abstraction for the sake of abstraction. They include many very important categorical constructions as special cases, so they unify a lot of ideas in category theory.

Example 3.55 (Initial and terminal objects as limits). Let $I$ be the empty set, and view $I$ as an "empty category" which has no objects and no morphisms. Let $\mathcal{C}$ be any category. Then there is a unique diagram of shape $I$ in $\mathcal{C}$, the "empty diagram" in $\mathcal{C}$.

The limit may of this diagram may or may not exist, but if it exists, it is an object $X$ in $\mathcal{C}$ such that for every object $Y$, there is a unique morphism $Y \rightarrow X$, that is, $X$ is the terminal object of $\mathcal{C}$. Dually, the colimit of the empty diagram, if it exists, is the initial object in $\mathcal{C}$.

Example 3.56 (Inverse and direct limits as general limits). Let $I$ be a filtered set, viewed as a category with morphisms given by $i \rightarrow j$ when $i \leq j$. For any category $\mathcal{C}$, a (covariant) functor $\mathcal{F}: I \rightarrow \mathcal{C}$ is a directed system $\{\mathcal{F}(i): i \in I\}$ and a contravariant functor $\mathcal{G}: I \rightarrow \mathcal{C}$ gives an inverse system $\{\mathcal{G}(i): i \in I\}$ in $\mathcal{C}$.

In this situation, the limit of $\mathcal{F}$, if it exists, is the direct limit of the system $\mathcal{F}(i)$. Dually, if the limit of $\mathcal{G}$ exists, it is the inverse limit of the system $\mathcal{G}(i)$.

Remark 3.57. The previous example shows how one might generalize the notion of direct and inverse limits, by loosening the description of the indexing set $I$. Instead of a filtered set $I$, one can take any small category $I$, and a functor $I \rightarrow \mathcal{C}$, and define the generalized direct limit over $I$ as the limit of the functor $I \rightarrow \mathcal{C}$.

Example 3.58 (Products and coproducts as limits). Let $I$ be a discrete category ${ }^{6}$ Then a diagram $\mathcal{F}: I \rightarrow \mathcal{C}$ of shape $I$ is just a collection of objects $\{\mathcal{F}(i): i \in \mathrm{Ob}(I)\}$ with no morphisms. The limit of $\mathcal{F}$, if it exists, is an object $\lim \mathcal{F}$ of $\mathcal{C}$ with the property that there exists morphisms

$$
\gamma_{i}: \lim _{\leftrightarrows} \mathcal{F} \rightarrow \mathcal{F}(i)
$$

for each $i \in \operatorname{Ob}(I)$, such that if $B$ is an object of $\mathcal{C}$ with maps

$$
\gamma_{i}^{\prime}: B \rightarrow \mathcal{F}(i)
$$

then there is a unique morphisms $\tau: B \rightarrow \varliminf_{\varliminf} \mathcal{F}$ such that $\gamma_{i}^{\prime}=\gamma_{i} \circ \tau$. That is to say, if $\lim _{\rightleftarrows} \mathcal{F}$ exists, it is the product.

$$
\lim _{\rightleftarrows} \mathcal{F} \cong \prod_{i \in I} \mathcal{F}(i)
$$

Dually, the colimit of $\mathcal{F}$, if it exists, is the coproduct in $\mathcal{C}$.

$$
\lim _{\longrightarrow} \mathcal{F} \cong \bigsqcup_{i \in I} \mathcal{F}(i)
$$

The next goal is to show that limits and colimits can also realize the categorical notion of equalizers. Since these are not as well known, first we define equalizers and give an example of another important concept, the kernel as an equalizer.

Definition 3.59. Let $\mathcal{C}$ be a category, and let $f, g: X \rightarrow Y$ be morphisms in $\mathcal{C}$. The equalizer of $f$ and $g$ is an object $E$ and a morphism $e: E \rightarrow X$ making the following diagram commute,

[^4]
and such that $E, e$ are universal in this diagram. Concretely, that means that for any object $Z$ with a morphism $h: Z \rightarrow X$ such that the analogous diagram commutes,

then there exists a unique morphisms $\theta: Z \rightarrow E$ making the following diagram commute.


Example 3.60 (Kernel as equalizer). Let $\mathcal{C}$ be the category of abelian groups, and let $f: A \rightarrow B$ be a homomorphism of abelian groups. We also have the zero morphism $0: A \rightarrow B$. The equalizer of $f$ and 0 is the kernel of $f$. More precisely, the object $E$ is the subobject of $A$ which is ker $f$, and the map $e: \operatorname{ker} f \rightarrow A$ is the inclusion map. We leave it to the reader to check the universal property.

Example 3.61 (Equalizers as limits). Let $I$ be the finite category


We have omitted the obvious identity arrows. Let $\mathcal{F}: I \rightarrow \mathcal{C}$ be a diagram of shape $I$ in a cateogory $\mathcal{C}$. That is, $\mathcal{F}$ is a parallel pair of morphisms

$$
A \underset{g}{\stackrel{f}{\rightrightarrows}} B
$$

A cone over this diagram consists of an object $C$ in $\mathcal{C}$ and morphisms $h: C \rightarrow A$ and $k: C \rightarrow B$ making the following diagram commute, meaning $f h=g h=k$.


Thus a cone is entirely determined by the morphism $h: C \rightarrow A$, and any morphism $h: C \rightarrow$ $A$ satisfying $f h=g h$ gives a cone. Thus, the limit $\lim \mathcal{F}$, if it exists, is the terminal object in the category of cones over $\mathcal{F}$ with this property, which is precisely the equalizer of $f$ and $g$.

Example 3.62 (Fiber product/pullback as limit). Let $I$ be the finite category


A diagram of shape $I$ in $\mathcal{C}$ is a pair of morphisms with common target object.


A cone over $\mathcal{F}$ is an object $D$ with morphisms $D \rightarrow A$ and $D \rightarrow C$ and $D \rightarrow B$, making suitable commutative diagrams. Basically, the morphisms all make commutative triangles, so it is sufficient that $D \rightarrow A \rightarrow B$ and $D \rightarrow C \rightarrow B$ both agree with $D \rightarrow B$. That is to say, a cone over $\mathcal{F}$ is simply an object $D$ with maps to $A$ and $C$ making a commutative square as below.


A limit of $\mathcal{F}$, if it exists, is an object $D$ as above with morphisms as above, such that $D$ is the terminal object with this property. This is known as the fiber product of $f$ and $g$, also known as the pullback. It is usually written as

$$
\lim _{\rightleftarrows}^{\mathcal{F}}=A \times_{B} C
$$

This notation is unfortunate, since it leaves the morphisms $f, g$ out, which are critically important. However, it is standard notation, so we should get used to it.

Example 3.63 (Concrete description of fiber product in the category of sets). We describe the fiber product concretely in the category of sets. The same construction will work for
groups, abelian groups, modules over a ring, and topological spaces. Given $f: A \rightarrow B$ and $g: C \rightarrow B$, the fiber product is

$$
A \times_{B} C=\{(a, c) \in A \times C: f(a)=g(c)\}
$$

with the obvious choice of maps to $A$ and $B$ given by projection onto the first or second coordinate, respectively.

Example 3.64 (Pushouts as colimits). Let $I$ be the finite category


The discussion for fiber products basically repeats. Give a diagram $\mathcal{F}: I \rightarrow \mathcal{C}$ of shape $I$, a colimit of $\mathcal{F}$, if it exists, coincides with the usual notion of a pushout of a diagram of this shape.


Example 3.65 (Concrete description of pushout in the category of topological spaces). Let $\mathcal{C}$ be the category of topological spaces. Consider a diagram of the shape considered above.


The pushout always exists in this category, and it is described concretely as the "gluing" of $X$ and $Y$ along $Z$ (more precisely, gluing along the images of $Z$ via $f$ and $g$ ).

$$
X \sqcup_{Z} Y=(X \sqcup Y) / \sim
$$

where $f(z) \sim g(z)$ for all $z \in Z$. The maps from $X$ and $Y$ to the glued space are the obvious ones, given by sending an element $x \in X$ or $y \in Y$ to its equivalence class in $(X \sqcup Y) / \sim$


Even more concretely, if $Z$ is a single point space $Z=\left\{z_{0}\right\}$, then the pushout is the wedge sum of $X$ and $Y$ at the points which are the respective images of $z_{0}$.

Example 3.66 (Concrete description of pushout in the category of groups). Consider a diagram in the category of groups


The pushout of the diagram is the free product of $G$ and $H$ "with amalgamation." That is,

$$
G *_{F} H=(G * H) / K
$$

where $G * H$ is the free product of $G$ and $H$, and $K$ is the smallest normal subgroup containing all elements of the form $g(x) f(x)^{-1}$ for $x \in F$.


Remark 3.67. The previous two examples explain (to some extent) why the free product of groups with amalgamation arises in the Seifert-Van Kampen theorem, which relates the fundamental group of a union of topological spaces to the fundamental groups of the individual spaces.

One some level, SVK just says that given a pushout diagram in the category of pointed topological spaces, applying the fundamental group functor $\pi_{1}$ gives a pushout diagram in the category of groups, so "of course" the resulting fundamental group is a free product with amalgamation. There is more to the story; SVK is not a purely formal result of abstract nonsense, but this is an important aspect of the story.

### 3.5.2 Representable and adjoint functors

Let $I, \mathcal{C}$ be categories. Suppose for the remainder of the following discussion that $I, \mathcal{C}$ are such that all diagrams $\mathcal{F}: I \rightarrow \mathcal{C}$ have a colimit. (This occurs, for example, if $\mathcal{C}$ is the category of sets and $I$ is small, so it is not an unreasonable assumption.)

Remark 3.68. Suppose $\mathcal{F}, \mathcal{F}^{\prime}: I \rightarrow \mathcal{C}$ are diagrams of shape $I$ in $\mathcal{C}$. A natural transformation $\eta: \mathcal{F} \rightarrow \mathcal{F}^{\prime}$ induces a morphism on the limits, $\lim \eta: \lim _{\rightleftarrows} \mathcal{F} \rightarrow \lim _{\longleftarrow} \mathcal{F}^{\prime}$. This is just a formal consequence of the universal property defining limits.

Definition 3.69. The functor category $\mathcal{C}^{I}$ is the category whose objects are functors (diagrams) $\mathcal{F}: I \rightarrow \mathcal{C}$ and whose morphisms are natural transformations. By the preceding remark, we may view $\varliminf_{\longleftarrow}$ as a functor

$$
\lim _{\leftrightarrows}: \mathcal{C}^{I} \rightarrow \mathcal{C} \quad \mathcal{F} \mapsto \lim _{\leftrightarrows} \mathcal{F}
$$

and uses the preceding remark to induce morphisms. On the other hand, for any object $A$ in $\mathcal{C}$, there is the constant diagram $c_{A}: I \rightarrow \mathcal{C}$, and for any morphism $\theta: A \rightarrow B$ there is the induced natural transformation $\widetilde{\theta}: c_{A} \rightarrow c_{B}$. We may think of this as a functor

$$
\Delta: \mathcal{C} \rightarrow \mathcal{C}^{I} \quad A \mapsto c_{A}
$$

Remark 3.70. Fix an object $B$ in $\mathcal{C}$. By definition of the universal property of colimits, we have a correspondence between homomorphisms (in $\mathcal{C}$ ) from $\underset{\mathcal{F}}{\lim } \mathcal{F}$ to $B$ and natural transformations $\mathcal{F} \rightarrow c_{B}$. Every natural transformation $\mathcal{F} \rightarrow c_{B}=\Delta B$ corresponds to a unique morphism $\lim \mathcal{F} \rightarrow B$, and vice versa. That is, there is a bijection (of sets)

$$
\operatorname{Hom}_{\mathcal{C}}\left(\lim _{\check{ }} \mathcal{F}, B\right) \cong \operatorname{Hom}_{\mathcal{C}^{I}}\left(\mathcal{F}, c_{B}=\Delta B\right)
$$

Moreover, this bijection is "natural," in the sense that a morphism $\eta: \mathcal{F} \rightarrow \mathcal{F}^{\prime}$ or a morphism $\phi: B \rightarrow B^{\prime}$ each make the respective diagram below commute.


The fact that this isomorphism is natural means that $\varliminf_{\longleftarrow}$ and $\Delta$ are respectively left and right adjoint functors to each other.

Remark 3.71. Given a functor/diagram $\mathcal{F}: I \rightarrow \mathcal{C}$ as above, consider the functor

$$
\mathscr{F}: \mathcal{C} \rightarrow \text { Set } \quad B \mapsto \operatorname{Hom}_{\mathcal{C}^{I}}(\mathcal{F}, \Delta B)
$$

Using the previous natural isomorphism, we can also write $\mathscr{F}$ as

$$
\mathscr{F}: \mathcal{C} \rightarrow \text { Set } \quad B \mapsto \operatorname{Hom}_{\mathcal{C}}\left(\lim _{\leftrightarrows} \mathcal{F}, B\right)
$$

That is to say, $\mathscr{F}$ is naturally isomorphic to the functor $\operatorname{Hom}_{\mathcal{C}}\left(\lim _{\longleftarrow} \mathcal{F},-\right)$, which means that $\mathscr{F}$ is a representable functor, with representing object $\lim _{\leftrightarrows} \mathcal{F}$.

## 4 Stalks of sheaves

We return from our venture into abstraction to slightly less abstract matters. At least, if you consider sheaves and stalks to be less abstract than the previous discussion of natural isomorphisms and such. Our goal is to use stalks to extract "global" information about sheaves and morphisms of sheaves, since in general sheaves have a lot of local structure but getting a handle on "global" considerations is much more difficult.

Remark 4.1. Let $X$ be a topological space and $\mathcal{F}$ be a presheaf of sets on $X$. Let $\mathcal{F}_{x}$ be the stalk of $\mathcal{F}$ at $x \in X$, recalling that

$$
\mathcal{F}_{x}=\underset{x \in U}{\lim } \mathcal{F}(U)
$$

By definition of direct limit, for each open neighborhood $U$ of $x$, we have a map

$$
\rho_{x}^{U}: \mathcal{F}(U) \rightarrow \mathcal{F}_{x}
$$

This behaves in many ways like a restriction map for $\mathcal{F}$, which is why we use the same notation $\rho$. In particular, if the set $\{x\}$ is open, then the stalk $\mathcal{F}_{x}$ may be identified with $\mathcal{F}(\{x\})$, and the faux restriction map $\rho_{x}^{U}$ may be identitified with the bonafide restriction map $\rho_{\{x\}}^{U}$.

If $\mathcal{F}$ is merely a presheaf, the maps $\rho_{x}^{U}$ defined above do not behave all that well. However, if $\mathcal{F}$ satisfies the sheaf axioms, they behave very nicely, as captured in the following proposition.

Proposition 4.2. Let $\mathcal{F}$ be a sheaf of sets on a space $X$. Let $U \subset X$ be an open set. The map

$$
\prod_{x \in U} \rho_{x}^{U}: \mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_{x} \quad s \mapsto\left(\rho_{x}^{U}(s)\right)
$$

is injective.
Proof. Suppose we have two sections $s, t \in \mathcal{F}(U)$ such that their images are the same under this map. That is, $\rho_{x}^{U}(s)=\rho_{x}^{U}(t)$ for all $x \in U$. This equalit is in $\mathcal{F}_{x}$, and by our previous concrete descriptions of the stalk and what it means for elements of the stalk to be equal, $\rho_{x}^{U}(s)=\rho_{x}^{U}(t)$ means that there is an open neighborhood $U_{x} \subset U$ such that

$$
\rho_{U_{x}}^{U}(s)=\rho_{U_{x}}^{U}(t)
$$

Since this holds for each $x \in U, U$ is covered by such neighborhoods.

$$
U=\bigcup_{x \in U} U_{x}
$$

Then by the separation axiom of sheaves (which $\mathcal{F}$ has), $s, t$ agree everywhere locally so they agree globally, which is to say, $s=t$ in $\mathcal{F}(U)$.

Recall that if we have a morphism of of presheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$ and fix $x \in X$, then $\phi$ induces morphisms $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ on stalks in such a way that the following diagram commutes whenever $x \in U$.


Corollary 4.3. Let $\phi_{1}, \phi_{2}: \mathcal{F} \rightarrow \mathcal{G}$ be morphism of presheaves on $X$, where $\mathcal{G}$ is a sheaf. If the induced morphisms on stalks agree for every $x \in X$, that is,

$$
\phi_{1, x}=\phi_{2, x}
$$

then $\phi_{1}=\phi_{2}$.
Proof. To show that two morphisms $\phi_{1}, \phi_{2}$ of presheaves are equal, we need to show that for every open set $U \subset X$, the morphisms $\phi_{1, U}$ and $\phi_{2, U}$ are equal as maps $\mathcal{F}(U) \rightarrow \mathcal{F}(G)$. Let $s \in \mathcal{F}(U)$. Using the commutative square right above this corollary, and our hypothesis that $\phi_{1, x}=\phi_{2, x}$, we have

$$
\rho_{x}^{U}(\mathcal{G})\left(\phi_{1, U}(s)\right)=\phi_{1, U}\left(\rho_{x}^{U}(\mathcal{F})(s)\right)=\phi_{2, U}\left(\rho_{x}^{U}(\mathcal{F})(s)\right)=\rho_{x}^{U}(\mathcal{G})\left(\phi_{2, U}(s)\right)
$$

Then using the fact that $\mathcal{G}$ is a sheaf and Proposition 4.2, the fact that these agree for all $x$ shows that $\phi_{1, U}(s)=\phi_{2, U}(s)$. That is, $\phi_{1}=\phi_{2}$.

Proposition 4.4. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on $X$, with $\mathcal{F}$ a sheaf. The following are equivalent.

1. The induced maps on stalks $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ are injective for all $x \in X$.
2. The maps $\phi_{U}: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ are injective for all $U \subset X$ open.

Before we go on to the proof, it is worth remarking that the "expected" analog of the above involving surjectivity is false. Essentially, the failure comes down to the failure of the image presheaf to be a sheaf. We will hopefully give concrete examples later.

Proof. First, we suppose all of the induced maps $\phi_{x}$ on stalks are injective. Let $U \subset X$ be an open set. We need to show that $\phi_{U}$ is injective. Let $s, t \in \mathcal{F}(U)$ such that $\phi_{U}(s)=\phi_{U}(t)$. Then for any $x \in U$, we have

$$
\begin{aligned}
\phi_{x}\left(\rho_{x}^{U}(\mathcal{F})(s)\right) & =\rho_{x}^{U}(\mathcal{G})\left(\phi_{U}(s)\right) \\
& =\rho_{x}^{U}(\mathcal{G})\left(\phi_{U}(t)\right) \\
& =\phi_{x}\left(\rho_{x}^{U}(\mathcal{F})(t)\right)
\end{aligned}
$$

Since $\phi_{x}$ is injective, this shows that $\rho_{x}^{U}(\mathcal{F})(s)=\rho_{x}^{U}(\mathcal{F})(t)$. Then by Proposition 4.2, it follows that $s=t$. Hence $\phi_{U}$ is injective.

Now for the converse, we assume that all maps $\phi_{U}$ are injective. Fix $x \in X$, and let $s_{x}, t_{x} \in \mathcal{F}_{x}$ such that $\phi_{x}\left(s_{x}\right)=\phi_{x}\left(t_{x}\right)$. Choose a neighborhood $U$ of $x$. Then we know that $s_{x}, t_{x} \in \mathcal{F}_{x}$ both have a representative in $\mathcal{F}(U)$, call them $s_{u}, t_{u}$. That is,

$$
s_{x}=\rho_{x}^{U}(\mathcal{F})\left(s_{u}\right) \quad t_{x}=\rho_{x}^{U}(\mathcal{F})\left(t_{u}\right)
$$

Then

$$
\begin{aligned}
\phi_{x}\left(s_{x}\right) & =\phi_{x}\left(\rho_{x}^{U}(\mathcal{F})\left(s_{u}\right)\right)=\rho_{x}^{U}(\mathcal{G})\left(\phi_{U}\left(s_{u}\right)\right) \\
\phi_{x}\left(t_{x}\right) & =\phi_{x}\left(\rho_{x}^{U}(\mathcal{F})\left(t_{u}\right)\right)=\rho_{x}^{U}(\mathcal{G})\left(\phi_{U}\left(t_{u}\right)\right)
\end{aligned}
$$

That is, $\phi_{U}\left(s_{u}\right), \phi_{U}\left(t_{u}\right) \in \mathcal{G}(U)$ are respectively representatives for $\phi_{x}\left(s_{x}\right), \phi_{x}\left(t_{x}\right)$. Since $\phi_{U}\left(s_{u}\right)$ and $\phi_{U}\left(t_{u}\right)$ represent the same element of the stalk, there is a neighborhood $V \subset U$ of $x$ such that

$$
\rho_{V}^{U}\left(\phi_{U}\left(s_{U}\right)\right)=\rho_{V}^{U}\left(\phi_{U}\left(t_{u}\right)\right)
$$

Since $\phi$ is a morphism of presheaves,

$$
\phi_{V}\left(s_{u}\right)=\rho_{V}^{U} \phi_{U}\left(s_{u}\right)=\rho_{V}^{U} \phi_{U}\left(t_{u}\right)=\phi_{V}\left(t_{u}\right)
$$

By hypothesis, $\phi_{V}$ is injective, so $s_{u}=t_{u}$. Then

$$
s_{x}=\rho_{x}^{U}(\mathcal{F})\left(s_{u}\right)=\rho_{x}^{U}(\mathcal{F})\left(t_{u}\right)=t_{x}
$$

so $s_{x}=t_{x}$. Hence $\phi_{x}$ is injective, as claimed.

### 4.1 Exactness properties of limits

Definition 4.5. Let $I$ be a filtered set, and suppose we have three directed systems of abelian groups, all indexed by $I$.

$$
\begin{aligned}
A & =\left\{A_{i}, \tau_{i}^{j}(A)\right\} \\
B & =\left\{B_{i}, \tau_{i}^{j}(B)\right\} \\
C & =\left\{B_{i}, \tau_{i}^{j}(C)\right\}
\end{aligned}
$$

Let $\phi: A \rightarrow B$ and $\psi: B \rightarrow C$ be morphisms of directed systems. Recall that this means that for each $i \in I$, we have morphisms $\phi_{i}: A_{i} \rightarrow B_{i}$ and $\psi_{i}: B_{i} \rightarrow C_{i}$, and for every $i \leq j$ we have the following commutative diagram.


We say $A \rightarrow B \rightarrow C$ is an exact sequence of directed systems if for each $i$, the sequence $A_{i} \rightarrow B_{i} \rightarrow C_{i}$ is exact.

Remark 4.6. We describe concretely the description of the induced map on direct systems for the category of abelian groups. Let $A \rightarrow B$ be a morphism of directed systems, with maps $\phi_{i}: A_{i} \rightarrow B_{i}$. Recall that

$$
\begin{aligned}
& \underset{\longrightarrow}{\lim } A_{i}=\left(\bigsqcup_{i \in I} A_{i}\right) / \sim \\
& \xrightarrow[\longrightarrow]{\lim } B_{i}=\left(\bigsqcup_{i \in I} B_{i}\right) / \sim
\end{aligned}
$$

where the relation $\sim$ is given by $a_{i} \sim \tau_{i}^{j}(A)\left(a_{i}\right)$. The induced map is

$$
\xrightarrow{\lim } \phi_{i}: \xrightarrow{\lim } A_{i} \rightarrow \xrightarrow{\lim } B_{i} \quad\left[a_{i}\right] \mapsto\left[\phi_{i}\left(a_{i}\right)\right]
$$

Proposition 4.7. If $A \rightarrow B \rightarrow C$ is an exact sequence of directed systems, then the induced sequence $\underset{\longrightarrow}{\lim } A_{i} \rightarrow \underset{\longrightarrow}{\lim } B_{i} \rightarrow \underset{\longrightarrow}{\lim } C_{i}$ is exact.

Proof. Let $\Phi=\underset{\longrightarrow}{\lim } \phi_{i}: \underset{\longrightarrow}{\lim } A_{i} \rightarrow \underset{\longrightarrow}{\lim } B_{i}$ and $\Psi=\underset{\longrightarrow}{\lim } \psi_{i}: B_{i} \rightarrow \underset{\longrightarrow}{\lim C_{i}}$ be the induced maps on the direct limits. The fact that $\operatorname{im} \Phi \subset \operatorname{ker} \Psi$ is routine to check, so we omit it. The reverse inclusion is more interesting, so we include a proof.

Let $b \in \operatorname{ker} \Psi$. Then it has a representative $b_{i} \in B_{i}$ for some fixed $i \in I$, and $\psi_{i}\left(b_{i}\right)=0$ in $C_{i}$, so $\psi_{i}\left(b_{i}\right)$ is a representative for the zero element of $\underset{\longrightarrow}{\lim } C_{i}$. So there exists $j \in I$ with $i \leq j$ such that

$$
\tau_{i}^{j}(C)\left(\psi_{i}\left(b_{i}\right)\right)=0
$$

Then using our commutative diagram (the fact that $\Psi$ is a morphism of directed systems), we have

$$
\psi_{j} \tau_{i}^{j}(B)\left(b_{i}\right)=\tau_{i}^{j}(C) \psi_{i}\left(b_{i}\right)=0
$$

That is, $\tau_{i}^{j}(B)\left(b_{i}\right) \in \operatorname{ker} \psi_{j}$. By exactness of $A_{i} \rightarrow B_{i} \rightarrow C_{i}$, $\operatorname{ker} \psi_{j}=\operatorname{im} \phi_{j}$, so there exists $a_{j} \in A_{j}$ such that $\phi_{j}\left(a_{j}\right)=\tau_{i}^{j}(B)\left(b_{i}\right)$. Let $a \in \lim A_{i}$ be the image of $a_{j}$. Then

$$
\Phi(a)=\Phi\left[a_{j}\right]=\left[\phi_{j}\left(a_{j}\right)\right]=\left[\tau_{i}^{j}(B)\left(b_{i}\right)\right]=\left[b_{i}\right]=b
$$

Thus $\operatorname{ker} \Psi \subset \operatorname{im} \Phi$, which completes the proof.
Remark 4.8. In contrast with the previous result, the analogous statement about inverse limits is not quite true. The inverse limit functor is "half exact," that is, exact on one side, and suitable hypotheses on the inverse systems can make the resulting sequence exact. The curious reader can look up the Mittag-Leffler condition to learn more.

Definition 4.9. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be presheaves of abelian groups on a space $X$. Let $\phi: \mathcal{F} \rightarrow$ $\mathcal{G}, \psi: \mathcal{G} \rightarrow \mathcal{H}$ be morphisms of presheaves. We say the sequence

$$
\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}
$$

is an exact sequence of presheaves if for each $U \subset X$ the sequence

$$
\mathcal{F}(U) \xrightarrow{\phi_{U}} \mathcal{G}(U) \xrightarrow{\psi_{U}} \mathcal{H}(U)
$$

is exact (in the category of abelian groups).
Corollary 4.10. If

$$
0 \rightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \rightarrow 0
$$

is a short exact sequence of presheaves of abelian groups on $X$, then the induced sequence on stalks is exact for every $x \in X$.

$$
0 \rightarrow \mathcal{F}_{x} \xrightarrow{\phi_{x}} \mathcal{G}_{x} \xrightarrow{\psi_{x}} \mathcal{H}_{x} \rightarrow 0
$$

Proof. This is immediate from Proposition 4.7 and the definitions.
Definition 4.11. A sequence of sheaves of abelian groups on $X$ is exact (in the category of sheaves) if the sequence of stalks is exact for every $x \in X$.

Remark 4.12. The previous definition introduces some unfortunate ambiguity. When given a sequence of sheaves, one may regard it as a sequence of sheaves or as a sequence of presheaves. It is exact as a sequence of presheaves if the corresponding sequences on sections over $U \subset X$ are all exact for all $U \subset X$ open. It is exact as a sequence of sheaves if the corresponding sequences on stalks at $x \in X$ are exact for all $x \in X$.

Thankfully, by Corollary 4.10, at least one of these implies the other. If it is exact as presheaves, then it is exact as sheaves. However, the reverse implication is NOT true. A sequence of sheaves may be exact as a sequence of sheaves, but not as a sequence of presheaves.

However, a sequence of sheaves which is exact as a sequence of sheaves is at least partially exact as a sequence of presheaves. This is made more precise in Theorem 4.15 below, which says that if a sequence of sheaves is exact as a sequence of sheaves, it is at least left exact as a sequence of presheaves.

However, it does (in general) fail to be right exact as a sequence of presheaves. This is the entire motivation for sheaf cohomology, to study the failure of this right exactness by extending such a sequence to the right to a long exact sequence, utilitizing derived functors and so on. Also see Remark 4.15 below for further discussion.

Definition 4.13. A morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism if it has a two sided inverse (which is a morphism of sheaves).

[^5]Remark 4.14. An immediate consequence of the previous definitions is that a morphism of sheaves is an isomorphism if and only if it is an isomorphism on all stalks.

Theorem 4.15. Let

$$
0 \rightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \rightarrow 0
$$

be a short exact sequence of sheaves of abelian groups on a space $X$. Then for every open set $U \subset X$, the sequence

$$
0 \rightarrow \mathcal{F}(U) \xrightarrow{\phi_{U}} \mathcal{G}(U) \xrightarrow{\psi_{U}} \mathcal{H}(U)
$$

is exact.
Proof. First, we prove exactness at the $\mathcal{F}(U)$ term. By assumption, all the maps on stalks $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ are injective. By Proposition 4.4, this implies that $\phi_{U}$ is injective for every open $U \subset X$. This proves exactness at $\mathcal{F}(U)$.

Now we prove exactness at the $\mathcal{G}(U)$ term. Let $\mathcal{K}$ be the kernel presheaf of $\psi$, defined by $\mathcal{K}(U)=\operatorname{ker} \psi_{U}$. From previous work, we know $\mathcal{K}$ is a sheaf, not just a presheaf. By construction, we have an exact sequence of presheaves

$$
0 \rightarrow \mathcal{K} \rightarrow \mathcal{G} \xrightarrow{\psi} \mathcal{H}
$$

Hence by Corollary 4.10, we have an exact sequence on stalks for every $x \in X$.

$$
0 \rightarrow \mathcal{K}_{x} \rightarrow \mathcal{G}_{x} \xrightarrow{\psi_{x}} \mathcal{H}_{x}
$$

In particular, $\mathcal{K}_{x}=\operatorname{ker} \psi_{x}$ for every $x$. We also know that the composition $\psi \phi: \mathcal{F} \rightarrow \mathcal{H}$ is zero as a morphism of presheaves, hence $(\psi \phi)_{x}=\psi_{x} \phi_{x}=0$ as morphisms on stalks. Since $\mathcal{H}$ is a sheaf, by Proposition 4.3 this implies that $\psi \phi=0$, meaning for any $U \subset X,(\psi \phi)_{U}=0$. Thus

$$
\phi_{U}(\mathcal{F}(U)) \subset \mathcal{K}(U)=\operatorname{ker} \psi_{U}
$$

That is, we have a morphism of presheaves

$$
\widetilde{\phi}: \mathcal{F} \rightarrow \mathcal{K} \quad(\widetilde{\phi})_{U}=\phi_{U}: \mathcal{F}(U) \rightarrow \mathcal{K}(U)
$$

Since the original sequence is an exact sequence of sheaves, we have an exact sequence on stalks

$$
0 \rightarrow \mathcal{F}_{x} \xrightarrow{\phi_{x}} \mathcal{G}_{x} \xrightarrow{\psi_{x}} \mathcal{H}_{x} \rightarrow 0
$$

that is, $\mathcal{F}_{x}=\operatorname{ker} \psi_{x}$. Hence the morphism $\widetilde{\phi}$ induces isomorphism on all stalks,

$$
(\widetilde{\phi})_{x}: \mathcal{F}_{x} \xrightarrow{\cong} \mathcal{K}_{x}
$$

Since it is a morphism of sheaves which induces isomorphisms on all stalks, it is an isomorphism of sheaves. Thus

$$
0 \rightarrow \mathcal{F}(U) \cong \mathcal{K}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)
$$

is exact, as claimed.

Remark 4.16. An exact sequence of sheaves $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ does not, in general, give a full exact sequence on sections $0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U) \rightarrow 0$. The map $\mathcal{G}(U) \rightarrow \mathcal{H}(U)$ is not always surjective. This may fail even for the case $X=U$, that is, it may fail for global sections. We will give an example later.

Remark 4.17. The failure of exactness on the right in this exact situation motivates the definition of sheaf cohomology. We will define this more rigorously later, but the general idea is to define functors $H^{i}(X,-)$ with $i \geq 0$ from sheaves of abelian groups on $X$ to abelian groups in such a way that given a short exact sequence of sheaves (of abelian groups on $X$ ) $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$, we get an induced long exact sequence
$0 \rightarrow H^{0}(X, \mathcal{F}) \rightarrow H^{0}(X, \mathcal{G}) \rightarrow H^{0}(X, \mathcal{H}) \rightarrow H^{1}(X, \mathcal{F}) \rightarrow H^{1}(X, \mathcal{G}) \rightarrow H^{1}(X, \mathcal{H}) \rightarrow H^{2}(X, \mathcal{F}) \rightarrow \cdots$
where $H^{0}(X, \mathcal{F})=\mathcal{F}(X), H^{0}(X, \mathcal{G})=\mathcal{G}(X), H^{0}(X, \mathcal{H})=\mathcal{H}(X)$. For those who already know something about derived functors, the functors $H^{i}(X,-)$ will be the right derived functors of the "global sections" functor $\mathcal{F} \mapsto \mathcal{F}(X)$.

Remark 4.18. Despite the fact that $\mathcal{G}(U) \rightarrow \mathcal{H}(U)$ may fail to be surjective in the situation above, by definition of exactness of a sequence of sheaves, we know that the morphisms $\mathcal{G}_{x} \rightarrow \mathcal{H}_{x}$ are surjective for every $x \in X$. So we may still say a litte about what is going on.

Given $U \subset X$ open with $h \in \mathcal{H}(U)$, there may not be $g \in \mathcal{G}(U)$ such that $\psi_{U}(g)=h$, but for any $x \in U$, we can consider the class represented by $h$ in the stalk $\mathcal{H}_{x}$ (recall $\mathcal{H}_{x}$ is a direct limit over a system involving $\mathcal{H}(U)$, which we denote $h_{x}=\rho_{x}^{U}(h)$. Since the map $\psi_{x}: \mathcal{G}_{x} \rightarrow \mathcal{H}_{x}$ is surjective, there exists $g_{x} \in \mathcal{G}_{x}$ such that $\psi_{x}\left(g_{x}\right)=h_{x}$. We may then choose a representative $g \in \mathcal{G}\left(U_{x}\right)$ for $g_{x}$, where $U_{x} \subset U$ is a neighborhood of $x$. That is,

$$
\psi_{x} \rho_{x}^{U}(g)=\psi_{x}\left(g_{x}\right)=h_{x}=\rho_{x}^{U}(h)
$$

Hence $\psi_{U}(g)$ and $h$ represent the same element of the stalk $\mathcal{H}_{x}$. Another way to think about this is that by definition of what it means for two sections to represent the same element of the stalk, there is a neighborhood $V_{x} \subset U_{x}$ such that the restrictions of $h$ and $\psi_{U_{x}}(g)$ are equal. That is,

$$
\rho_{V_{x}}^{U}(\mathcal{H})(h)=\rho_{V_{x}}^{U_{x}}(\mathcal{H})\left(\psi_{U_{x}}(g)\right)=\psi_{V_{x}}\left(\rho_{V_{x}}^{U_{x}}(\mathcal{G})(g)\right)
$$

Summarizing, given $h \in \mathcal{H}(U)$, there exists a neighborhood $V$ with $x \in V \subset U$ and a section $s \in \mathcal{G}(V)$ such that

$$
\psi_{V}(s)=\rho_{V}^{U}(h)
$$

When $\psi_{U}: \mathcal{G}(U) \rightarrow \mathcal{H}(U)$ has this property, we say that $\psi_{U}$ is locally surjective. More broadly, this property of $\psi$ is called local surjectivity.

Example 4.19 (Concrete failure of surjectivity on sections). We give a specific example of a short exact sequence of sheaves where the right term of global sections fails to be surjective. Let $X$ be an open subset of $\mathbb{C}$, and let $\mathcal{O}$ be the sheaf of holomorphic functions on $X$. That is, for $U \subset X, \mathcal{O}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions $U \rightarrow \mathbb{C}$. The restriction maps
are literal function restrictions. This makes $\mathcal{O}$ a sheaf on $X$. ( $\mathcal{O}$ is called the structure sheaf on $X$.)

Let $\mathcal{C}$ be the sheaf of locally constant functions on $X$, that is, $\mathcal{C}(U)$ is the $\mathbb{C}$-algebra of locally constant functions $U \rightarrow \mathbb{C}$. Once again, restriction maps are literal function restrictions. $\mathcal{C}$ is in fact a sheaf.

Let $\psi: \mathcal{O} \rightarrow \mathcal{O}$ be the morphism of sheaves defined by $\psi_{U}: \mathcal{O}(U) \rightarrow \mathcal{O}(U)$ where $\psi_{U}$ is the differentiation operator $\frac{d}{d z}$. That is, $\psi_{U}(f)=\frac{d f}{d z}$. This gives a morphism of sheaves, and $\mathcal{C}$ is exactly the kernel sheaf. Then we claim that we have an exact sequence of sheaves

$$
0 \rightarrow \mathcal{C} \rightarrow \mathcal{O} \xrightarrow{\psi} \mathcal{O} \rightarrow 0
$$

Exactness at the first two terms is reasonably plausible and not terribly complicated, so we'll just justify exactness at the right side. That is, we will justify that each morphism on stalks $\psi_{x}: \mathcal{O}_{x} \rightarrow \mathcal{O}_{x}$ is surjective.

Given any open neighborhood $U$ of $x$, and a holomorphic function $g: U \rightarrow \mathbb{C}$, we just need to find a smaller neighborhood of $x$ on which $g$ has an antiderivative. We can always find a small neighborhood $U_{x}$ of $x$ which is simply-connected, and then using theorems of Cauchy and Morrera, a holomorphic function on a simply-connected region has an antiderivative. That is, there exists $f \in \mathcal{O}\left(U_{x}\right)$ such that $\frac{d f}{d z}=\psi_{U_{x}}(f)=\rho_{U_{x}}^{U}(g)$. Thus $\psi_{x}$ is surjective.

If $X$ is a simply-connected region, then the same argument as above shows that $\mathcal{O}_{X}$ : $\mathcal{O}(X) \rightarrow \mathcal{O}(X)$ is surjective. However, if $X$ is not simply-connected, then there are holomorphic functions on $X$ which do not possess a global antiderivaitve. For example, take $X$ to be the punctured plane, and consider $f(z)=\frac{1}{z}$. By various results in complex analysis, this does not have a global antiderivative, so in this situation, $\mathcal{O}(X) \rightarrow \mathcal{O}(X)$ is NOT surjective, even though all of the maps on stalks are surjective. Hence the induced sequence on sections need not be exact at the right side.

Definition 4.20. A sheaf $\mathcal{F}$ on $X$ is flasque or flabby if the restriction maps $\rho_{V}^{U}: \mathcal{F}(U) \rightarrow$ $\mathcal{F}(V)$ are surjective for all $V \subset U \subset X$. Equivalently, the maps $\rho_{U}^{X}: \mathcal{F}(X) \rightarrow \mathcal{F}(U)$ are surjective for all $U \subset X$.

Example 4.21. A skyscraper sheaf is flasque.
Remark 4.22. For those who know something about derived functors, we are going to show that flasque sheaves are acyclic objects with respect to the global sections functor.

Theorem 4.23. Let

$$
0 \rightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \rightarrow 0
$$

be a short exact sequence of sheaves of abelian groups on $X$.

1. If $\mathcal{F}$ is flasque, then all induced sequences on sections are exact.

$$
0 \rightarrow \mathcal{F}(U) \xrightarrow{\phi_{U}} \mathcal{G}(U) \xrightarrow{\psi_{U}} \mathcal{H}(U) \rightarrow 0
$$

2. If $\mathcal{F}$ and $\mathcal{G}$ are flasque, then $\mathcal{H}$ is also flasque.

Proof. To prove (1), it suffices to prove that the sequence on global sections is exact. We already know it is exact on the left, so we just need to show that $\psi_{X}: \mathcal{G}(X) \rightarrow \mathcal{H}(X)$ is surjective. This will be a somewhat convoluted argument involving Zorn's lemma.

Take an arbitrary global section $t \in \mathcal{H}(X)$, and let $x \in X$. By assumtion, the map on stalks $\psi_{x}: \mathcal{G}_{x} \rightarrow \mathcal{H}_{x}$ is surjective, so there exists a neighborhood $U$ of $x$ and a section $s \in \mathcal{G}(U)$ such that $\psi_{U}(s)=\rho_{U}^{X}(\mathcal{H})(t)$. Our goal is to show that we can choose $U=X$, in which case we have found a global section $s$ so that $\psi_{X}(x)=t$.

Consider all pairs $(U, s)$ with $U$ an open neighborhood of $x$, and $s \in \mathcal{G}(U)$ such that $\psi_{U}(s)=\rho_{U}^{X}(\mathcal{H})(t)$. Let $S$ be the set of all such pairs. We then partially order $S$ by "inclusion," meaning that $\left(U_{1}, s_{1}\right) \leq\left(U_{2}, s_{2}\right)$ whenever $U_{1} \subset U_{2}$ and $\rho_{U_{1}}^{U_{2}}\left(s_{2}\right)=s_{1}$. In order to apply Zorn's lemma to $S$, we need to show that every chain in $S$ has an upper bound. Suppose we have a chain in $S$,

$$
\left(U_{1}, s_{2}\right) \leq\left(U_{2}, s_{2}\right) \leq \cdots
$$

This has an upper bound given by $(U, s)$ where

$$
U=\bigcup_{i} U_{i}
$$

and $s \in \mathcal{G}(U)$ is the section obtained by gluing all of the $s_{i}$. This gluing is possible because $\rho_{U_{j}}^{U_{i}}\left(s_{i}\right)=s_{j}$ by definition of $\leq \operatorname{in} S$.

Hence we may apply Zorn's lemma to $S$, to conclude that there is a maximal element $\left(U, s_{U}\right)$. Now our goal is to prove that the maximal element is $\left(U=X, s_{U}\right)$ so that $s$ is a global section for $\mathcal{G}$ which maps to our given section $t \in \mathcal{H}(X)$. In order to prove that $X$ is the maximal subset, we will prove that anything other than $X$ cannot be maximal, by extending it to a larger open subset of $X$.

To that end, suppose $U \subset X$ and $U \neq X$, where $\left(U, s_{U}\right)$ is the maximal element of $S$. Then choose $x \in X \backslash U$. By local surjectivity of $\psi$, there exists an open neighborhood $V$ of $x$ and a section $s_{V} \in \mathcal{G}(V)$ such that $\psi_{V}\left(s_{V}\right)=\rho_{V}^{x}(\mathcal{H})(t)$. Then

$$
\psi_{U \cap V}\left(\rho_{U \cap V}^{U}(\mathcal{G})\left(s_{U}\right)-\rho_{U \cap V}^{V}(\mathcal{G})\left(s_{V}\right)\right)=\rho_{U \cap V}^{X}(\mathcal{H})(t)-\rho_{U \cap V}^{X}(\mathcal{H}(t)=0
$$

By left exactness of the sequence on sections, this means that

$$
\rho_{U \cap V}^{U}(\mathcal{G})\left(s_{U}\right)-\rho_{U \cap V}^{V}(\mathcal{G})\left(s_{V}\right) \in \operatorname{ker} \psi_{U \cap V}=\operatorname{im} \phi_{U \cap V}=\phi_{U \cap V}(\mathcal{F}(U \cap V)
$$

Since $\mathcal{F}$ is flasque, $\rho_{U \cap V}^{V}: \mathcal{F}(V) \rightarrow \mathcal{F}(V \cap U)$ is surjective, so there exists $r \in \mathcal{F}(V)$ such that

$$
\phi_{U \cap V} \rho_{U \cap V}^{V}(r)=\rho_{U \cap V}^{U}(\mathcal{G})\left(s_{U}\right)=\rho_{U \cap V}^{V}(\mathcal{G})\left(s_{v}\right)
$$

Now set $s_{V}^{\prime}=S_{V}+\phi_{V}(r)$. By construction, $s_{V}^{\prime}$ agrees with $s_{U}$ on the intersetion, i.e.

$$
\rho_{U \cap V}^{V}\left(s_{V}^{\prime}\right)=\rho_{U \cap V}^{V}\left(s_{U}\right)
$$

Hence we may glue $s_{V}^{\prime}$, $s_{U}$ to obtain a section $s_{U \cup V} \in \mathcal{G}(U \cup V)$. Thus

$$
\left(U \cup V, s_{U \cup V}\right)
$$

is an element of $S$, which is strictly bigger than $\left(U, s_{U}\right)$. This contradicts maximality of $\left(U, s_{U}\right)$. So we reject the possibility that $U \neq X$, and conclude that $X=U$ and $s \in \mathcal{G}(U)$ is a global section which satisfies $\psi_{X}(s)=t$. This finishes the proof of part (1).

Now we prove part (2). This would be entirely obvious after developing a bit of sheaf cohomology, but we can also prove it by a straightforward diagram chase. Let $U \subset X$ be open. Then we have the following commutative diagram with exact rows. (The rows are exact on the right because $\mathcal{F}$ is flasque.)


We need to show that $\rho_{U}^{X}(\mathcal{H})$ is surjective, and we know that $\rho_{U}^{X}(\mathcal{G})$ is surjective. Choose $t \in \mathcal{H}(U)$, then lift to an element of $\mathcal{G}(U)$, then lift to an element of $\mathcal{G}(X)$, then take the image of this lift under $\psi_{X}$. This element maps (under $\left.\rho_{U}^{X}(\mathcal{H})\right)$ to $t$.

### 4.2 Summary of definitions of exactness

Since the various definitions and results about exactness for presheaves and sheaves have lots of subtle differences, we summarize them here as a reference. Fix a space $X . U$ denotes an open subset of $X$ and $x$ denotes a point in $X$.

Exact sequence of presheaves $\Longleftrightarrow$ Exact sequence on sections on every $U$
Exact sequence of presheaves $\Longrightarrow$ Exact sequence on stalks at every $x$
Exact sequence of presheaves $\Longrightarrow$ Exact sequence of sheaves
Exact sequence of sheaves
Exact sequence of sheaves
Exact sequence of sheaves, with left sheaf flasque $\quad \Longrightarrow \quad$ Exact sequence on sections on every $U$

We give another graphic to represent the same information. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be presheaves on $X$, and $\mathcal{F}^{+}, \mathcal{G}^{+}, \mathcal{H}^{+}$sheaves on $X$.
$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ exact $\quad \Longleftrightarrow 0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U) \rightarrow 0$ exact $\forall U$
$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ exact $\quad \Longrightarrow \quad 0 \rightarrow \mathcal{F}_{x} \rightarrow \mathcal{G}_{x} \rightarrow \mathcal{H}_{x} \rightarrow 0$ exact $\forall x$
$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ exact as presheaves $\quad \Longrightarrow 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ exact as sheaves
$0 \rightarrow \mathcal{F}^{+} \rightarrow \mathcal{G}^{+} \rightarrow \mathcal{H}^{+} \rightarrow 0$ exact $\quad \Longleftrightarrow 0 \rightarrow \mathcal{F}_{x}^{+} \rightarrow \mathcal{G}_{x}^{+} \rightarrow \mathcal{H}_{x}^{+} \rightarrow 0$ exact $\forall x$
$0 \rightarrow \mathcal{F}^{+} \rightarrow \mathcal{G}^{+} \rightarrow \mathcal{H}^{+} \rightarrow 0$ exact $\quad \Longrightarrow \quad 0 \rightarrow \mathcal{F}^{+}(U) \rightarrow \mathcal{G}^{+}(U) \rightarrow \mathcal{H}^{+}(U)$ exact $\forall U$
$0 \rightarrow \mathcal{F}^{+} \rightarrow \mathcal{G}^{+} \rightarrow \mathcal{H}^{+} \rightarrow 0$ exact, $\mathcal{F}^{+}$flasque $\quad \Longrightarrow \quad 0 \rightarrow \mathcal{F}^{+}(U) \rightarrow \mathcal{G}^{+}(U) \rightarrow \mathcal{H}^{+}(U) \rightarrow 0$ exact $\forall U$

I think the one thing that remains consistently confusing is that saying " $0 \rightarrow \mathcal{F}^{+} \rightarrow \mathcal{G}^{+} \rightarrow$ $\mathcal{H}^{+} \rightarrow 0$ is exact" is ambiguous, in the case where they are sheaves. Is it meant that it is exact in the category of sheaves, or in the category of presheaves? Both are possible interpretations, since every sheaf is an object in the category of presheaves, and every morphism of sheaves is a morphism in the category of presheaves.

The convention demanded by the definitions is that this only means that the sequence is exact in the category of sheaves, since in general, an exact sequence of sheaves does NOT make an exact sequence on sections, which would have to be the case if it were exact as a sequence of presheaves. On the other hand, an exact sequence of presheaves, if all the presheaves are sheaves, is an exact sequence of sheaves.

Another way to phrase this is that being exact as a sequence of presheaves is a stronger requirement than being exact as a sequence of sheaves.

### 4.3 Epimorphisms

This section spends some time to justify the slightly asymmetrical definitions of exactness of sequences for sheaves and presheaves. Essentially, the justification is in the fact that the definitions as given coincide with more general categorical notions, primarily in terms of epimorphisms.

Definition 4.24. A morphism $f: X \rightarrow Y$ in a category $\mathcal{C}$ is an epimorphism if for any two morphisms $g_{1}, g_{2}: Y \rightarrow Z$ we have the implication

$$
g_{1} f=g_{2} f \Longrightarrow g_{1}=g_{2}
$$

Example 4.25. In the category of abelian groups, a morphism is an epimorphism if and only if it surjective. We give a proof of this fact. It is clear that a surjective morphism is an epimorphism. Conversely, suppose $f: X \rightarrow Y$ is an epimorphism, and consider $Z=Y / f(X)=$ coker $f$, and the following maps.

$$
X \xrightarrow{f} Y \xrightarrow[0]{\stackrel{\pi}{\rightrightarrows}} Z=Y / f(X)
$$

where $\pi$ is the quotient map and 0 is the zero map. Suppose $f$ is not surjective. Then $\pi, 0$ are not the same map, bu $\pi f=0 f$, contradicting the epimorphism property. Hence $f$ is surjective.

Lemma 4.26. Let $X$ be a space, and $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups on $X$. A morphism of presheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is surjective as a morphism of presheaves if and only if it is an epimorphism in the category of presheaves.

Proof. This is essentially the same proof as for abelian groups. Consider the cokernel presheaf

$$
\text { coker } \phi=\mathcal{C} \quad \mathcal{C}(U)=\mathcal{G}(U) / \phi_{U}(\mathcal{F}(U))
$$

and repeat the argument of the previous example.

This is the point where we can finally explain why "surjective as a morphism of sheaves" is not defined in the same way that "surjective as a morphism of presheaves" is defined. Recall that a morphism of presheaves is surjective if the maps on open sets are surjective, but a morphism of sheaves is surjective (as a morphism of sheaves) if the maps on stalks are surjective. If we had defined a morphism of sheaves to be surjective if the maps on open sets were surjective, then the previous lemma would not generalize to sheaves, because the cokernel presheaf is not in general a sheaf.

Because the cokernel presehaf is not in general a sheaf, we need to instead look at the sheafification of the cokernel presheaf.

Proposition 4.27. Let $X$ be a space, and let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on $X$. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Let $\mathcal{C}$ be the cokernel presheaf on $\phi$, and let $\theta: \mathcal{C} \rightarrow \mathcal{C}^{+}$ be the sheafification of $\mathcal{C}$. The following are equivalent.

1. $\phi$ is an epimorphism in the category of sheaves.
2. $\mathcal{C}^{+}$is the zero sheaf.
3. $\phi$ is surjective as a morphism of sheaves (all induced maps on stalks are surjective).

Proof. First, we prove $(2) \Longleftrightarrow(3)$. First, recall that the zero sheaf has stalks which are trivial, and all stalks being trivial forces a sheaf to be the zero sheaf. By definition of exactness for presheaves, we have an exact sequence of presheaves

$$
\mathcal{F} \xrightarrow{\phi} \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0
$$

which induces an exact sequence on stalks

$$
\mathcal{F}_{x} \xrightarrow{\phi_{x}} \mathcal{G}_{x} \rightarrow \mathcal{C}_{x} \rightarrow 0
$$

Also recall that the sheafification map $\theta$ induces isomorphisms on all stalks, $\theta_{x}: \mathcal{C}_{x} \rightarrow \mathcal{C}_{x}^{+}$. So if $\phi$ is surjective as a morphism of sheaves, all $\phi_{x}$ are surjective, hence all $\mathcal{C}_{x}$ are zero, hence all $\mathcal{C}^{+}$are zero, hence $\mathcal{C}^{+}$is the zero sheaf. This proves $(3) \Longrightarrow(2)$. Conversely, if $\mathcal{C}^{+}$is zero, then all stalks are zero, and $\phi_{x}$ is surjective for every $x$, which is to say, $\phi$ is surjective as a morphism of sheaves. This proves $(2) \Longrightarrow(3)$.

Now we prove that (2), (3) together imply (1). Consider morphisms of sheaves $g_{1}, g_{2}$ : $\mathcal{G} \rightarrow \mathcal{H}$ such that $g_{1} \phi=g_{2} \phi$. To prove $\phi$ is an epimorphism, we need to show $g_{1}=g_{2}$. Considering the morphisms on stalks, we get that for any $x \in X$,

$$
g_{1, x} \phi_{x}=g_{2, x} \phi_{x}
$$

By (3), $\phi_{x}$ is surjective, so $g_{1, x}=g_{2, x}$. Since $\mathcal{G}, \mathcal{H}$ are sheaves, by Corollary 4.3 $g_{1}=g_{2}$. Hence $\phi$ is an epimorphism, proving (1).

Finally, we prove (1) $\Longrightarrow$ (3), which completes the equivalence. Let $\phi$ be an epimorphism. Suppose to the contrary that for some $x_{0} \in X$, the induced map on stalks $\phi_{x_{0}}$ is not surjective. Consider the diagram

$$
\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow[0]{\pi} \mathcal{C} \xrightarrow{\theta} \mathcal{C}^{+}
$$

where $\pi$ is the canonical quotient map with $\pi_{U}: \mathcal{G}(U) \rightarrow \mathcal{C}(U)=\mathcal{G}(U) / \phi(\mathcal{F}(U))$ being the quotient map, and zero refers to the map which is the zero map on each $\mathcal{G}(U)$. It is clear that

$$
\theta \circ \pi \circ \phi=\theta \circ 0 \circ \phi
$$

Since $\phi$ is an epimorphism, it follows that $\theta \circ \pi=\theta \circ 0$. In particular, on the stalks at $x_{0}$, we get

$$
\theta_{x_{0}} \circ \pi_{x_{0}}=\theta_{x_{0}} \circ 0_{x_{0}}
$$

Since $\theta_{x_{0}}$ is an isomorphism, $\pi_{x_{0}}=0_{x_{0}}$. This implies that $\phi_{x_{0}}$ is surjective, which is a contradiction. So we conclude that $\phi$ is surjective as a morphism of sheaves. Thus (1) $\Longrightarrow$ (3).

### 4.4 Image presheaf

We start by recalling the definition which we introduced earlier.
Definition 4.28. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. The image presheaf is the presheaf on $X$ given by

$$
\mathcal{I}(U)=\phi_{U}(\mathcal{F}(U))
$$

It is a subpresheaf of $\mathcal{G}$, meaning the restriction maps are induced by the restriction maps for $\mathcal{G}$.

Since we are more interested in sheaves than presheaves, we would like it if the image presheaf was always a sheaf. Unfortunately, this is not generally the case, as demonstrated in the following example.

Example 4.29 (Failure of image presheaf to be a sheaf). Let $X \subset \mathbb{C}$ be an open subset which is not simply-connected. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $X$, and let $\phi: \mathcal{O} \rightarrow \mathcal{O}$ be the differentiation operator.

$$
\phi_{U}: \mathcal{O}(U) \rightarrow \mathcal{O}(U) \quad f \mapsto \frac{d f}{d z}
$$

Then we see that the sections of the image presheaf, $\mathcal{I}(U)=\phi(\mathcal{O}(U))$ is the $\mathbb{C}$-algebra of holomorphic functions on $U$ which have an antiderivative on $U$. Since $X$ is not simply connected, there exist holomorphic functions on $X$ which do not possess an antiderivative. Let $f: X \rightarrow \mathbb{C}$ be such a function. However, it is possible to cover $X$ by simply connected neighborhoods,

$$
X=\bigcup_{\alpha} U_{\alpha}
$$

Then set $f_{\alpha}=\left.f\right|_{U_{\alpha}}$. Since $U_{\alpha}$ is simply connected, by various powerful theorems in complex analysis, $f_{\alpha}$ has an antiderivative on $U_{\alpha}$. It is then possible to glue the holomorphic functions
$f_{\alpha}$ to obtain a holomorphic function on $X$, namely $f$, using the fact that $\mathcal{O}$ is a sheaf. But by construction, $f$ is a section of the image presheaf $\mathcal{I}(X)$. So $\mathcal{I}$ does NOT have the gluing property, hence it is not a sheaf.

Definition 4.30. For a morphism of sheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$, the inclusion of the image presheaf $i: \mathcal{I} \hookrightarrow \mathcal{G}$ factors through the sheafification $\mathcal{I}^{+}$, by the universal property of sheafification.


Since $i$ is injective (as a morphism of presheaves), $i^{+}$is injective (as a morphism of sheaves and/or presheaves). The sheaf $\mathcal{I}^{+}$is the image sheaf of $\phi$. It is a subsheaf of $\mathcal{G}$.

Remark 4.31. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves, and $\mathcal{I}$ the image presheaf and $\mathcal{I}^{+}$ the image sheaf. By construction of $\mathcal{I}$, the sequence

$$
\mathcal{F} \xrightarrow{\phi} \mathcal{I} \rightarrow 0
$$

is an exact sequence of presheaves, so the induced sequence on stalks is also exact.

$$
\mathcal{F}_{x} \xrightarrow{\phi_{x}} \mathcal{I}_{x} \rightarrow 0
$$

Since $\theta_{x}$ gives an isomorphism $\mathcal{I}_{x} \cong \mathcal{I}_{x}^{+}$, we get an exact sequence

$$
\mathcal{F}_{x} \xrightarrow{\theta_{x} \circ \phi_{x}} \mathcal{I}_{x}^{+} \rightarrow 0
$$

Hence $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ is surjective if and only if $\mathcal{I}_{x}^{+} \rightarrow \mathcal{G}_{x}$ is surjective.
Corollary 4.32. A morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of abelian groups on $X$ is surjective if and only if the image sheaf coincides with $\mathcal{G}$.

Proof. Following the discussion in the previous remark, $\phi_{x}$ is surjective for all $x$ if and only if $\mathcal{I}_{x}^{+} \rightarrow \mathcal{G}_{x}$ is surjective for all $x$, so $\phi$ is surjective as a morphism of sheaves if and only if $\mathcal{I}^{+}$and $\mathcal{G}$ have isomorphic stalks everywhere, which happens if and only if $\mathcal{I}^{+}$and $\mathcal{G}$ are isomorphic as sheaves.

## 5 Sheafification

We have already stated the main result/universal property/existence of sheafification, and we have already been using it without reservation, but we have not given a proof/construction of this result. The goal of this section is to do that. Given a presheaf $\mathcal{F}$ on $X$, the strategy is to construct a topological space $E$ with a local homeomorphism $\pi: E \rightarrow X$ with the following properties.

1. The fiber $\pi^{-1}(x)=E_{x}$ is isomorphic to the stalk $\mathcal{F}_{x}$.
2. For $U \subset X$ open, let $\Gamma(U, \pi)$ be the set of continuous sections $s: U \rightarrow E$ of $\pi$, meaning $\pi s=\operatorname{Id}_{U}$. We want $\Gamma(-, \pi)$ to be a sheaf on $X$.

In the end, the sheaf $\Gamma$ will be the sheafification $\mathcal{F}^{+}$. The space $E$ constructed along the way is known as the étale space of the presheaf $\mathcal{F}$.

### 5.1 Local homeomorphisms

First, in order to motivate the construction, we start with some generalities about local homeomorphisms. Most of the proofs will be omitted.

Definition 5.1. A map $\pi: E \rightarrow X$ of topological paces is a local homeomorphism if for every $e \in E$, there exist open neighborhoods $O_{e} \subset E$ and $U_{x}=U_{\pi(e)} \subset X$ with $e \in O_{e}, x \in U_{x}$ such that

$$
\left.\pi\right|_{O_{e}}: O_{e} \rightarrow U_{x}
$$

is a homeomorphism.
Example 5.2. Let $X$ be any space, and $U \subset X$ a proper open subset. The inclusion $U \hookrightarrow X$ is a local homeomorphism.

Example 5.3. Recall that a covering map is a surjective continuous map $\pi: E \rightarrow X$ such that for every $x \in X$, there exists a neighborhood $U$ of $x$ such that the preimage of $U$ under $\pi$ is a disjoint union of homeomorphic copies of $U$. More precisely,

$$
\pi^{-1}(U)=\bigsqcup_{\alpha} V_{\alpha}
$$

and for each $\alpha,\left.\pi\right|_{V_{\alpha}}: V_{\alpha} \rightarrow U$ is a homoeomorphism. A covering map is an example of a local homeomorphism. Note that not every local homeomorphism is a covering space, as the previous example of a simple inclusion shows.

Definition 5.4. Let $X$ be a topological space. A pair $(E, \pi)$ of a space $E$ and a local homeomorphism $\pi: E \rightarrow X$ is a étale space over $X$. In this situation, $E$ is called the total space, and $X$ is called the base space.

Definition 5.5. Let $\pi: E \rightarrow X$ be a continuous map, and let $U \subset X$ be an open subset. A section of $\pi$ over $U$ is a continuous map $s: U \rightarrow E$ such that $\pi s=\operatorname{Id}_{U}$. The set of all such sections is denoted $\Gamma(U, \pi)$.

Proposition 5.6. Let $\pi: E \rightarrow X$ be a local homeomorphism, and let $E_{x}=\pi^{-1}(x)$ be the fiber over $x$. Then

1. $\pi$ is an open map.
2. $E$ is the union of the fibers,

$$
E=\bigcup_{x \in X} \pi^{-1}(x)=\bigcup_{x \in X} E_{x}
$$

and each fiber $E_{x}$ is discrete under the subspace topology.
3. If $s_{1}: U_{1} \rightarrow E$ and $s_{2}: U_{2} \rightarrow E$ are sections such that there is a point $x \in U_{1} \cap U_{2} \subset X$ with $s_{1}(x)=s_{2}(x)$, then there exists an open neighborhood $V$ of $x$ such that $\left.s_{1}\right|_{V}=\left.s_{2}\right|_{V}$.
4. For any $U \subset X$ open and any section $s: U \rightarrow E$, the image $s(U)$ is open in $E$, and homeomorphic to $U$, via s. That is, $s=\left(\left.\pi\right|_{s(U)}\right)^{-1}$.
5. Sets of the form $s(U)$ where $U$ ranges over open subsets of $X$ and s ranges over sections of $U$ form a basis of the topology on $E$. Philosophically, the topology on $E$ is determined by the topology on $X$ and sections of $\pi$.

Proof. This is all just point-set topology, which is not the focus of this class. See a resource such as Munkres book on topology.

Proposition 5.7. Let $\pi: E \rightarrow X$ be a local homeomorphism. Define

$$
\mathcal{F}(U)=\Gamma(U, \pi)
$$

and for $V \subset U \subset X$ open subsets define

$$
\rho_{V}^{U}:\left.\mathcal{F}(U) \rightarrow \mathcal{F}(V) \quad s \mapsto s\right|_{V}
$$

Then $\mathcal{F}$ is a sheaf of sets on $X$. Moreover, for each $x \in X$, the stalk $\mathcal{F}_{x}$ is isomorphic to (in bijeciton with) the stalk $E_{x}=\pi^{-1}(x)$.

Before we get to the proof, note that the important part of the previous proposition is the isomorphism $\mathcal{F}_{x} \cong E_{x}$. The fact that $\mathcal{F}$ is a sheaf is not so profound, it is mostly obvious. In fact, $\mathcal{F}$ is a sheaf if $\pi$ is any continuous map; it need not be a local homeomorphism. The important and interesting part is the fact that $\pi$ being a local homeomorphism gives a lot of structure to the stalks.

Proof. First we verify that $\mathcal{F}$ is a sheaf. Note that $\mathcal{F}$ is a subpresheaf of the sheaf of continuous $E$-valued functions on $X$, so we know automatically that $\mathcal{F}$ is separated. To prove gluing, note that it is always possible to glue sections to obtain a continuous map, it just suffices to show that the resulting glued function is a section. Let $U \subset X$ be open, and take an open cover,

$$
U=\bigcup_{\alpha} U_{\alpha}
$$

and suppose we have sections $s_{\alpha} \in \Gamma\left(U_{\alpha}, \pi\right)=\mathcal{F}\left(U_{\alpha}\right)$. We glue them together to obtain a continuous map $s: U \rightarrow E$ using the gluing property for continuous functions. Since $s_{\alpha}$ is a section of $\pi$, we know $\pi s_{\alpha}=\operatorname{Id}_{U_{\alpha}}$ for each $\alpha$, so for $x \in U_{\alpha}$,

$$
\pi s(x)=\left.\pi s\right|_{U_{\alpha}}(x)=x
$$

Thus $\pi s=\operatorname{Id}_{U}$, so $s$ is a section. Hence $\mathcal{F}$ is a sheaf. Now we prove the statement regarding stalks. Let $x \in X$, and define

$$
\eta_{U}: \mathcal{F}(U)=\Gamma(U, \pi) \rightarrow E_{x} \quad s \mapsto s(x)
$$

We want to use the maps $\eta_{U}$ to induce a map on the direct limit $\eta: \mathcal{F}_{x} \rightarrow E_{x}$. That is, we want to define

$$
\eta: \mathcal{F}_{x}=\underset{x \in U}{\lim } \mathcal{F}(U) \rightarrow E_{x} \quad[s] \mapsto s(x)
$$

To verify that this is well defined, we need to check that if $s_{1}, s_{2}$ are sections with $\left[s_{1}\right]=\left[s_{2}\right]$, then $s_{1}(x)=s_{2}(x)$. Suppose we have sections $s_{1}: U_{1} \rightarrow E$ and $s_{2}: U_{2} \rightarrow E$ which represent the same element in the stalk $\mathcal{F}_{x}$. Then we know that there is a neighborhood of $x$ on which $s_{1}, s_{2}$ agree; in particular, $s_{1}(x)=s_{2}(x)$. Hence $\eta$ is well defined.

We claim that $\eta$ is a bijection. First we prove $\eta$ is surjective. Because $\pi$ is a local homeomorphism, lgiven $e \in E_{x}=\pi^{-1}(x)$, we can find open neighborhoods $O_{e}$ of $e$ and $U_{x}$ of $x=\pi(e)$ such that

$$
\left.\pi\right|_{O_{e}}: O_{e} \rightarrow U_{x}
$$

is a homoemorphism. Then

$$
\left(\left.\pi\right|_{O_{e}}\right)^{-1}: U_{x} \rightarrow O_{e}
$$

is a section of $\pi$, and

$$
\eta\left[\left(\left.\pi\right|_{O_{e}}\right)^{-1}\right]=\left(\left.\pi\right|_{O_{e}}\right)^{-1}(x)=\left(\left.\pi\right|_{O_{e}}\right)^{-1}(\pi(e))=e
$$

Hence $\eta$ is surjective. Now we prove $\eta$ is injective. Suppose two elements of the stalk have the same value under $\eta, \eta\left[s_{1}\right]=\eta\left[s_{2}\right]$. Then $s_{1}(x)=s_{2}(x)$. By part (3) of Proposition 5.6, there is an open neighborhood of $x$ on which $s_{1}, s_{2}$ agree. That is, $\left[s_{1}\right]=\left[s_{2}\right]$. Thus $\eta$ is injective. This completes the required bijection

$$
\mathcal{F}_{x} \cong E_{x}
$$

## 5.2 Étale space of a presheaf

In order to construct the sheafification of a presheaf, we start by constructing the étale space of the presheaf, which is basically reverse engineered by examining the proof of the previous proposition. We know that if we have a local homeomorphism $\pi: E \rightarrow X$ gives a sheaf whose stalks are precisely the fibers, so we take a presheaf $\mathcal{F}$ and start by constructing a space $E$ and map $\pi: E \rightarrow X$ such that the fibers are the stalks $\mathcal{F}_{x}$.

Definition 5.8. Let $X$ be a topological space, and let $\mathcal{F}$ be a presheaf of sets on $X$. The étale space of $\mathcal{F}$ is

$$
E=\bigsqcup_{x \in X} \mathcal{F}_{x}
$$

We define a map $\pi: E \rightarrow X$ by sending all points in the stalk $\mathcal{F}_{x}$ to the point $x$.
By definition, we now have a space $E$ and a map $\pi: E \rightarrow X$ which captures sufficient information about the stalks, that is, the (set-theoretic) fibers of $\pi$ are precisely the stalks of $\mathcal{F}$. However, $E$ as yet has no topology, so it doesn't even make sense to ask if $\pi$ is continuous, let alone whether $\pi$ is a local homeomorphism. The next goal is to remedy this, by equipping $E$ with a topology which makes $\pi$ a (continuous) local homeomorphism.

To define the topology on $E$, we again reverse the process of our previous work. We previously noted that if $\pi: E \rightarrow X$ is a local homeomorphism, then the sets $s(U)$ for $U \subset X$ open and $s \in \Gamma(U, \pi)$ give a basis for the topology on $U$. So now we follow that prompting, and define the topology on $E$ in this way.

Proposition 5.9. Let $\mathcal{F}$ be a presheaf of sets on a space $X$, and let $\pi: E \rightarrow X$ be the étale space of $\mathcal{F}$. For $U \subset X$ open and $s \in \mathcal{F}(U)$, define

$$
\widetilde{s}: U \rightarrow E \quad x \mapsto \rho_{x}^{U}(s) \in \mathcal{F}_{x} \subset E
$$

Then

1. The sets $\widetilde{s}(U)$ for $U \subset X$ open and $s \in \mathcal{F}(U)$ form a basis for a topology on $E$.
2. Giving $E$ the above topology makes $\pi: E \rightarrow X$ a local homeomorphism, and each $\widetilde{s}: U \rightarrow E$ is a continuous section of $\pi$.

Proof. It is clear that $E$ is covered by such sets $\widetilde{s}(U)$, since an element of the stalk $F_{x}$ is of the form $\rho_{x}^{U}(s)$ for some $U \subset X, s \in \mathcal{F}(U)$.

$$
E=\bigcup_{\substack{U \subset X \\ s \in \mathcal{F}(U)}} \widetilde{s}(U)
$$

In order to show that the sets $\widetilde{s}(U)$ can form a basis, we need to show that an intersection of two such sets contains a third such set. In fact, we will show that the intersection is anothe set of this form.

Consider two arbitrary alleged basis sets for $E$; namely, take arbitrary open subsets $U_{1}, U_{2} \subset X$ and sections $s_{1} \in \mathcal{F}\left(U_{1}\right), s_{2} \in \mathcal{F}\left(U_{2}\right)$, and consider the intersection $\widetilde{s}_{1}\left(U_{1}\right) \cap$ $\widetilde{s}_{2}\left(U_{2}\right)$.

$$
\widetilde{s}_{1}\left(U_{1}\right) \cap \widetilde{s}_{2}\left(U_{2}\right)=\left\{e \in E: \exists x_{1} \in U_{1}, x_{2} \in U_{2} \text { such that } e=\widetilde{s}_{1}\left(x_{1}\right)=\widetilde{s}_{2}\left(x_{2}\right)\right\}
$$

The condition that

$$
\rho_{x_{1}}^{U_{1}}\left(s_{1}\right)=\widetilde{s}_{1}\left(x_{1}\right)=\widetilde{s}_{2}\left(x_{2}\right)=\rho_{x_{2}}^{U_{2}}\left(s_{2}\right)
$$

only makes sense when they are all in the same stalk, which is to say, when $x_{1}=x_{2}$, so we use $x$ to denote this common value (note that $x \in U_{1} \cap U_{2}$ ). It is also clear that if $\widetilde{s}_{1}(x)=e$, then $\pi(e)=x$. So we may rewrite the intersection as

$$
\begin{aligned}
\widetilde{s}_{1}\left(U_{1}\right) \cap \widetilde{s}_{2}\left(U_{2}\right) & =\left\{e \in E: \exists x \in U_{1} \cap U_{2} \text { such that } x=\pi(e)=\widetilde{s}_{1}(x)=\widetilde{s}_{2}(x)\right\} \\
& =\left\{\widetilde{s}_{1}(x): x \in U_{1} \cap U_{2}, \widetilde{s}_{1}(x)=\widetilde{s}_{2}(x)\right\}
\end{aligned}
$$

Now define

$$
U=\left\{x \in U_{1} \cap U_{2}: \rho_{x}^{U_{1}}\left(s_{1}\right)=\widetilde{s}_{1}(x)=\widetilde{s}_{2}(x)=\rho_{x}^{U_{2}}\left(s_{2}\right)\right\}
$$

We claim that $U$ is open in $X$. If $y \in U$, then by definition of equality in stalks, there exists an open neighborhood $V$ with $x \in V \subset U$ such that $\left.s_{1}\right|_{V}=\left.s_{2}\right|_{V}$. All we care about is that $x$ has an open neighborhood inside of $U$, which proves $U$ is open. Now define $s \in \mathcal{F}(U)$ by $s=\rho_{U_{1}}^{U}\left(s_{1}\right)$. We claim that

$$
\widetilde{s}_{1}\left(U_{1}\right) \cap \widetilde{s}_{2}\left(U_{2}\right)=\widetilde{s}(U)
$$

which will finish the proof of (1).

$$
\begin{aligned}
\widetilde{s}(U) & =\{\widetilde{s}(x): x \in U\} \\
& =\left\{\rho_{x}^{U} \rho_{U}^{U_{1}}\left(s_{1}\right): x \in U\right\} \\
& =\left\{\rho_{x}^{U_{1}}\left(s_{1}\right): x \in U_{1} \cap U_{2}, \widetilde{s}_{1}(x)=\widetilde{s}_{2}(x)\right\} \\
& =\left\{\widetilde{s}_{1}(x): x \in U_{1} \cap U_{2}, \widetilde{s}_{1}(x)=\widetilde{s}_{2}(x)\right\} \\
& =\widetilde{s}_{1}\left(U_{1}\right) \cap \widetilde{s}_{2}\left(U_{2}\right)
\end{aligned}
$$

This finishes the proof of (1). Now we prove (2). Using (1), the sets $\widetilde{s}(U)$ cover $E$ and form a basis for the topology, and it is immediate that $\pi \widetilde{s}=\operatorname{Id}_{U}$. Thus $\pi$ is an open map and the restriction of $\pi$ to $\widetilde{s}(U)$ is a bijection. We also need to verify that $\pi$ is continuous, which we do now. Let $U \subset X$ be open. Then

$$
\pi^{-1}(U)=\bigcup_{x \in U} \mathcal{F}_{x}
$$

Each $e \in \mathcal{F}_{x}$ is of the form $e=\rho_{x}^{V}(s)$ for some $V \subset U$ open and some $s \in \mathcal{F}(V)$, so this can be rewritten as

$$
\pi^{-1}(U)=\bigcup_{x \in U} \mathcal{F}_{x}=\bigcup_{\substack{V \subset U \text { open } \\ s \in \mathcal{F}(V)}} \widetilde{s}(V)
$$

Each $\widetilde{s}(V)$ is open by definition, hence $\pi^{-1}(U)$ is open, hence $\pi$ is continuous. Using what we said before, $\pi$ is open and gives a continuous and open bijection between $\widetilde{s}(U)$ and $U$, for $U \subset X$ arbitrary and open. Hence $\pi$ is a local homeomorphism.

All that remains is to verify that $\widetilde{s}$ is a continuous section of $\pi$. We already know it is a set-theoretic section of $\pi$, and the maps $\widetilde{s}: U \rightarrow \widetilde{s}(U)$ and $\left.\pi\right|_{\tilde{s}(U)}: \widetilde{s}(U) \rightarrow U$ are inverse, so the fact that $\pi$ is open means $\widetilde{s}$ is continuous.

Before we can get to the main result of sheafification, we need one somewhat techni$\mathrm{cal} /$ categorical lemma regarding "functoriality" of the étale space.

Lemma 5.10 (Functoriality of étale space). Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on $X$. Let

$$
\pi_{\mathcal{F}}: E_{\mathcal{F}} \rightarrow X \quad \pi_{\mathcal{G}}: E_{\mathcal{G}} \rightarrow X
$$

be the respective étale spaces. Then $\phi$ induces a continuous map

$$
\widetilde{\phi}: E_{\mathcal{F}} \rightarrow E_{\mathcal{G}} \quad e \mapsto \phi_{\pi_{\mathcal{F}}(e)}(e) \in \mathcal{G}_{x}
$$

Another way to say the above is that if $e \in \mathcal{F}_{x} \subset E_{\mathcal{F}}$, then $\widetilde{\phi}(e)=\phi_{x}(e)$. Also, the following diagram commutes.


Proof. The final commutative diagram is obvious from the definition of $\widetilde{\phi}$, all that we need to verify is that $\widetilde{\phi}$ is continuous. It suffices to show that the preimage (under $\widetilde{\phi}$ ) of a basic open set of $E_{\mathcal{G}}$ is open in $E_{\mathcal{F}}$. To fix notation, for $s \in \mathcal{F}(U)$ or $t \in \mathcal{G}(U)$, define

$$
\begin{array}{ll}
\widetilde{s}: U \rightarrow E_{\mathcal{F}} & \widetilde{s}(x)=\rho_{x}^{U}(\mathcal{F})(s) \\
\widehat{t}: U \rightarrow E_{\mathcal{G}} & \widehat{t}(x)=\rho_{x}^{U}(\mathcal{G})(t)
\end{array}
$$

Let $\widehat{t}(U) \subset E_{\mathcal{G}}$ be a basic open subset. Then

$$
\begin{aligned}
\widetilde{\phi}^{-1}(\widehat{t}(U)) & =\left\{e \in E_{\mathcal{F}}: \widetilde{\phi}(e) \in \widehat{t}(U)\right\} \\
& =\left\{e \in E_{\mathcal{F}}: \phi_{\pi_{\mathcal{F}}(e)}(e) \in \widehat{t}(U)\right\} \\
& =\left\{e \in E_{\mathcal{F}}: \exists x \in U \text { such that } \phi_{\pi_{\mathcal{F}}(e)}(e)=\widehat{t}(x) \text { in } \mathcal{G}_{x}\right\} \\
& =\left\{e \in E_{\mathcal{F}}: \exists x \in U \text { such that } \phi_{\pi_{\mathcal{F}}(e)}(e)=\rho_{x}^{U}(\mathcal{G})(t) \text { in } \mathcal{G}_{x}\right\} \\
& =\left\{e \in E_{\mathcal{F}}: \phi_{\pi_{\mathcal{F}}(e)}(e)=\rho_{\pi_{\mathcal{F}}(e)}^{U}(\mathcal{G})(t) \text { in } \mathcal{G}_{x}\right\}
\end{aligned}
$$

In order to show this is open, given $e \in \widetilde{\phi}^{-1}(\widehat{t}(U))$, it suffices to show that there is a basic open subset of $E_{\mathcal{F}}$ containing $e$ contained in $\widetilde{\phi}^{-1}(\widehat{t}(U))$. Let $e \in \widetilde{\phi}^{-1}(\widehat{t}(U))$, and let $x=\pi_{\mathcal{F}}(e) \in X$.

By the above, $x \in U$. Choose a section $s \in \mathcal{F}(U)$ representing $e$, that is, $\rho_{x}^{U}(\mathcal{F})(s)=e$. Then we have $\phi_{U}(s) \in \mathcal{G}(U)$, and because $\phi$ is a morphism of presheaves,

$$
\rho_{x}^{U}(\mathcal{G})\left(\phi_{U}(s)\right)=\phi_{x}\left(\rho_{x}^{U}(\mathcal{F})(s)\right)=\phi_{x}(e)=\rho_{x}^{U}(\mathcal{G})(t)
$$

we see that $\phi(s), t$ represent the same element of the stalk $\mathcal{G}_{x}$. Then by definition of equality of elements of stalks, there is an open neighborhood $V \subset U$ of $x=\pi_{\mathcal{F}}(e)$ on which $\phi_{U}(s), t$ agree.

$$
\rho_{V}^{U}(\mathcal{G})\left(\phi_{U}(s)\right)=\rho_{V}^{U}(\mathcal{G})(t)
$$

Now let $s^{\prime}=\rho_{V}^{U}(\mathcal{F})(s)$. We claim that that $\widetilde{s}^{\prime}(V)$ is the required basic open subset of $E_{\mathcal{F}}$ containing $e$ which is contained in $\widetilde{\phi}^{-1}(\widehat{t}(U))$. It is clear that $e \in \widetilde{s}^{\prime}(V)$, and that $\widetilde{s}^{\prime}(V)$ is open, we just need to show $\widetilde{s}(V) \subset \widetilde{\phi}(\widehat{t}(U))$. To see this, take an arbitrary element $y \in V$ and consider $\rho_{y}^{V}(\mathcal{F})\left(s^{\prime}\right) \in \widetilde{s}^{\prime}(V) \subset E_{\mathcal{F}}$. We want to show $\widetilde{\phi}\left(\rho_{y}^{V}(\mathcal{F})\left(s^{\prime}\right)\right) \in \widehat{t}(U)$.

$$
\begin{aligned}
\widetilde{\phi}\left(\rho_{y}^{V}(\mathcal{F})\left(s^{\prime}\right)\right) & =\phi_{y}\left(\rho_{y}^{V}(\mathcal{F})\left(s^{\prime}\right)\right) \\
& =\phi_{y}\left(\rho_{y}^{V}(\mathcal{F}) \circ \rho_{V}^{U}(\mathcal{F})(s)\right) \\
& =\rho_{y}^{V}(\mathcal{G})\left(\phi_{V}\left(\rho_{V}^{U}(\mathcal{F})(s)\right)\right) \\
& =\rho_{y}^{V}(\mathcal{G}) \circ \rho_{V}^{U}(\mathcal{G})\left(\phi_{U}(s)\right) \\
& =\rho_{y}^{V}(\mathcal{G})\left(\rho_{V}^{U}(\mathcal{G})(t)\right) \\
& =\rho_{y}^{U}(\mathcal{G})(t)
\end{aligned}
$$

This last line is clearly an element in $\widehat{t}(U)$, so we are done.
Remark 5.11. Another way to rephrase the previous lemma is that the assignment $\mathcal{F} \mapsto E_{\mathcal{F}}$ is a covariant functor from the category of presheaves on $X$ to the category of étale spaces over $X$, where objects are pairs $(E, \pi)$ with $\pi: E \rightarrow X$ a local homeomorphism, and morphisms are morphisms over $X$.

$$
E: \operatorname{PSh}(X) \rightarrow\{\text { étale spaces over } X\} \quad \mathcal{F} \mapsto\left(E_{\mathcal{F}}, \pi_{\mathcal{F}}\right)
$$

There is some mild verification to check that this respects compositions and the identity, but these are relatively obvious.

### 5.3 Sheafification - main result

We have already stated the main fact of sheafification, although at that point we had not yet developed the language of stalks, so this formulation will be more complete. More significantly, now that we have developed results about the étale space of a presheaf, we can prove the theorem.

Theorem 5.12 (Existence of sheafification). Let $\mathcal{F}$ be a presheaf of sets on a topological space $X$. There exists a sheaf $\mathcal{F}^{+}$on $X$ and a morphism of presheaves $\theta: \mathcal{F} \rightarrow \mathcal{F}^{+}$such that

1. For every $x \in X$, the induced map on stalks $\theta_{x}: \mathcal{F}_{x} \rightarrow \mathcal{F}_{x}^{+}$is a bijection.
2. If $\mathcal{G}$ is a sheaf and $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, then $\phi$ factors uniquely through $\mathcal{F}^{+}$. Explicitly, there exists a unique morphism of sheaves $\psi: \mathcal{F}^{+} \rightarrow \mathcal{G}$ making the following diagram commute.


Before the proof, we make a few observations/remarks/interpretations.

1. The condition that $\theta_{x}$ be a bijection says that $\mathcal{F}^{+}$is "as close as possible" to $\mathcal{F}$.
2. The universal property (condition 2 ) implies that $\mathcal{F}^{+}$is unique up to isomorphism.
3. Either of the two conditions implies that $\theta$ is an isomorphism if and only if $\mathcal{F}$ is already a sheaf.

Proof. Let $\pi: E \rightarrow X$ be the étale space of $\mathcal{F}$. We will define $\mathcal{F}^{+}$to be the sheaf of continuous sections of $\pi$.

$$
\mathcal{F}^{+}(U)=\Gamma(U, \pi)=\left\{s: U \rightarrow E \mid \pi s=\operatorname{Id}_{U}, s \text { is continuous }\right\}
$$

We know from a general example that this gives a sheaf, so we already know $\mathcal{F}^{+}$is a sheaf. We will define the morphism $\theta$ by

$$
\theta_{U}: \mathcal{F}(U) \rightarrow \mathcal{F}^{+}(U) \quad s \mapsto \widetilde{s}
$$

recalling that $\widetilde{s}: U \rightarrow E$ is defined by $x \mapsto \rho_{x}^{U}(s)$. There are several things to show:

1. $\theta$ defined by the maps $\theta_{U}$ on sections gives a morphism of presheaves.
2. $\theta_{x}$ gives a bijection on stalks.
3. $\mathcal{F}^{+}, \theta$ satisfy the universal property.

First, we verify that $\theta$ is a morphism of sheaves, which requires commutativity of the following diagram for $V \subset U \subset X$ open sets. The map $\tilde{\rho}_{V}^{U}$ is the restriction map associated with the sheaf $\mathcal{F}^{+}$, which is just literal function restriction, since $\mathcal{F}^{+}$is a sheaf of continuous functions.


Let $s \in \mathcal{F}(U)$. The image (either way around the square) in $\mathcal{F}^{+}(V)$ is a continuous section $V \rightarrow E$, so to check that the images are equal we evaluate them on an arbitrary element $x \in V$.

$$
\begin{aligned}
& \widetilde{\rho}_{V}^{U} \theta_{U}(s)(x)=\widetilde{\rho}_{V}^{U}(\widetilde{s})(x)=\left.\widetilde{s}\right|_{V}(x)=\widetilde{s}(x)=\rho_{x}^{U}(s) \\
& \theta_{V} \rho_{V}^{U}(s)(x)=\rho_{x}^{V} \rho_{V}^{U}(s)=\rho_{x}^{U}(s)
\end{aligned}
$$

These are equal, so the diagram commutes, so $\theta$ is a morphism of presheaves. Now we verify that $\theta_{x}$ is a bijection on stalks. Previously, we showed that we have a bijection

$$
\mathcal{F}_{x}^{+} \cong \pi^{-1}(x)=\mathcal{F}_{x}
$$

which is induced by

$$
\sigma_{x}^{U}: \mathcal{F}^{+}(U) \rightarrow \mathcal{F}_{x} \quad t \mapsto t(x)
$$

and then passing to the direct limit. We claim that this bijection is in fact induced by $\theta$, which follows if we can show commutativity of the following diagram.


This is commutative because for $s \in \mathcal{F}(U)$, we have

$$
\sigma_{x}^{U} \theta_{U}(s)=\sigma_{x}^{U}(\widetilde{s})=\widetilde{s}(x)=\rho_{x}^{U}(s)=\theta_{x} \rho_{x}^{U}(s)
$$

Hence $\theta_{x}$ is a bijection on stalks. Finally, we need to prove the universal property, which is essentially a consequence of Lemma 5.10. Let $\mathcal{G}$ be a sheaf, and $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. Using the lemma, $\phi$ induces a continuous map $\widetilde{\phi}$ on the étale spaces.


In particular, for $U \subset X$ open, $\phi_{U}^{+}$is given by

$$
\phi_{U}^{+}: \Gamma\left(U, \pi_{\mathcal{F}}\right)=\mathcal{F}^{+}(U) \rightarrow \Gamma\left(U, \pi_{\mathcal{G}}\right)=\mathcal{G}^{+}(U) \quad \widetilde{s} \mapsto \widetilde{\phi} \widetilde{s}
$$

We then have the following commutative diagram, where $\theta_{\mathcal{G}}$ is the analogous morphism of presheaves defined as we defined $\theta=\theta_{\mathcal{F}}$ for $\mathcal{F}$.


This commutes because for $s \in \mathcal{F}(U)$ and $x \in U$, we have

$$
\begin{aligned}
& \theta_{U}^{+} \theta_{\mathcal{F}}(s)(x)=\theta_{U}^{+}(\widetilde{s})(x)=\widetilde{\phi} \widetilde{s}(x)=\widetilde{\phi} \rho_{x}^{U}(\mathcal{F})(s)=\phi_{x} \rho_{x}^{U}(\mathcal{F})(s) \\
& \theta_{\mathcal{G}} \phi_{U}(s)(x)=\widetilde{\phi_{U}(s)}(x)=\rho_{x}^{U}(\mathcal{G}) \phi_{U}(s)=\phi_{x} \rho_{x}^{U}(\mathcal{F})(s)
\end{aligned}
$$

Because the above square commutes, the corresponding square of morphisms of presheaves commutes (depicted below).


Since $\mathcal{G}$ is a sheaf and $\theta_{\mathcal{G}}$ is a morphism of sheaves which induces bijections on all stalks (by previous parts of this proof), $\theta_{G}$ is an isomorphism of sheaves. Hence $\theta_{\mathcal{G}}^{-1} \phi^{+}: \mathcal{F}^{+} \rightarrow \mathcal{G}$ is the required morphism making the triangle commute.


Finally, we need to check that this morphism is unique. Suppose $\psi, \psi^{\prime}: \mathcal{F}^{+} \rightarrow \mathcal{G}$ are two morphisms of sheaves which make the diagram commute. Then they also make the induced diagram on stalks commutes for each $x \in X$.


Since $\theta_{x}$ is a bijection, $\psi_{x}$ and $\psi_{x}^{\prime}$ are both determined by $\phi_{x}$, that is, $\psi_{x}=\psi_{x}^{\prime}=\phi_{x} \theta_{x}^{-1}$. Since $\psi, \psi^{\prime}$ are morphisms of sheaves which induce all the same morphisms on stalks, they must be the same. This completes the proof of uniqueness, which finishes proving the universal property.

Remark 5.13. It is possible, in priniciple, to describe the sheafification of a presheaf without explicit use of the étale space, although the construction then appears very strange and unmotivated on a first reading. For such a description, see Hartshorne's book on algebraic geometry.

Remark 5.14. One of the more useful concrete details to come out of the previous proof is that we do not merely have an existence statement about sheafification, but we have a concrete construction in terms of the étale space. A further consequence of this the following: given any sheaf $\mathcal{F}$, by uniqueness $\mathcal{F}$ is isomorphic to its own sheafification, so $\mathcal{F}$ is isomorphic to the sheaf of continuous sections of its étale space. That is to say, we can think of any sheaf $\mathcal{F}$ on a space $X$ as a sheaf of continuous sections of some local homeomorphism $\pi: E \rightarrow X$.

Example 5.15 (Sheafification of the constant presheaf). Let $X$ be a space and $S$ be a set. Let $\mathcal{F}$ be the constant sheaf on $X$ with values in $S$, so

$$
\mathcal{F}(U)= \begin{cases}\text { constant functions } U \rightarrow S & U \neq \emptyset \\ \{*\} & U=\emptyset\end{cases}
$$

When $U$ is not empty, we have an obvious identification of the constant functions $U \rightarrow S$ with elements of $S$, though we prefer to think of them as constant functions for this example. The restriction maps are literal function restriction. Recall that for $x \in X$, the stalk $\mathcal{F}_{x}$ is in bijection with $S$, via

$$
\mathcal{F}_{x} \rightarrow S \quad[f] \mapsto f(x)
$$

where $[f]$ is the class in the stalk of a (constant) function $f: U \rightarrow S$. The étale space of $\mathcal{F}$ is

$$
E=\bigsqcup_{x \in X} \mathcal{F}_{x}
$$

which we identify with $X \times S$ via

$$
\bigsqcup_{x \in X} \mathcal{F}_{x} \rightarrow X \times S \quad s \in \mathcal{F}_{x} \mapsto(x, s)
$$

How can we describe the topology on the étale space then? We claim that it corresponds to the product topology on $X \times S$, after giving $S$ the discrete topology. Recall that a basic open set for the topology on $E$. is one of the form $\widetilde{s}(U)$ for some $U \subset X$ and $s \in \mathcal{F}(U)$.

$$
\widetilde{s}(U)=\left\{\rho_{x}^{U}(s)=s: x \in U\right\}
$$

Under the correspondence between $E$ and $X \times S$ above, this corresponds to

$$
\{(x, s) \in X \times S: x \in U\}=U \times\{s\}
$$

That is, the basis of $\widetilde{s}(U)$ for the topology on $E$ corresponds to the basis for the topology on $X \times S$ given by sets $U \times\{s\}$ where $s \in S$, which is precisely a basis for the product topology. So we henceforth identify the étale space with $X \times S$, and note that the associated map of the étale space is just the projection

$$
\pi: X \times S \rightarrow X \quad(x, s) \mapsto x
$$

Now how can we describe the sheafification $\mathcal{F}^{+}$? Concretely, we described the sections as continuous sections of $\pi$.

$$
\mathcal{F}^{+}(U)=\Gamma(U, \pi)=\left\{\sigma: U \rightarrow X \times S \mid \sigma \text { is continuous, and } \pi \sigma=\operatorname{Id}_{U}\right\}
$$

The second condition says that $\sigma$ has the form

$$
\sigma: U \rightarrow X \times S \quad \sigma(x)=(x, f(x))
$$

where $f: U \rightarrow S$ is some function. The fact that $\sigma$ needs to be continuous places some restrictions on $f$, which we now describe. For $\sigma$ to be continuous, it is equivalent that the preimage of a basic open subset $V \times\{s\}$ is open in $X$, where $V \subset U$ is open in $X$.

$$
\begin{aligned}
\sigma^{-1}(V \times\{s\}) & =\{x \in X: \sigma(x)=(x, f(x)) \in V \times\{s\}\} \\
& =\{x \in X: x \in U \cap V, f(x)=s\} \\
& =\{x \in U \cap V: f(x)=s\}
\end{aligned}
$$

Let $W=\sigma^{-1}(V \times\{s\})$. By the above, $W$ is open in $X$ if and only if for every $x \in W$, there is an open neighborhood of $x$ such that $f$ is constant (with value $s$ ) on that neighborhood. That is, $W$ is open if and only if $f$ is locally constant. Thus $\sigma$ is continuous if and only if $f$ is locally constant. To sum up the previous discussion, there is a bijection

$$
\begin{aligned}
\Gamma(U, \pi) & \rightarrow\{\text { locally constant functions } U \rightarrow S\} \\
\sigma=(x \mapsto(x, f(x))) & \mapsto f: U \rightarrow S
\end{aligned}
$$

Thus the sheafification $\mathcal{F}^{+}$is the sheaf of locally constant functions on $X$ with values in $S$.
Corollary 5.16. Let $\mathcal{F}, \mathcal{G}$ be sheaves (of abelian groups) on $X$. Let $\pi_{\mathcal{F}}: E_{\mathcal{F}} \rightarrow X, \pi_{\mathcal{G}}$ : $E_{\mathcal{G}} \rightarrow X$ be the respective étale spaces. Suppose we have a homeomorphism $\widetilde{\phi}: E_{\mathcal{F}} \rightarrow E_{\mathcal{G}}$ over $X$.


Then $\mathcal{F}, \mathcal{G}$ are isomorphic as sheaves.
Proof. First note that the fact that $\widetilde{\phi}$ is map over $X$ means that we have maps

$$
\phi_{x}=\left.\widetilde{\phi}\right|_{\mathcal{F}_{x}}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{X}
$$

for every $x \in X$. Since $\widetilde{\phi}$ is a homeomorphism, each $\phi_{x}$ is an isomorphism. So $\mathcal{F}, \mathcal{G}$ have isomorphic stalks everywhere. As we have seen in an example, this is not sufficient to get an
isomorphism, but it does say that as long as we have a morphism of sheaves inducing these maps on stalks, that will be an isomorphism (of sheaves).

Following the proof of the sheafification result, since $\mathcal{F}, \mathcal{G}$ are already sheaves, we have isomorphisms of sheaves

$$
\mathcal{F} \cong \Gamma\left(-, \pi_{\mathcal{F}}\right) \quad \mathcal{G} \cong \Gamma\left(-, \pi_{\mathcal{G}}\right)
$$

where $\Gamma\left(U, \pi_{\mathcal{F}}\right)$ is the group of continuous sections of $\pi_{\mathcal{F}}$ on $U$. That is, we may identify $\mathcal{F}(U)$ with $\Gamma\left(U, \pi_{\mathcal{F}}\right)$ and likewise for $\mathcal{G}$. Next, we will define a morphism of sheaves $\Gamma\left(-, \pi_{\mathcal{F}}\right) \rightarrow$ $\Gamma\left(-, \pi_{\mathcal{G}}\right)$. For $U \subset X$, define

$$
\psi_{U}: \Gamma\left(U, \pi_{\mathcal{F}}\right) \rightarrow \Gamma\left(U, \pi_{\mathcal{G}}\right) \quad s \mapsto \phi_{U}(s)=\widetilde{\phi} \circ s
$$

It is clear that $\tilde{\phi} \circ s$ maps from $U$ to $E_{\mathcal{G}}$ and is continuous, we just need to verify that it is a section of $\pi_{\mathcal{G}}$. It is basically immediate from the fact that $\widetilde{\phi}$ is a map over $X$.

$$
\pi_{\mathcal{G}} \circ \psi_{U}(s)=\pi_{\mathcal{G}} \circ \tilde{\phi} \circ s=\pi_{\mathcal{F}} \circ s=\operatorname{Id}_{U}
$$

After some verification that these $\psi_{U}$ are compatible with restriction maps, we see that these $\psi_{U}$ give a morphism of sheaves $\psi: \mathcal{F}=\Gamma\left(-, \pi_{\mathcal{F}}\right) \rightarrow \mathcal{G}=\Gamma\left(-, \pi_{\mathcal{G}}\right)$. Now we claim tha the induced map on stalks $\psi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ is exactly $\phi_{x}$, which will complete the proof.

Given $e \in \mathcal{F}_{x}$, to compute $\psi_{x}(e)$, we take a representing section $s \in \Gamma\left(U, \pi_{\mathcal{F}}\right)$ where $U$ is some neighborhood of $x$, then restrict the image of $\psi_{U}(s)$ to the stalk $\mathcal{G}_{x}$. Since $\Gamma\left(-, \pi_{\mathcal{F}}\right)$ is a sheaf of continuous functions, choosing such $s$ just means choosing $s$ such that $s(x)=e$, and restricting to the image in the stalk just means evaluating a function at the point $x$. Hence

$$
\psi_{x}(e)=\left.\psi_{U}(s)\right|_{x}=\left.(\widetilde{\phi} \circ s)\right|_{x}=\widetilde{\phi}(s(x))=\widetilde{\phi}(e)=\phi_{x}(e)
$$

Thus the morphism of sheaves $\psi$ induces isomrphisms $\phi_{x}$ on all stalks, so it is an isomorphism.

Remark 5.17. In the language of Lemma 5.10 and the proceeding remark, the previous corollary says that the functor

$$
E: \operatorname{PSh}(x) \rightarrow\{\text { étale spaces over } X\} \quad \mathcal{F} \mapsto\left(E_{\mathcal{F}}, \pi_{\mathcal{F}}\right)
$$

has the following property: the restriction of $E$ to the full subcategory of sheaves is functor which is "injective on objects." That is, consider

$$
E: \operatorname{Sh}(X) \rightarrow\{\text { étale spaces over } X\} \quad \mathcal{F} \mapsto\left(E_{\mathcal{F}}, \pi_{\mathcal{F}}\right)
$$

If $E_{\mathcal{F}} \cong E_{\mathcal{G}}$ (over $X$, then $\mathcal{F} \cong \mathcal{G}$.
Next, we want to describe how to extend the main sheafification result to the case where $\mathcal{F}$ is a presheaf of abelian groups. First, we introduce some terminology. This is not exactly standard, but we will only use it for a short while anyway.

Definition 5.18. A local homeomorphism $\pi: E \rightarrow X$ is an étale space of abelian groups if

1. Each fiber $\pi^{-1}(x)$ is an abelian group.
2. The maps

$$
\begin{array}{rlrl}
E \times_{X} E & \rightarrow E & \left(e_{1}, e_{2}\right) & \mapsto e_{1}+e_{2} \\
E & \rightarrow E & e & \mapsto-e
\end{array}
$$

are continuous.
Remark 5.19. Let $\mathcal{F}$ be a presheaf of abelian groups. Recall that the stalks $\mathcal{F}_{x}$ are then abelian groups. Let

$$
E=\bigsqcup_{x \in X} \mathcal{F}_{x}
$$

be the (set-theoretic) étale space of $\mathcal{F}$, with the usual projection $\pi$. Then $\pi: E \rightarrow X$ is an étale space of abelian groups.

Remark 5.20. Given an étale space of abelian groups, the continuous sections $\Gamma(U, \pi)$ is a an abelian group by pointwise addition, so the sheaf $\mathcal{F}^{+}$they define is a sheaf of abelian groups. Hence the sheafification of a presheaf of abelian groups is a sheaf of abelian groups.

[^6]
## 6 Čech cohomology

The next major topic is Čech cohomology. This is the first step towards sheaf cohomology. Sheaf cohomology is probably easier to define, and more general, but lacks the computability of Čech cohomology, so we start there. Eventually, we will get to Leray's theorem, which establishes an isomorphism between Čech cohomology and sheaf cohomology, under some assumptions.

### 6.1 The Mittag-Leffler problem

In order to motivate the strange definition of Čech cohomology, we start by describing how it arises somewhat naturally from the Mittag-Leffler problem, which comes from complex analysis. Here is the setup for the problem:

Let $X$ be a Riemann surface (a one-dimensional complex manifold), and let $E \subset X$ be a closed, discrete set. For $a \in E$, let $U_{a}$ be a neighborhood of $a$ which is homeomorphic to $\mathbb{C}$, and let

$$
z_{a}: U_{a} \rightarrow \mathbb{C}
$$

be a homeomorphism (sometimes $z_{a}$ is called a "local coordinate" at a), and normalize $z_{a}$ so that $z_{a}(a)=0$. Let

$$
p_{a}(z)=c_{m} z_{a}^{-m}+c_{m-1} z_{a}^{-m+1}+\cdots+c_{2} z_{a}^{-2}+c_{1} z_{a}^{-1}
$$

be a polynomial in $z_{a}^{-1}$ with no constant term. We think of $p_{a}$ as the "principal part" or "Laurent part" of $z_{a}$. Then the question is as follows.

Mittag-Leffler problem: Under the setup above, does there exist a meromorphic function $f: X \rightarrow \mathbb{C}$ which is holomorphic outside of $E$, and such that for each $a \in E$, the function

$$
f\left(z_{a}\right)-p_{a}\left(z_{a}\right)
$$

(on $U_{a}$ ) has a removable singularity at $a$ ? Roughly speaking, is it possible to find a global meromorphic function with specificed poles (the set $E$ ) and specified principal parts (the polynomials $p_{a}$ ) at each pole?

We will see in a moment how to translate this question into a "cohomological condition," which leads to the definition of Čech cohomology. Before going on, we should mention that Mittag-Leffler himself proved that the question above has the answer "yes" in the case where $X$ is an open subset of $\mathbb{C}$, and later other people extended this work to an arbitrary noncompact Riemann surface $X$.

Now we transition to a general discussion of the Mittag-Leffler problem, retaining all of the setup above.

Choose an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$ such that for each $i, U_{i} \cap E$ is either empty or a single point $a_{i}$. Then choose functions $f_{i}: U_{i} \rightarrow \mathbb{C}$ such that $f_{i}$ is either holomorphic, or
has a single pole $a_{i} \in U_{i} \cap E$. Note that by construction, $a_{i} \notin U_{j}$ for $i \neq j$. In particular, no intersection $U_{i} \cap U_{j}$ contains any pole for any $f_{i}$. That is, we are choose $f_{i}$ to have the specified principal parts near $a_{i}$.

Now, we want to "glue" the functions $f_{i}$ to obtain a meromorphic function $f$ on $X$. If we knew that for all $i, j$,

$$
\left.\left.f_{i}\right|_{U_{i} \cap U_{j}} \stackrel{?}{=} f_{j}\right|_{U_{i} \cap U_{j}}
$$

then we would be able to glue to obtain $f$, but the above may fail. However, under certain circumstances, it may be possible to modify each $f_{i}$ by a holomorphic function in order to get this compatibility. That is, we seek holomorphic functions $h_{i}: U_{i} \rightarrow \mathbb{C}$ such that

$$
\left.\left(f_{i}+h_{i}\right)\right|_{U_{i} \cap U_{j}}=\left.\left(f_{j}+h_{j}\right)\right|_{U_{i} \cap U_{j}}
$$

Since $h_{i}$ is holomorphic, $f_{i}$ and $f_{i}+h_{i}$ have the same principal part, so if we glue the $f_{i}+h_{i}$ to obtain a global meromorphic function, it will have the right poles and principal parts. For $i, j \in I$, set $t_{i j}=f_{i}-f_{j}$, and rewrite the previous equation as

$$
\left.t_{i j}\right|_{U_{i} \cap U_{j}}=\left.\left(f_{i}-f_{j}\right)\right|_{U_{i} \cap U_{j}}=\left.\left(h_{j}-h_{i}\right)\right|_{U_{i} \cap U_{j}}
$$

Assuming the $h_{i}$ exist, since $t_{i j}=h_{j}-h_{i}$ on $U_{i} \cap U_{j}, t_{i j}$ is holomorphic on that intersection. It is clear also that

$$
t_{i j}-t_{i k}+t_{j k}=0
$$

on the triple intersection $U_{i} \cap U_{j} \cap U_{k}$. The equation above is hopefully reminiscent of previous encounters with homology and cohomology. It is called a "cocycle condition." This is what we meant when we said we would translate the Mittag-Leffler problem into a cohomological condition. We have now reduced the Mittag-Leffler problem to the following:

Cohomological Mittag-Leffler problem: Given meromorphic functions $t_{i j}: U_{i} \cap U_{j} \rightarrow \mathbb{C}$ as above, which satisfy the cocycle condition

$$
t_{i j}-t_{i k}+t_{j k}=0
$$

when do there exist holomorphic functions $h_{i}: U_{i} \rightarrow \mathbb{C}$ such that $t_{i j}=h_{j}-h_{i}$ on $U_{i} \cap U_{j}$ ? This leads into the definition of Cohomology. We will next define Cech cohomology (just the first cohomology group), and then return to this interpretation of the Mittag-Leffler problem.

### 6.2 Defining Čech cohomology

Definition 6.1. Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space $X$. Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $X$. A Čech 1-cocycle with respect to $\mathcal{U}$ is a collection of elements $t_{i j} \in \mathcal{F}\left(U_{i} \cap U_{j}\right)$ for all $i, j \in I$ such that

$$
t_{i j}-t_{i k}+t_{j k}=0 \quad \text { on } U_{i} \cap U_{j} \cap U_{k}
$$

To make the previous relation more precise, it should be written as

$$
\rho_{U_{i} \cap U_{j} \cap U_{k}}^{U_{i} \cap U_{j}}\left(t_{i j}\right)-\rho_{U_{i} \cap U_{j} \cap U_{k}}^{U_{i} \cap U_{k}}\left(t_{i k}\right)+\rho_{U_{i} \cap U_{j} \cap U_{k}}^{U_{j} \cap U_{k}}\left(t_{j k}\right)=0 \quad \text { in } \mathcal{F}\left(U_{i} \cap U_{j} \cap U_{k}\right)
$$

We think of a Čech 1-cocycle $\left\{t_{i j}\right\}$ as living in the direct product

$$
\prod_{i, j \in I} \mathcal{F}\left(U_{i} \cap U_{j}\right)
$$

The set of Čech 1-cocycles forms a subgroup of this direct product. This subgroup is denoted

$$
\check{Z}^{1}(\mathcal{U}, \mathcal{F})
$$

Definition 6.2. Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space $X$. Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $X$. A Čech 1-coboundary with respect to $\mathcal{U}$ is a collection of elements $t_{i j} \in \mathcal{F}\left(U_{i} \cap U_{j}\right)$ such that there exists a collection $h_{i} \in \mathcal{F}\left(U_{i}\right)$ with $t_{i j}=h_{i}-h_{j}$ on $U_{i} \cap U_{j}$. The group of coboundaries is denoted

$$
\check{B}^{1}(\mathcal{U}, \mathcal{F})
$$

Remark 6.3. It is relatively easy to show that

$$
\check{B}^{1}(\mathcal{U}, \mathcal{F}) \subset \check{Z}^{1}(\mathcal{U}, \mathcal{F})
$$

To verify it directly, suppose $t_{i j}=h_{i}-t_{j}$. Then

$$
t_{i j}-t_{i k}+t_{j k}=h_{i}-h_{j}-\left(h_{i}-h_{k}\right)+h_{j}-h_{k}=0
$$

Definition 6.4. Let $\mathcal{F}, X, \mathcal{U}$ be as above. The first Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ is the quotient

$$
\check{H}^{1}(\mathcal{U}, \mathcal{F})=\frac{\check{Z}^{1}(\mathcal{U}, \mathcal{F})}{\check{B}^{1}(\mathcal{U}, \mathcal{F})}
$$

Remark 6.5. Note that all of the above definitions are with reference to a fixed (but arbitrary) open cover of $X$. It is quite undesirable to have such a dependence, since a cohomology theory which gives different computations for different open covers would not be a very good cohomology theory. We will address this issue a little later.

At this point, we can return to the Mittag-Leffler problem and give an equivalent formulation as the vanishing of $\check{H}^{1}(\mathcal{U}, \mathcal{F})$ in a particular circumstance. To be a bit more specific, if $\check{H}^{1}(\mathcal{U}, \mathcal{F})=0$ under the right setup, then the Mittag-Leffler problem has a positive solution.

Proposition 6.6. Let $X$ be a Riemann surface, with $E \subset X$ closed and discrete, and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ an open cover of $X$. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $X$. Suppose that for each $i \in I$, we have a function $f_{i}: U_{i} \rightarrow \mathbb{C}$ such that

1. Each $f_{i}$ is either holomorphic, or has a single pole at $a_{i} \in U_{i} \cap E$.
2. For each $i, j \in I, f_{i}-f_{j}$ is holomorphic on $U_{i} \cap U_{j}$.

Assume that $\check{H}^{1}(\mathcal{U}, \mathcal{O})=0$. Then there exists a global meromorphic function $f$ on $X$ such that $\left.\left(f-f_{i}\right)\right|_{U_{i}}$ is holomorphic for all $i$.

Proof. Given $f_{i}$ as in the statement, let $t_{i j}=f_{i}-f_{j}$. Then $\left\{t_{i j}\right\}$ is an element of $\check{Z}^{1}(\mathcal{U}, \mathcal{O})$, by a simple calculation as in Remark 6.3. ${ }^{9}$ Since $\check{H}^{1}(\mathcal{U}, \mathcal{O})=0$ by assumption, $\check{B}^{1}(\mathcal{U}, \mathcal{O})=$ $\check{Z}^{1}(\mathcal{U}, \mathcal{O})$, hence $\left\{t_{i j}\right\} \in \check{B}^{1}(\mathcal{U}, \mathcal{O})$, That is, there exist holomorphic functions $h_{i} \in \mathcal{O}\left(U_{i}\right)$ for all $i$ such that

$$
t_{i j}=h_{i}-h_{j} \quad \text { on } U_{i} \cap U_{j}
$$

Rearranging this,

$$
f_{i}-h_{i}=f_{j}-h_{j} \quad \text { on } U_{i} \cap U_{j}
$$

Then we may glue the functions $f_{i}-h_{i}$ to obtain a global meromorphic function $f$ on $X$ such that $\left.f\right|_{U_{i}}=f_{i}-h_{i}$, which is to say, $\left.\left(f-f_{i}\right)\right|_{U_{i}}$ is holomorphic.

### 6.3 Resolving dependence on the cover

Now we attempt to deal with the dependency on the choice of open cover in defining Čech cohomology.

Definition 6.7. Let $X$ be a topological space, and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ two open covers of $X$. A refinement map is a map $\tau: J \rightarrow I$ such that

$$
V_{j} \subset U_{\tau(j)}
$$

for all $j \in J$. The cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ is there is a refinement map $\tau: J \rightarrow I$.
Definition 6.8. Let $X$ be a topological space, and $\mathcal{F}$ a presheaf of abelian groups on $X$. Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}, \mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ be open covers of $X$, and let $\tau: J \rightarrow I$ be a refinement map. Then define

$$
\tau^{1}: \check{Z}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{Z}^{1}(\mathcal{V}, \mathcal{F}) \quad\left\{g_{i_{1} i_{2}}\right\} \mapsto\left\{g_{j_{1} j_{2}}\right\}=\left\{\left.g_{\tau\left(j_{1}\right) \tau\left(j_{2}\right)}\right|_{V_{j_{1}} \cap V_{j_{2}}}\right\}
$$

It is not that hard to verify that $\tau^{1}$ does in fact map into $\check{Z}^{1}(\mathcal{V}, \mathcal{F})$, and it is also not too hard to check that $\tau^{1}$ maps $\check{B}^{1}(\mathcal{U}, \mathcal{F})$ to $\check{B}^{1}(\mathcal{V}, \mathcal{F})$. Hence $\tau^{1}$ induces a map

$$
\tau_{\mathcal{U}}^{\mathcal{V}}: \check{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{1}(\mathcal{V}, \mathcal{F})
$$

Lemma 6.9. Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Then the homomorphism $\tau_{\mathcal{U}}^{\mathcal{V}}$ does not depend on the choice of refinement map $\tau$.

Proof. Not very interesting, but technical.

[^7]Definition 6.10. We give a preorder on the collection of open covers of a topological space $X$ by refinement. That is, $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. If $\mathcal{U} \leq \mathcal{V}$ and $\mathcal{V} \leq \mathcal{U}$, we say they are equivalent.

Corollary 6.11. If $\mathcal{U}, \mathcal{V}$ are equivalent covers on $X$, then the map

$$
\tau_{\mathcal{U}}^{\mathcal{V}}: \check{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{1}(\mathcal{V}, \mathcal{F})
$$

is an isomorphism.
Proof. Since $\mathcal{U}, \mathcal{V}$ are equivalent, there are refinements both ways. Composing them, we obtain a map $\grave{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{1}(\mathcal{U}, \mathcal{F})$. By the previous lemma, these maps are independent of the choice of refinement map, so they are the same as if we had chosen the "identity refinement map" for $\mathcal{U}$, which clearly induces the identity on $\check{H}^{1}(\mathcal{U}, \mathcal{F})$. Similarly, the composition the other direction induces the identity on $\check{H}^{1}(\mathcal{V}, \mathcal{F})$.

Definition 6.12. Let $X$ be a topological space and $\mathcal{F}$ a presheaf of abelian groups on $X$. The collection of open covers of $X$ with refinement maps forms a direct system, which gives rise to a direct system of the groups $\check{H}^{1}(\mathcal{U}, \mathcal{F})$, with induced maps $\tau_{\mathcal{U}}^{\mathcal{U}}$. We define

$$
\check{H}^{1}(X, \mathcal{F})=\underset{\longrightarrow}{\lim } \check{H}^{1}(\mathcal{U}, \mathcal{F})
$$

to be the direct limit of this system.
In order to complete our discussion of the Mittag-Leffler problem, we include the following proposition without proof.

Proposition 6.13. Let $\mathcal{F}$ be a sheaf. Then the canonical map

$$
\check{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{1}(X, \mathcal{F})
$$

is injective for any open cover $\mathcal{U}$.
This is not important at the moment, but note that the above proposition does NOT generalize to $\check{H}^{n}$. More importantly, the significant for the Mittag-Leffler problem is that if $\check{H}^{1}(X, \mathcal{O})$ vanishes for a Riemann surface $X$, then the Mittag-Leffler problem always has a solution.

Theorem 6.14. Let $X$ be a noncompact Riemann surface. Then $\check{H}^{1}(X, \mathcal{O})=0$.
Combining these two results, we get:
Corollary 6.15. Let $X$ be a noncompact Riemann surface. Then the Mittag-Leffler problem always has a positive solution, for any open cover $\mathcal{U}$ of $X$.

### 6.4 Classification of vector bundles via Čech cohomology

In this section, we give another motivation for Čech cohomology, by giving an interpretation of $\check{H}^{1}(\mathcal{U}, \mathcal{F})$ in terms of vector bundles.

Definition 6.16. Let $X$ be a smooth real manifold. A vector bundle $E$ of rank $k$ on $X$ is a smooth real manifold $E$ together with a smooth, surjective map $\pi: E \rightarrow X$ with the following properties.

1. For each $x \in X$, the fiber $\pi^{-1}(x)$ is a $k$-dimensional real vector space.
2. For each $x \in X$, there is an open neighborhood $U_{x}$ of $x$ and a diffeomorphism

$$
\Phi_{x}: \pi^{-1}\left(U_{x}\right) \rightarrow U_{x} \times \mathbb{R}^{k}
$$

making the following diagram commute

where $p$ is projection onto $U_{x}$, and additionally $\Phi_{x}$ has the property that for each $y \in U_{x}$, the induced map

$$
\left.\Phi_{x}\right|_{\pi^{-1}(y)}: \pi^{-1}(y) \rightarrow\{y\} \times \mathbb{R}^{k} \cong \mathbb{R}^{k}
$$

is $\mathbb{R}$-linear. The map $\Phi_{x}$ is called a local trivialization.
Definition 6.17. The trivial vector bundle of rank $k$ on $X$ is the product $E=X \times \mathbb{R}^{k}$ together with the projection

$$
\pi: X \times \mathbb{R}^{k} \rightarrow X \quad(x, v) \mapsto x
$$

It is clear that each fiber is an $\mathbb{R}$-vector space, and for any $U \subset X$, there is a local trivialization

$$
\Phi_{U}: U \times \mathbb{R}^{k} \rightarrow X \quad(x, v) \mapsto x
$$

which obviously has the needed properties.
Remark 6.18. Using the language of the trivial vector bundle, we can rephrase the definition of a general vector bundle of rank $k$ as a map $\pi: E \rightarrow X$ which is locally the trivial vector bundle. That is, there is an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ such that for all $i \in I$, there is a diffeomorphism between $\pi^{-1}\left(U_{i}\right)$ and $U_{i} \times \mathbb{R}^{k}$ over $X$.


Also, $\Phi_{i}$ must have the property that it restricts to an $\mathbb{R}$-linear map on each fiber. As before, the map $\Phi_{i}$ is called the local trivialization of $E$ over $U_{i}$.

Now we begin our attempt to connect vector bundles to Čech cohomology. Let $\pi: E \rightarrow X$ be a vector bundle of rank $k$ on a smooth real manifold $X$. Fix an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$, and fix local trivializations $\Phi_{i}$. For any $i, j \in I$, we can consider the composition

$$
\Phi_{i} \circ \Phi_{j}^{-1}:\left(U_{i} \cap U_{j}\right) \times \mathbb{R}^{k} \rightarrow\left(U_{i} \cap U_{j}\right) \times \mathbb{R}^{k}
$$

which is a diffeomorphism, and gives an $\mathbb{R}$-linear map when restricted to fibers. Note that since $\Phi_{j}, \Phi_{i}$ are boths maps over $X$, we can write $\Phi_{i} \circ \Phi_{j}^{-1}$ as

$$
(x, v) \mapsto\left(x, \phi_{i j}^{x}(v)\right)
$$

for some function

$$
\phi_{i j}^{x}: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}
$$

Since $\Phi_{i} \circ \Phi_{j}^{-1}$ restricts to an $\mathbb{R}$-linear map on fibers, the maps $\phi_{i j}^{x}$ are $\mathbb{R}$-linear. They are also clearly invertible since $\Phi \circ \Phi_{j}^{-1}$ is, so we may view $\phi_{i j}^{x}$ as an element of $\mathrm{GL}_{k}(\mathbb{R})$.

We want to understand what sort of object $\phi_{i j}$ is, as a function of $x$. It takes in $x \in U_{i} \cap U_{j}$, and then outputs $\phi_{i j}^{x} \in \mathrm{GL}_{k}(\mathbb{R})$. We may think of $\phi_{i j}$ as a matrix whose entries are functions of $x$, and evaluating at $x \in U_{i} \cap U_{j}$ gives a particular $\phi_{i j}^{x} \in \mathrm{GL}_{k}(\mathbb{R})$. That is,

$$
\phi_{i j}=\left(a_{i j}(x)\right)
$$

where each $a_{i j}$ is a function $U_{i} \cap U_{j} \rightarrow \mathbb{R}$. We should expect the functions $a_{i j}$ to be smooth since we are working with smooth manifolds, but we won't go into exactly why they are smooth.

We can now phrase the above in terms of sheaves. Let $\mathcal{S}$ to be the sheaf of $\mathbb{R}$-valued smooth functions on $X$, that is, for $U \subset X, \mathcal{S}(U)$ is the set of smooth functions $U \rightarrow \mathbb{R}$. Then $a_{i j} \in \mathcal{S}\left(U_{i} \cap U_{j}\right)$, so we realize we should view $\phi_{i j}$ as an element of $\mathrm{GL}_{k}\left(\mathcal{S}\left(U_{i} \cap U_{j}\right)\right.$. This prompts us to consider the sheaf $\mathcal{F}$ on $X$, defined by

$$
\mathcal{F}(U)=\mathrm{GL}_{k}(\mathcal{S}(U))
$$

In these terms, $\phi_{i j} \in \mathcal{F}\left(U_{i} \cap U_{j}\right)$. Note that $\mathcal{F}(U)$ is a sheaf of nonabelian groups on $X$. This is how we will think of $\phi_{i j}$. Returning to local trivializations, it is a simple computation to realize that

$$
\left(\Phi_{j} \circ \Phi_{k}^{-1}\right) \circ\left(\Phi_{i} \circ \Phi_{k}^{-1}\right)^{-1} \circ\left(\Phi_{i} \circ \Phi_{j}^{-1}\right)=\mathrm{Id}
$$

on the triple intersection $U_{i} \cap U_{j} \cap U_{k}$ for any $i, j, k \in I$, which induces the relation

$$
\phi_{j k} \phi_{i k}^{-1} \phi_{i j}=1
$$

in $\mathcal{F}\left(U_{i} \cap U_{j} \cap U_{k}\right)$. After some thinking and translating between additive and multiplicative notation, one should hopefully recognize this as the Cech cocycle condition, which is to say,

$$
\left\{\phi_{i j}\right\} \in \check{Z}^{1}(\mathcal{U}, \mathcal{F})
$$

However, we have slightly overstepped our bounds, since our previous definition of Čech cocycles, coboundaries, and cohomology used, to a small extend, the fact that the sheaf $\mathcal{F}$ involved was a sheaf of abelian groups. Since we are now dealing with a sheaf of nonabelian groups, we need to be a little more careful.

It turns out that we can still talk about Čech cocycles and cohomology, the only subtletly is that the resulting cohomology set $\check{H}^{1}(\mathcal{U}, \mathcal{F})$ no longer is a group, but merely a pointed set. After all this motivation, we now give the formal definition of nonabelian Čech cocycles and cohomology.

Definition 6.19. Let $\mathcal{F}$ be a sheaf (or just a presheaf, no big deal) of (possibly nonabelian) groups on a space $X$. Fix an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$. A nonabelian Čech 1-cocycle with respect to $\mathcal{U}$ is a collection of sections

$$
\left\{\phi_{i j}\right\}
$$

with $\phi_{i j} \in \mathcal{F}\left(U_{i} \cap U_{j}\right)$ satisfying the cocycle condition

$$
\phi_{j k} \phi_{i k}^{-1} \phi_{i j}=1
$$

on the triple intersection $\mathcal{F}\left(U_{i} \cap U_{j} \cap U_{k}\right)$. The set of such Čech 1-cocycles is denoted

$$
\check{Z}^{1}(\mathcal{U}, \mathcal{F})
$$

Note that unlike in the case where $\mathcal{F}$ is a sheaf of abelian groups, $\check{Z}^{1}$ does not have a natural group structure. It is merely a set, with a special element, the trivial cocycle, with $\phi_{i j}=1$ for all $i, j$.

Example 6.20. Following all of the motivating discussion leading up to the definition, given a vector bundle $\pi: E \rightarrow X$ of rank $k$ on a smooth real manifold $X$, and a fixed open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and fixed trivializations $\Phi_{i}$, we obtain a nonabelian Čech 1-cocycle $\left\{\phi_{i j}\right\} \in \check{Z}^{1}(\mathcal{U}, \mathcal{F})$, where $\mathcal{F}$ is the sheaf

$$
\mathcal{F}(U)=\mathrm{GL}_{k}(\mathcal{S}(U))
$$

So we have an "assignment"
$\{$ vector bundles of rank $k$ on $X$ with fixed open cover $\mathcal{U}\} \rightarrow \check{Z}^{1}(\mathcal{U}, \mathcal{F})$
This is not really a function, since producing a cocycle from a vector bundle requires chooseing local trivializations $\Phi_{i}$. However, if we can resolve the dependence on choice of trivializations, and then also remove dependence on the cover $\mathcal{U}$, this gives us some hope that we might establish some sort of correspondence betwen vector bundles (perhaps up to isomorphism) on $X$ of rank $k$, and $\check{Z}^{1}(X, \mathcal{F})$, or perhaps $\check{H}^{1}(X, \mathcal{F})$.

Remark 6.21. First, let us address the issue of choice of trivializations. What happens to the resulting cocycle $\left\{\phi_{i j}\right\}$ when different local trivializations are chosen? Note that we keep the same open cover $\mathcal{U}=\left\{U_{i}\right\}$ of $X$. Suppose $\left\{\Phi_{i}\right\},\left\{\Phi_{i}^{\prime}\right\}$ are different choices of local trivializations for this cover. Then define $\phi_{i j}, \phi_{i j}^{\prime}$ via

$$
\begin{array}{rlrl}
\Phi_{i} \circ \Phi_{j}^{-1}:\left(U_{i} \cap U_{j}\right) \times \mathbb{R}^{k} & \rightarrow\left(U_{i} \cap U_{j}\right) & \rightarrow \mathbb{R}^{k} & \\
\Phi_{i}^{\prime} \circ(x, v) & \mapsto\left(x, \phi_{i j}^{x}(v)\right) \\
)^{-1}:\left(U_{i} \cap U_{j}\right) \times \mathbb{R}^{k} & \rightarrow\left(U_{i} \cap U_{j}\right) & \rightarrow \mathbb{R}^{k} & \\
(x, v) & \mapsto\left(x,\left(\phi_{i j}^{\prime}\right)^{x}(v)\right)
\end{array}
$$

For $i \in I$, define $\psi_{i}^{x} \in \mathrm{GL}_{k}(\mathbb{R})$ by

$$
\Phi_{i}^{\prime} \circ \Phi_{i}^{-1}: U_{i} \times \mathbb{R}^{k} \rightarrow U_{i} \times \mathbb{R}^{k} \quad(x, v) \mapsto\left(x, \psi_{i}^{x}(v)\right)
$$

As before, let $\mathcal{S}$ be the sheaf of smooth $\mathbb{R}$-valued functions on $X$. Similar to our discussion of how we may view $\phi_{i j}$ as element of $\mathrm{GL}_{k}\left(\mathcal{S}\left(U_{i} \cap U_{j}\right)\right)$, we view $\psi_{i}$ as an element of $\mathrm{GL}_{k}\left(\mathcal{S}\left(U_{i}\right)\right)$. Now we need to do a somewhat tedious calculation, in order to determine an algebraic relation between the $\psi$ and $\phi$ maps. The basic idea is that the $\psi$ maps should be a sort of "nonabelian Čech coboundary," whatever that means. First, note that

$$
\Phi_{i}^{\prime} \circ\left(\Phi_{j}^{\prime}\right)^{-1}=\left(\Phi_{i}^{\prime} \circ \Phi_{i}^{-1}\right) \circ\left(\Phi_{i} \circ \Phi_{j}^{-1}\right) \circ\left(\Phi_{j}^{\prime} \circ \Phi_{j}^{-1}\right)
$$

Thus, evaluating both sides at $(x, v) \in\left(U_{i} \cap U_{j}\right) \times \mathbb{R}^{k}$ gives the same result.

$$
\begin{aligned}
\left(x,\left(\phi_{i j}^{\prime}\right)^{x}(v)\right) & =\Phi_{i}^{\prime} \circ\left(\Phi_{j}^{\prime}\right)^{-1}(x, v) \\
& =\left(\Phi_{i}^{\prime} \circ \Phi_{i}^{-1}\right) \circ\left(\Phi_{i} \circ \Phi_{j}^{-1}\right) \circ\left(\Phi_{j}^{\prime} \circ \Phi_{j}^{-1}\right) \\
& =\left(x, \psi_{i}^{x} \phi_{i j}^{x}\left(\psi_{j}^{x}\right)^{-1}(v)\right)
\end{aligned}
$$

The resulting equality that we actually want out of all of this mess is that

$$
\left(\phi_{i j}^{\prime}\right)^{x}=\psi_{i}^{x} \phi_{i j}^{x}\left(\psi_{j}^{x}\right)^{-1} \quad \text { on } U_{i} \cap U_{j}
$$

which in turn implies the real "coboundary" condition that we were looking for:

$$
\phi_{i j}^{\prime}=\psi_{i} \phi_{i j} \psi_{j}^{-1} \quad \text { in } \operatorname{GL}_{k}\left(\mathcal{S}\left(U_{i} \cap U_{j}\right)\right)
$$

This motivates us to use this exact equation as our definition for equivalence of cocycles in nonabelian $\check{Z}^{1}(\mathcal{U}, \mathcal{F})$.

Definition 6.22. Let $\mathcal{F}$ be a sheaf of groups on $X$, and $\mathcal{U}=\left\{U_{i}\right\}$ an open cover of $X$. Two Cech 1-cocycles $\left\{\phi_{i j}\right\},\left\{\phi_{i j}^{\prime}\right\}$ are equivalent if there exists a collection of sections $\psi_{i} \in \mathcal{F}\left(U_{i}\right)$ such that

$$
\phi_{i j}^{\prime}=\psi_{i} \phi_{i j} \psi_{j}^{-1}
$$

It is not terribly difficult to verify that this is an equivalence relation.
Definition 6.23. Let $\mathcal{F}, X, \mathcal{U}$ be as above. The quotient of $\check{Z}^{1}(\mathcal{U}, \mathcal{F})$ by the above equivalence relation is denoted $\check{H}^{1}(\mathcal{U}, \mathcal{F})$.

Example 6.24. Let $X$ be a smooth real manifold, and $\mathcal{F}$ be the sheaf $\mathcal{F}(U)=\mathrm{GL}_{k}(\mathcal{S}(U))$, where $\mathcal{S}$ is the sheaf of smooth $\mathbb{R}$-valued functions. Fix an open cover $\mathcal{U}=\left\{U_{i}\right\}$ of $X$. Previously, we defined an "assignment"
$\{$ vector bundles of rank $k$ on $X$ with fixed open cover $\mathcal{U}\} \rightarrow \check{Z}^{1}(\mathcal{U}, \mathcal{F})$
which was not a well defined function, since different choices of local trivializations could give rise to different Cech 1-cocycles. However, basically by construction, two different sets of choices of local trivializations give two equivalent Čech 1-cocycles, so the above "assignment" gives rise to a well defined function
$\{$ vector bundles of rank $k$ on $X$ with fixed open cover $\mathcal{U}\} \rightarrow \check{H}^{1}(\mathcal{U}, \mathcal{F})$
We have now dealt with one ambiguity, the dependence on the choice of local trivializations, by taking an appropriate quotient of $\breve{Z}^{1}(\mathcal{U}, \mathcal{F})$. Now, as in the commutative case, we want to also deal with the dependence on the choice of the open cover $\mathcal{U}$, in the same manner using refinements of covers, and eventually taking a direct limit.

Recall that if $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ are open covers of $X$, a refinement map between them is a map $\tau: J \rightarrow I$ such that $V_{j} \subset U_{\tau(j)}$. As in the abelian case, a refinement map $\tau$ induces a map on $\check{Z}^{1}$.

$$
\tau_{\mathcal{U}}^{\mathcal{V}}: \check{Z}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{Z}^{1}(\mathcal{V}, \mathcal{F}) \quad\left\{\phi_{i_{1} i_{2}}\right\} \mapsto\left\{\phi_{j_{1} j_{2}}^{\prime}\right\}
$$

where

$$
\phi_{j_{1} j_{2}}^{\prime}=\left.\phi_{\tau\left(j_{1}\right) \tau\left(j_{2}\right)}\right|_{j_{j_{1}} \cap V_{j_{2}}}
$$

As before, one can verify that $\tau_{\mathcal{U}}^{\mathcal{V}}$ respects the equivalence, and hence may be viewed as a map on the quotient $\check{H}^{1}$.

$$
\tau_{\mathcal{U}}^{\mathcal{V}}: \check{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{1}(\mathcal{V}, \mathcal{F})
$$

Less obviously, $\tau_{\mathcal{U}}^{\mathcal{V}}$ is independent of the refinement map $\tau$, so the open covers of $X$, partially ordered by refinement, form a direct system, so the sets $\breve{H}^{1}(\mathcal{U}, \mathcal{F})$, ordered by refinement of $\mathcal{U}$ form a direct system, so we may take the direct limit (in the category of sets, or pointed sets if you wish).

Definition 6.25. The first nonabelian Čech cohomology set of $X, \mathcal{F}$ is

$$
\check{H}^{1}(X, \mathcal{F})=\underset{\mathcal{U}}{\lim } \check{H}^{1}(\mathcal{U}, \mathcal{F})
$$

Returning to our discussion of vector bundles, we now have the necessary tools for dispensing with our unfortunate dependence on the choice of open cover. A vector bundle of rank $k$ on $X$ gives rise to an element of $\check{H}^{1}(\mathcal{U}, \mathcal{F})$ for any $\mathcal{U}$, and (proof omitted) these various cohomology classes form a "coherent sequence," so that they give an element of the direct limit. That is, we have a map

$$
\{\text { vector bundles of rank } k \text { on } X\} \rightarrow \check{H}^{1}(X, \mathcal{F})
$$

We have not defined a map of vector bundles, but hopefully you can guess (or look it up somewhere), and from there guess the appropriate definition of isomorphism of vector bundles. Whatever the precise definition is, it is clear that if the definition is at all reasonable, then isomorphic vector bundles ought to give rise to the same Čech cohomology class. So we may more precisely state the previous map as

$$
\{\text { isomorphism classes of vector bundles of rank } k \text { on } X\} \rightarrow \check{H}^{1}(X, \mathcal{F})
$$

We make no attempt to justify this, but in fact this map is a bijection. In order to state the theorem succinctly, we note that the traditional notation for the sheaf we have been calling $\mathcal{F}$ is $\mathcal{G} \mathcal{L}_{k}$.

$$
\mathcal{G} \mathcal{L}_{k}(U)=\mathrm{GL}_{k}(\mathcal{S}(U))
$$

where $\mathcal{S}$ is the sheaf of smooth $\mathbb{R}$-valued functions.
Theorem 6.26. Let $X$ be a real smooth manifold, and $\mathcal{G} \mathcal{L}_{k}$ the sheaf above. There is a bijection

$$
\{\text { isomorphism classes of vector bundles of rank } k \text { on } X\} \rightarrow \check{H}^{1}\left(X, \mathcal{G} \mathcal{L}_{k}\right)
$$

Remark 6.27. Taking the case $k=1$, we get vector bundles of rank 1 , which are more commonly known as line bundles. On the cohomology side, we note that

$$
\mathcal{G} \mathcal{L}_{1}(U)=\mathrm{GL}_{1}(\mathcal{S}(U))=\mathcal{S}(U)^{\times}
$$

In particular, $\mathcal{S}(U)^{\times}$is an abelian group, so we are back to the case where $\check{H}^{1}$ is an abelian group. In this case, the above theorem says that line bundles on $X$ are in bijection with $H^{1}\left(X, \mathcal{S}^{\times}\right)$, which is more commonly known as the Picard group of $X$.

Remark 6.28. The previous theorem has many generalizations. For example, if we replace the sheaf $\mathcal{G} \mathcal{L}_{k}$ with the corresponding sheaf associated with $\mathrm{PGL}_{k}$, then we get another Čech cohomology group

$$
\check{H}^{1}\left(X, \mathcal{P G} \mathcal{L}_{k}\right)
$$

which is then in bijection with certain objects called "Azumaya algebras" on $X$, which, as you may guess, bear a passing resemblence to vector bundles. Of course, many more subtleties are involved.

### 6.5 The Čech cochain complex

At this point, we have only discussed $\check{H}^{1}$, but the notation clearly suggests that there should be objects $\check{H}^{n}$ for, at least, $n \geq 0$. In order to define all of these groups, and put our understanding of $\check{H}^{1}$ into this context, we define the Čech cochain complex.

Definition 6.29. Let $X$ be a topological space, and $\mathcal{F}$ a presheaf of abelian groups on $X$. Fix an open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$. Given $n \in \mathbb{Z}_{\geq 0}$, and an $(n+1)$-tuple

$$
\left(i_{0}, \ldots, i_{n}\right) \in I^{n+1}
$$

define the shorthand

$$
U_{i_{0} \cdots i_{n}}=\bigcap_{k=0}^{n} U_{i_{k}}=U_{i_{0}} \cap \cdots \cap U_{i_{n}}
$$

The group of $n$-cochains with respect to $\mathcal{U}$ is the (abelian) group

$$
\check{C}^{n}(\mathcal{U}, \mathcal{F})=\prod_{\left(i_{0}, \ldots, i_{n}\right) \in I^{n+1}} \mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)
$$

Example 6.30. In the case of $n=0$, we get $\check{C}^{0}(\mathcal{U}, \mathcal{F})$, which is the product over all $\mathcal{F}\left(U_{i}\right)$, where $U_{i}$ ranges over the open sets of the covering.
Definition 6.31. For $n \in \mathbb{Z}_{\geq 0}$, we define the $n$th differential $d^{n}: \check{C}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$ by

$$
\left(d^{n}(s)\right)_{i_{0}, \ldots, i_{n+1}}=\left.\sum_{j=0}^{n+1}(-1)^{j} s_{i_{0} \ldots \hat{i}_{j} \cdots i_{n+1}}\right|_{U_{i_{0} \cdots i_{n+1}}}
$$

The above definition is difficult to parse. What is happening is that since $d^{n}(s)$ is in the product over $(n+2)$-tuples, it suffices to describe the component for each $(n+2)$-tuple $\left(i_{0}, \ldots, i_{n+1}\right)$. Since $s \in \check{C}^{n}(\mathcal{U}, \mathcal{F}), s$ has a component for each $(n+1)$-tuple. Since we're given an $(n+2)$-tuple, we can form various ( $n+1$ )-tuples by removing each index, one at a time. Then we have to restrict it (using the restriction map associated with $\mathcal{F}$ ) to a smaller intersection, and we throw in a sign $(-1)^{j}$ just to make your life harder ${ }^{10}$.
Lemma 6.32. Let $d^{n}$ be the differentials defined above. Then $d^{n+1} \circ d^{n}=0$ for all $n \geq 0$.
Proof. Do it yourself, if you're a masochist.
Definition 6.33. For $n \in \mathbb{Z}_{\geq 0}$, the Čech $n$-cocyles is the kernel of $d^{n}$, the Čech $n$ boundaries is the image of $d^{n-1}$.

$$
\check{Z}^{n}(\mathcal{U}, \mathcal{F})=\operatorname{ker} d^{n} \quad \check{B}^{n}(\mathcal{U}, \mathcal{F})=\operatorname{im} d^{n-1}
$$

The shift of index $n-1$ is so that both are subgroups of $\check{C}^{n}(\mathcal{U}, \mathcal{F})$. Since $d^{n-1} \circ d^{n}=0$, we get a cochain complex, called the Čech cochain complex.

$$
0 \rightarrow \check{C}^{0}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{0}} \check{C}^{1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{1}} \check{C}^{2}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{2}} \cdots
$$

The equation $d^{n-1} \circ d^{n}=0$ is sometimes written sloppily as $d^{2}=0$, and it also implies that

$$
\check{B}^{n}(\mathcal{U}, \mathcal{F}) \subset \check{Z}^{n}(\mathcal{U}, \mathcal{F})
$$

[^8]So we may form the quotient, which is the cohomology of the Čech cochain complex.

$$
\check{H}^{n}(\mathcal{U}, \mathcal{F}):=\frac{\check{Z}^{n}(\mathcal{U}, \mathcal{F})}{\check{B}^{n}(\mathcal{U}, \mathcal{F})}
$$

Remark 6.34. There are two common variations of the definition of the Čech cochain complex, which end up giving the same cohomology groups, although this is not obvious. We will not go into the details of why they give the same cohomology, but we will describe the variants, so that the reader can compare with other sources.

The first variation is based on the recognition that it shouldn't be necessary to look at all the $(n+1)$-tuples in $I^{n+1}$, since permuting the order of the indices doesn't change the intersection $U_{i_{0} \cdots i_{n}}$, and so throwing away the "redundant copies" of $\mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)$ should be a reasonable thing to do, which in fact it is. More precisely, the symmetric group $S_{n+1}$ acts by permutation on $I^{n+1}$. Define the alternating $n$-cochains,

$$
\check{C}_{a}^{n}(\mathcal{U}, \mathcal{F}) \subset \check{C}^{n}(\mathcal{U}, \mathcal{F})
$$

to be the cochains which are "invariant under $S_{n+1}$ up to sign," meaning that a cochain $\left(f_{i}\right) \in \check{C}^{n}(\mathcal{U}, \mathcal{F})$ is in $\check{C}_{a}^{n}(\mathcal{U}, \mathcal{F})$ if for $\sigma \in S_{n+1}$,

$$
\left(f_{\sigma(i)}\right)=\left(\operatorname{sgn}(\sigma) f_{i}\right)
$$

Note that this implies that if two distinct indices $i_{r}, i_{s}$ are equal, then $\left(f_{i}\right)=(0)$. One can show that the differentials $d^{n}$ as defined above restrict to maps between the alternating cochains, giving rise to cohomology groups $\check{H}_{a}^{n}(\mathcal{U}, \mathcal{F})$. One can also show that

$$
\check{H}_{a}^{n}(\mathcal{U}, \mathcal{F}) \cong \check{H}^{n}(\mathcal{U}, \mathcal{F})
$$

For the second variation on the definition, we start by fixing a total order on $I$, and define the ordered $n$-cochains

$$
\check{C}_{o}^{n}(\mathcal{U}, \mathcal{F})=\prod_{i_{0}<\cdots<i_{n}} \mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)
$$

Using the same formula, define a differential $d^{n}$ on the ordered $n$-cochains, which also forms a cochain complex, whose homology we denote by $\check{H}_{o}^{1}(\mathcal{U}, \mathcal{F})$. As with the first variation, one may show that

$$
\check{H}_{o}^{n}(\mathcal{U}, \mathcal{F}) \cong \check{H}^{n}(\mathcal{U}, \mathcal{F})
$$

Example $6.35\left(\check{H}^{0}(\mathcal{U}, \mathcal{F})\right)$. In this example, we attempt to say what we can about the zeroth Čech cohomology group $\check{H}^{0}(\mathcal{U}, \mathcal{F})$, for an arbitrary space $X$ and arbitrary presheaf $\mathcal{F}$, and arbitrary open cover $\mathcal{U}$. By definition,

$$
\begin{aligned}
\check{C}^{0}(\mathcal{U}, \mathcal{F}) & =\prod_{i \in I} \mathcal{F}\left(U_{i}\right) \\
\check{C}^{1}(\mathcal{U}, \mathcal{F}) & =\prod_{i, j \in I} \mathcal{F}\left(U_{i} \cap U_{j}\right)
\end{aligned}
$$

and the zeroth differential

$$
d^{0}: \check{C}^{0}(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{1}(\mathcal{U}, \mathcal{F})
$$

can be described concretely by

$$
\left(d^{0}\left(f_{i}\right)\right)_{i j}=\left.f_{j}\right|_{U_{i} \cap U_{j}}-\left.f_{i}\right|_{U_{i} \cap U_{j}}
$$

So the zero cocycles, $\operatorname{ker} d^{0}=\check{Z}^{0}(\mathcal{U}, \mathcal{F})$ is the set of $\left(f_{i}\right)$ which are compatible on all intersections (of the given open covering sets). The zero coboundaries are just zero, so $\check{H}^{0}(\mathcal{U}, \mathcal{F})=\check{Z}^{0}(\mathcal{U}, \mathcal{F})$.

If $\mathcal{F}$ is a sheaf, then $\check{H}^{0}(\mathcal{U}, \mathcal{F})$ is the set of $\left(f_{i}\right)$ which are compatible on intersections, which means that they glue to a global section $f \in \mathcal{F}(X)$. By the separatedness axiom, this $f$ is unique, so we obtain an isomorphism

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)
$$

To emphasize, this is only true when $\mathcal{F}$ is a sheaf. In fact, if $\mathcal{F}$ is an arbitrary presheaf, the failure of these two to be isomorphic is a "measurement" of how much $\mathcal{F}$ fails to be a sheaf. More precisely, if $\mathcal{F}$ is any presheaf, there is a map

$$
\phi_{\mathcal{U}}: \mathcal{F}(X) \rightarrow \check{H}^{0}(\mathcal{U}, \mathcal{F}) \quad f \mapsto\left(\left.f\right|_{U_{i}}\right)
$$

This is well defined since a global section always gives rise to local sections which are compatible. If $\mathcal{F}$ satisfies separatedness, then $\phi_{\mathcal{U}}$ is injective; that is, ker $\phi_{\mathcal{U}}$ is an obstruction to $\mathcal{F}$ being separated. If $\mathcal{F}$ satsifies gluing, then $\phi$ is surjective; that is, coker $\phi_{\mathcal{U}}$ is an obstruction to $\mathcal{F}$ having gluing.

Conversely, the separatedness axiom for $\mathcal{F}$ may be rephrased precisely as the requirement that $\operatorname{ker} \phi_{\mathcal{U}}=0$ for all open covers $\mathcal{U}$, and the gluing axiom for $\mathcal{F}$ may be rephrased precisely as the requirement that coker $\phi_{\mathcal{U}}=0$ for all open covers $\mathcal{U}$. If we are careful, then passing to the direct limit, we might hope that we obtain a map

$$
\phi: \mathcal{F}(X) \rightarrow \check{H}^{0}(X, \mathcal{F})
$$

such that $\operatorname{ker} \phi=0$ if and only if $\mathcal{F}$ is separated, and $\operatorname{coker} \phi=0$ if and only if $\mathcal{F}$ satisfies gluing. Combining these, $\mathcal{F}$ is a sheaf if and only if $\phi$ is an isomorphism. (Note: we have not justified any of this discussion, that will come later.)

Example 6.36 (Čech cohomology of the circle). Let $X=S^{1}$ be the circle (unit circle in $\mathbb{R}^{2}$ if you like), and let $\mathcal{F}$ be the sheaf of locally constant $\mathbb{R}$-valued functions on $X$. We will compute $\check{H}^{0}(\mathcal{U}, \mathcal{F})$ using the C Cech cochain complex for a particularly simple open cover of $X$. Let $\mathcal{U}=\left\{U_{1}, U_{2}\right\}$ be an open cover of two connected pieces, as depicted below.


The intersection $U_{1} \cap U_{2}$ has two connected components, as we depict below.


Note that $\mathcal{F}\left(U_{1}\right) \cong \mathbb{R} \cong \mathcal{F}\left(U_{2}\right)$ since $U_{1}, U_{2}$ are connected, and $\mathcal{F}\left(U_{1} \cap U_{2}\right) \cong \mathbb{R}^{2}$, since it has two components. The restriction maps are essentially the diagonal map $\mathbb{R} \rightarrow \mathbb{R}^{2}$.

$$
\rho_{U_{1} \cap U_{2}}^{U_{i}}: \mathcal{F}\left(U_{i}\right) \cong \mathbb{R} \rightarrow \mathcal{F}\left(U_{1} \cap U_{2}\right) \cong \mathbb{R}^{2} \quad a \mapsto(a, a)
$$

The first few terms of the Čech cochain complex look like

$$
\begin{array}{rl}
0 \longrightarrow \check{C}^{0}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{0}} \xrightarrow{d^{1}} \check{C}^{2}(\mathcal{U}, \mathcal{F}) \rightarrow \cdots \\
& \cong \\
\mathcal{F}\left(U_{1}\right) & \times \mathcal{F}\left(U_{2}\right) \\
& \mathcal{F}\left(U_{11}\right) \times \mathcal{F}\left(U_{12}\right) \\
\cong & \times \mathcal{F}\left(U_{21}\right) \times \mathcal{F}\left(U_{22}\right) \\
& \cong \\
\mathbb{R}^{2} & \mathbb{R} \times \mathbb{R}^{2} \times \mathbb{R}^{2} \times \mathbb{R}
\end{array}
$$

Now we try to explicitly describe the first differential $d^{0}$. For example, if we take $(a, b) \in$ $\mathcal{F}\left(U_{1}\right) \times \mathcal{F}\left(U_{2}\right) \cong \mathbb{R}^{2}$ and look at the $U_{12}$-component of $d^{0}(a, b)$ in $\mathcal{F}\left(U_{1} \cap U_{2}\right)$, we get

$$
\left(d^{0}(a, b)\right)_{12}=\left.a\right|_{U_{1} \cap U_{2}}-\left.b\right|_{U_{2} \cap U_{2}}=(a, a)-(b, b)=(a-b, a-b)
$$

Working the rest out similarly, we describe $d^{0}$ as

$$
\begin{aligned}
d^{0}: \mathcal{F}\left(U_{1}\right) \times \mathcal{F}\left(U_{2}\right) \cong \mathbb{R}^{2} & \rightarrow \mathcal{F}\left(U_{11}\right) \times \mathcal{F}\left(U_{12} \times \mathcal{F}\left(U_{21}\right) \times \mathcal{F}\left(U_{22}\right) \cong \mathbb{R} \times \mathbb{R}^{2} \times \mathbb{R}^{2} \times \mathbb{R}\right. \\
(a, b) & \mapsto(0,(b-a, b-a),(a-b, a-b), 0)
\end{aligned}
$$

From this description, we can see that the kernel of $d^{0}$ is $(a, b)$ such that $a=a$, that is,

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F}) \cong \check{Z}^{0}(\mathcal{U}, \mathcal{F})=\operatorname{ker} d^{0}=\left\{(a, a) \in \mathbb{R}^{2}: a \in \mathbb{R}\right\} \cong \mathbb{R}
$$

Based on our previous general example, we already knew this should be the answer, since the global sections $\mathcal{F}(X)$ are isomorphic to $\mathbb{R}$, since a locally constant function on the connected space $X=S^{1}$ is just a constant function $S^{1} \rightarrow \mathbb{R}$, which are obviously identified with $\mathbb{R}$.

We could, in principle, work out an explicit description for $d^{1}$, although the target space $\check{C}^{2}(\mathcal{U}, \mathcal{F})$ is rather large, having 8 terms, six of which are $\mathbb{R}^{2}$ and two of which are $\mathbb{R}$, so $\check{C}^{2} \cong \mathbb{R}^{14}$. The answer should come out to be that $\check{H}^{2}(\mathcal{U}, \mathcal{F})=0$.

Remark 6.37. Notice that in the previous example, the Čech cohomology groups ended up being isomorphic to the corresponding singular cohomology groups.

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F}) \cong H_{\text {sing }}^{0}\left(S^{1}, \mathbb{R}\right) \quad \check{H}_{\text {sing }}^{1}(\mathcal{U}, \mathcal{F}) \cong H^{1}\left(S^{1}, \mathbb{R}\right)
$$

This is not a coincidence, and the pattern should continue for higher groups, meaning that $\breve{H}^{n}(\mathcal{U}, \mathcal{F})=0$ for $n \geq 2$. Later, we will explore more fully the connection between $\breve{H}^{n}(X, \mathcal{F})$ and singular cohomology of $X$.

As you should expect, these may not be the same for every open cover $\mathcal{U}$ and ever sheaf $\mathcal{F}$, but it would be wonderful if we could establish something like $\check{H}^{n}(X, \mathcal{F}) \cong H^{n}(X, \mathbb{R})$ in general for a particular sheaf $\mathcal{F}$, since then singular homology would just be a special case of Čech cohomology. This turns out to be correct, although it will take until the very end of the course to establish this.

### 6.6 Resolving dependence on the cover, in more generality

We previously worked out in a somewhat brute force way how to remove dependency on the choice of open cover $\mathcal{U}$, in the case of $\breve{H}^{1}(\mathcal{U}, \mathcal{F})$ and obtain a cohomology group which is more "intrinsic" to $X$ and $\mathcal{F}$, by taking the direct limit over refinement maps.

$$
\check{H}^{1}(X, \mathcal{F})=\underset{\longrightarrow}{\lim } \check{H}^{1}(\mathcal{U}, \mathcal{F})
$$

We now want to do a similar process for the entire Čech cochain complex, to obtain an "intrinsic" cohomology group $\check{H}^{n}(X, \mathcal{F})$.

Definition 6.38. Let $A^{\bullet}=\left\{A^{n}, d_{A}^{n}\right\}$ and $B^{\bullet}=\left\{B^{n}, d_{B}^{n}\right\}$ be cochain complexes. A morphism of cochain complexes $f^{\bullet}: A^{\bullet} \rightarrow B^{\bullet}$ is a family of maps $f^{n}: A^{n} \rightarrow B^{n}$ which commutes with the differentials. That is, the following diagram commutes.


A morphism of cochain complexes is also called a chain map. A morphism of cochain complexes induces a map on cohomology, by a simple diagram chase.

$$
H^{n}(f): H^{n}(A) \rightarrow H^{n}(B) \quad \bar{a} \mapsto \overline{f^{n}(a)}
$$

Definition 6.39. Let $X$ be a space and $\mathcal{F}$ be a presheaf of abelian groups on $X$. Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}, \mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ be two open covers of $X$, and suppose we have a refinement map $\tau: J \rightarrow I$, which means $V_{j} \subset U_{\tau(j)}$. The induced map on cochains is

$$
\tau^{n}: \check{C}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{n}(\mathcal{V}, \mathcal{F}) \quad\left(\left(\tau^{n}(s)\right)_{j_{0} \cdots j_{n}}=\left.s_{\tau\left(j_{0} \cdots \tau\left(j_{n}\right)\right.}\right|_{V_{j_{0} \cdots j_{n}}}\right.
$$

Lemma 6.40. The maps $\tau^{n}$ form a chain $\operatorname{map} \check{C} \bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{C} \bullet(\mathcal{V}, \mathcal{F})$, which is to say, we have the following commutative diagram.


Consequently, $\tau^{n}$ induces a map on Čech cohomology,

$$
\check{\tau}^{n}: \check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{n}(\mathcal{V}, \mathcal{F})
$$

Proof. Technical details.
We want to show that the map $\check{\tau}^{n}$ is actually independent of the choice of refinement $\tau$. First, we recall the notion of chain homotopies.

Definition 6.41. Let $g^{\bullet}, f^{\bullet}: A^{\bullet} \rightarrow B^{\bullet}$ be two morphisms of cochain complexes. They are chain homotopic if there exist morphisms $h^{n}: A^{n} \rightarrow B^{n-1}$ such that

$$
f^{n}-g^{n}=d_{B}^{n-1} \circ h^{n}+h^{n+1} \circ d_{A}^{n}
$$

We depict the situation with the following diagram.


The diagram above does not commute, but at least all of the maps involved in the above equation appear in the diagram. The equation says that every downward vertical arrow is equal to the sum of the two triangular paths from $A^{n}$ to $B^{n}$. If $f^{\bullet}$ is chain homotopic to the zero chain map, we say $f^{\bullet}$ is nullhomotopic.

Lemma 6.42. Chain homotopic maps induce the same maps on homology.
Proof. Suppose $f^{\bullet}, g^{\bullet}: A^{\bullet} \rightarrow B^{\bullet}$ are chain homotopic via $h^{n}$. Then for $x \in Z^{n}\left(A^{\bullet}\right)=\operatorname{ker} d_{A}^{n}$,

$$
f^{n}(x)-g^{n}(x)=d_{B}^{n-1} \circ h^{n}(x)+0
$$

The RHS is then clearly in $B^{n}\left(B^{\bullet}\right)=\operatorname{im} d_{B}^{n-1}$, which is to say, it is zero in $H^{n}\left(B^{\bullet}\right)$. Hence

$$
\overline{f^{n}(x)}=\overline{g^{n}(x)} \quad \text { in } H^{n}\left(B^{\bullet}\right)
$$

Remark 6.43. If $A^{\bullet}$ is a (co)chain complex, it is exact if and only if $H^{n}\left(A^{\bullet}\right)=0$ for all $n$. One sufficient condition for this is that the identity chain map on $A$ is nullhomotopic, using the previous lemma. However, this is not a necessary condition, as the following example shows.

Example 6.44 (Exact chain complex with non-nullhomotopic identity chain map). Let $A$ be the chain complex of abelian groups

$$
0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} / n \mathbb{Z} \rightarrow 0
$$

where $\pi$ is the usual quotien map, $\bmod n$. It is clear that $A$ is exact; we claim that the identity map is not nullhomotopic. Suppose $h^{n}$ gives a chain homotopy between the identity and zero maps on $A^{\bullet}$.


This already forces $h^{0}=h^{2}=h^{3}=0$.


If this is a chain homotopy, then it the identity $\operatorname{map} \mathbb{Z} / n \mathbb{Z} \rightarrow \mathbb{Z} / n \mathbb{Z}$ is the sum of two maps which are zero, which is impossible. One can also run into an issue with the identity map for the middle $\mathbb{Z}$ term, but that's not necessary.

Now we return to Čech cohomology.
Proposition 6.45. If $\tau, \tau^{\prime}: J \rightarrow I$ are refinement maps for open covers $\mathcal{U}, \mathcal{V}$ on $X$, then the induced chain maps $\check{\tau}, \check{\tau}^{\prime}$ are chain homotopic.

Proof. We just describe the definition of the chain homotopy, and leave it to the reader to verify. To be precise, $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$, and then define

$$
h^{n+1}: \check{C}^{n+1}(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{n}(\mathcal{V}, \mathcal{F}) \quad\left(h^{n+1}(s)\right)_{j_{0} \cdots j_{n}}=\left.\sum_{k=0}^{n}(-1)^{k} S_{\tau\left(j_{0}\right) \cdots \tau\left(j_{k}\right) \tau^{\prime}\left(j_{k}\right) \cdots \tau\left(j_{n}\right)}\right|_{V_{j_{0} \cdots j_{n}}}
$$

Corollary 6.46. The map $\check{\tau}^{n}$ is independent of the refinement map $\tau$.

Definition 6.47. As before, we have a partial order on the set of open covers of $X$ by refinement. By the previous corollary, the induced map $\check{H}^{n}(\mathcal{V}, \mathcal{F}) \rightarrow \check{H}^{n}(\mathcal{U}, \mathcal{F})$ is independent of the choice of $\tau$ provided $\mathcal{V} \leq \mathcal{U}$, so we obtain a directed system of abelian groups indexed by open covers. We define the $n$th Čech cohomology group of the presheaf $\mathcal{F}$ is the direct limit of this system.

$$
\check{H}^{n}(X, \mathcal{F}):=\underset{\longrightarrow}{\lim } \check{H}^{n}(\mathcal{U}, \mathcal{F})
$$

Proposition 6.48 (Functoriality of Čech cohomology). Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves on a space $X$. There is an induced map

$$
\phi^{n}: \check{H}^{n}(X, \mathcal{F}) \rightarrow \check{H}^{n}(X, \mathcal{G})
$$

which makes $\check{H}^{n}(X,-)$ into a (covariant) functor from the category of presheaves of abelian groups on $X$ to the category of abelian groups ${ }^{11}$

Proof. For a fixed cover $\mathcal{U}$ of $X$, we get

$$
\phi_{\mathcal{U}}^{n}: \check{C}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^{n}(\mathcal{U}, \mathcal{G}) \quad\left(\phi_{\mathcal{U}}^{n}(s)\right)_{i_{0} \cdots i_{n}}=\phi_{U_{i_{0} \cdots i_{n}}}\left(s_{i_{0} \cdots i_{n}}\right)
$$

These give a morphism of chain complexes, so they induce maps on homology.

$$
\check{\phi}_{\mathcal{U}}^{n}: \check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{n}(\mathcal{U}, \mathcal{G})
$$

These maps are compatible with refinements, so passing to the direct limit we obtain a map

$$
\phi^{n}=\underset{\longrightarrow}{\lim } \check{\phi}^{n}: \check{H}^{n}(X, \mathcal{F})=\underset{\longrightarrow}{\lim } \check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{n}(X, \mathcal{G})=\underline{\lim } \check{H}^{n}(\mathcal{U}, \mathcal{G})
$$

which is precisely the claimed map. We leave the verification of functoriality properties to the reader.

### 6.7 Sheafified Čech complex

Previously, we defined the group of $n$-cochains of the Čech complex, which we denoted $\check{C}^{n}(\mathcal{U}, \mathcal{F})$. When $\mathcal{F}$ is a presheaf of abelian groups on $X$, this is an abelian group. In this section, we up the abstraction by defining a modified version of $\check{C}^{n}(\mathcal{U}, \mathcal{F})$ which is not just an abelian group, but instead a presheaf of abelian groups on $X$.

Definition 6.49. Let $X$ be a topological space and $\mathcal{F}$ a presheaf of abelian groups on $X$. Let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $X$. Recall the shorthand notation

$$
U_{i_{0} \cdots i_{n}}=\bigcap_{k=0}^{n} U_{i_{k}}
$$

[^9]For an arbitrary open subset $U \subset X$, define

$$
\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(U)=\prod_{\left(i_{0}, \ldots, i_{n}\right) \in I^{n+1}} \mathcal{F}\left(U \cap U_{i_{0} \cdots i_{n}}\right)
$$

For $V \subset U$ open subset of $X$, define restriction maps

$$
\operatorname{res}_{V}^{U}: \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(U) \rightarrow \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(V)
$$

using the restriction maps for $\mathcal{F}$ on each component of the product, that is, on the ( $i_{0}, \ldots, i_{n}$ ) component, $\operatorname{res}_{V}^{U}$ is $\operatorname{res}_{V \cap U_{i_{0}} \ldots i_{n}}^{U \cap U_{i} \cdots i_{n}}(\mathcal{F})$. This makes $\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ into a presheaf of abelian groups on $X$. We call it the presheaf of Čech $n$-cochains.

Remark 6.50. If we take global sections of the presheaf of Cech $n$-cochains, we recover the original definition of the group of Čech $n$-cochains.

$$
\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(X)=\prod_{\left(i_{0}, \ldots, i_{n}\right)} \mathcal{F}\left(X \cap U_{i_{0} \cdots i_{n}}\right)=\prod_{\left(i_{0}, \ldots, i_{n}\right)} \mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)=\check{C}^{n}(\mathcal{U}, \mathcal{F})
$$

Lemma 6.51. If $\mathcal{F}$ is a sheaf of abelian groups, then $\breve{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ is a sheaf (of abelian groups). If $\mathcal{F}$ is flasque, then $\check{C}^{n}(\mathcal{U}, \mathcal{F})$ is also flasque.
Proof. The fact that $\breve{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ is a sheaf follows from a more general discussion we will have later about pushforward sheaves. For the moment, notice that for any $V \subset X$, we have a sheaf $\mathcal{F}_{V}$ on $X$ given by

$$
\mathcal{F}_{V}(U)=\mathcal{F}(U \cap V)
$$

Then observe that $\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ is the product over all $(n+1)$-tuples of pushforward sheaves with $V=U_{i_{0} \cdots i_{n}}$. This is all we will say about $\check{C}^{n}(\mathcal{U}, \mathcal{F})$ being a sheaf at this point.

Now suppose $\mathcal{F}$ is flasque. Recall that this means all restriction maps for $\mathcal{F}$ are surjective. Since the restriction maps for $\breve{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ are component-wise restriction maps for $\mathcal{F}$, they are surjective on each component, so they are surjective. So $\check{C}^{n}(\mathcal{U}, \mathcal{F})$ is flasque.

Remark 6.52. Let $X, \mathcal{F}, \mathcal{U}$ be as above. For the moment, $\mathcal{F}$ is merely a presheaf. For $U \subset X$, the product of restriction maps gives a map

$$
\mathcal{F}(U) \xrightarrow{\prod_{i \in I} \operatorname{res}_{U \cap U_{i}}^{U}} \prod_{i \in I} \mathcal{F}\left(U \cap U_{i}\right)=\check{\mathcal{C}}^{0}(\mathcal{U}, \mathcal{F})(U)
$$

If $\mathcal{F}$ is a sheaf, or even if it is just a separated presheaf, then this map is injective. Since this is injective for all $U$, we may alternatively phrase this as saying that we have an exact sequence

$$
0 \rightarrow \mathcal{F} \rightarrow \check{\mathcal{C}}^{0}(\mathcal{U}, \mathcal{F})
$$

If $\mathcal{F}$ is merely a separated presheaf, then we may think of this as an exact sequence in the category of presheaves. If $\mathcal{F}$ is a sheaf, then we may also view this as an exact sequence in the category of sheaves.

For the Čech cochains, we had differential maps which made them into a cochain complex of abelian groups. For the presheaf of cochains, we have analogous differentials, which we define next.

Definition 6.53. Let $X, \mathcal{U}, \mathcal{F}, \breve{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ be as above, with $\mathcal{F}$ a sheaf. For $U \subset X$ open, define

$$
d_{U}^{n}: \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(U) \rightarrow \check{\mathcal{C}}^{n+1}(\mathcal{U}, \mathcal{F})(U)
$$

by the same formula as $d^{n}$ was defined for the Čech complex. Then by the same sort of tedious calculation as was omitted before, $d_{U}^{n+1} \circ d_{U}^{n}=0$. These differentials are clearly compatible with restriction maps, so they give a morphism of sheaves

$$
d^{n}: \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}}^{n+1}(\mathcal{U}, \mathcal{F})
$$

which satisfies $d^{n+1} \circ d^{n}=0$, since the same equation holds everywhere locally. That is, we get a cochain complex of sheaves on $X$.

$$
0 \longrightarrow \mathcal{F} \longrightarrow \check{\mathcal{C}}^{0}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{0}} \check{\mathcal{C}}^{1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{1}} \check{\mathcal{C}}^{2}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{2}} \cdots
$$

This cochain complex is called the sheafified Čech complex of $\mathcal{F}$. In the next proposition, we will show that this is in fact an exact sequence of sheaves, and then we will call it the Čech resolution of $\mathcal{F}$.

Remark 6.54. In the special case of the above where we take $U=X$, the map $d_{X}^{n}$ is just the original differential on Cech $n$-cochains. Using this and Remark 6.50, if we evaluate the entire sheafified Čech complex by taking global sections, we get the original Čech complex, with an extra $\mathcal{F}(X)$ term at the beginning.

$$
0 \longrightarrow \mathcal{F}(X) \longrightarrow \check{C}^{0}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{0}} \check{C}^{1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{1}} \cdots
$$

Remark 6.55. Later we will define sheaf cohomology groups $H^{n}(X, \mathcal{F})$ using injective resolutions of $\mathcal{F}$. Then using the abstract category theory, we will obtain a map $\check{H}^{n}(X, \mathcal{F}) \rightarrow$ $H^{n}(X, \mathcal{F})$ by using the Čech resolution, where $H^{n}(X, \mathcal{F})$ refers to the as-yet-undefined $n$th sheaf cohomology group.

Proposition 6.56. Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. The sheafified Čech complex is an exact sequence of sheaves.

Proof. We already discussed in Remark 6.52 that the induced sequence on sections over any open $U \subset X$ is exact, so the sequence on stalks is also exact, so we have exactness at the $\mathcal{F}$ term.

At the $\check{\mathcal{C}}^{0}$ term, exactness of the induced sequence on sections over $U$ is essentially equivalent to the axioms that $\mathcal{F}$ is a sheaf. The map $d_{U}^{0}$ is given by

$$
d_{U}^{0}\left(\left(s_{i}\right)\right)=\left.s_{i}\right|_{U \cap U_{i} \cap U_{j}}-\left.s_{j}\right|_{U \cap U_{i} \cap U_{j}}
$$

so the kernel of $d_{U}^{0}$ is exactly collections of sections $\left(s_{i}\right)$ with $s_{i} \in \mathcal{F}\left(U \cap U_{i}\right)$ which agree on all double intersections $U_{i} \cap U_{j}$. Since $\mathcal{F}$ is a sheaf, any such $\left(s_{i}\right)$ glue to a global section $s \in \mathcal{F}(U)$, so $s$ is in the image of the previous map. Since the induced sequence on sections is exact for all $U$, the induced sequence on stalks is also exact, so we have exactness (in the category of sheaves) at the $\check{C}^{0}$ term.

In order to prove exactness for the higher degree terms, we consider the induced cochain complex on stalks $\check{\mathcal{C}} \bullet(\mathcal{U}, \mathcal{F})_{x}$. To show it is exact, we will construct a chain homotopy between the identity map and the zero map. That is, we will define

$$
\theta^{n}: \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})_{x} \rightarrow \check{\mathcal{C}}^{n-1}(\mathcal{U}, \mathcal{F})_{x}
$$

satisfying

$$
d_{x}^{n-1} \circ \theta^{n}+\theta^{n+1} \circ d_{x}^{n}=\operatorname{Id}_{\mathscr{\mathcal { C }}^{n}(\mathcal{U}, \mathcal{F})_{x}}
$$

Now we describe how to construct $\theta^{n}$. Given $x \in X$ and given an element of the stalk $f_{x} \in \breve{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})_{x}$, we need to define $\theta^{n}\left(f_{x}\right)$.

First, pick $j \in l^{12}$ so that $x \in U_{j}$. Then choose a section $f \in \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(V)$, where $V \subset X$ is an open neighborhood of $x$. Since the stalk is the direct limit over shrinking neighborhoods, we may shrink $V$ if necessary so that $V \subset U_{j}$. Now define

$$
\begin{gathered}
\widetilde{\theta}^{n}: \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(V) \rightarrow \check{\mathcal{C}}^{n-1}(\mathcal{U}, \mathcal{F})(V) \\
\left(\widetilde{\theta}^{n}(f)\right)_{i_{0} \cdots i_{n-1}}=\left.f_{j i_{0} \cdots i_{n-1}}\right|_{V \cap U_{i_{0} \cdots i_{n-1}}}
\end{gathered}
$$

This defines $\widetilde{\theta}^{n}(f)$ on each component, so it defines an output in $\breve{\mathcal{C}}^{n-1}(\mathcal{U}, \mathcal{F})(V)$. Then define $\theta^{n}\left(f_{x}\right)$ to be the image of $\widetilde{\theta}^{n}(f)$ in the stalk $\check{\mathcal{C}}^{n-1}(\mathcal{U}, \mathcal{F})_{x}$.

$$
\theta^{n}\left(f_{x}\right)=\rho_{x}^{V}\left(\widetilde{\theta}^{n}(f)\right)
$$

It is not immediately clear that $\theta^{n}$ is well defined, since it might in priciple depend on the choice of $j, f$, or $V$, but we omit these details. We give some justification for the claimed equation which we wanted $\theta^{n}$ to satisfy. We just discuss the equality on the level of sections. For $f \in \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(V)$, we have

$$
\begin{aligned}
\left(\widetilde{\theta}^{n+1}\left(d_{V}^{n}(f)\right)\right)_{i_{0} \cdots i_{n}} & =d_{V}^{n}(f)_{j i_{0} \cdots i_{n}} \\
& =f_{i_{0} \cdots i_{n}}-\sum_{k \neq 1}^{n}(-1)^{k} f_{j i_{0} \cdots \tilde{i}_{k} \cdots i_{n}} \\
& =f_{i_{0} \cdots i_{n}}-d_{V}^{n-1}\left(\widetilde{\theta}^{n}(f)\right)
\end{aligned}
$$

which we may rearrange to

$$
\widetilde{\theta}^{n+1} \circ d_{V}^{n}+d_{V}^{n-1} \circ \widetilde{\theta}^{n}=\operatorname{Id}_{\mathcal{C}^{n}(\mathcal{U}, \mathcal{F})(V)}
$$

[^10]Then taking the image in the stalks, we obtain the desired equality.

$$
d_{x}^{n-1} \circ \theta^{n}+\theta^{n+1} \circ d_{x}^{n}=\operatorname{Id}_{\tilde{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})_{x}}
$$

Thus $\theta^{n}$ is a nullhomotopy of the identity for the chain complex on stalks $\check{\mathcal{C}}^{\bullet}(\mathcal{U}, \mathcal{F})_{x}$, so that sequence of abelian groups is exact. Since this is exact for all $x \in X$, the chain complex of sheaves $\check{\mathcal{C}} \bullet(\mathcal{U}, \mathcal{F})$ is exact.

### 6.8 Sheafification revisited, using Čech cohomology

Now with our powerful tool of Čech cohomology, we can give another approach to sheafification which does not involve the étale space. The main advantage of this approach is not visible at first, which is the possibility of generalizing to situations where $X$ does not have a true topology, but instead has a generalized thing called a Grothendieck topology.

Let $X$ be a topological space, $\mathcal{F}$ a presheaf of abelian groups on $X$, and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ an open cover of $X$. Recall that

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F})=\check{Z}^{0}(\mathcal{U}, \mathcal{F})=\operatorname{ker} d^{0}=\left\{\left(f_{i}\right) \in \prod_{i \in I} \mathcal{F}\left(U_{i}\right):\left.f_{i}\right|_{U_{i} \cap U_{j}}=\left.f_{j}\right|_{U_{i} \cap U_{j}}, \forall i, j \in I\right\}
$$

Also recall that if $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ is a refinement of $\mathcal{U}$, with refinement map $\tau: J \rightarrow I$, we obtain a map

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{0}(\mathcal{V}, \mathcal{F})
$$

which leads to defining

$$
\check{H}^{0}(X, \mathcal{F})=\underset{\longrightarrow}{\lim } \check{H}^{0}(\mathcal{U}, \mathcal{F})
$$

Definition 6.57. Let $X, \mathcal{F}$ be as above. For $U \subset X$, define

$$
\left.\mathcal{F}\right|_{U}(W)=\mathcal{F}(W \cap U)
$$

which makes $\left.\mathcal{F}\right|_{U}$ a presheaf on $U$. So we obtain a group

$$
\mathcal{F}^{\#}(\mathcal{U}):=\check{H}^{0}\left(U,\left.\mathcal{F}\right|_{U}\right)
$$

Definition 6.58. Let $X, \mathcal{F}, \mathcal{F}^{\#}$ be as above. Suppose we have $V \subset U$ open subset of $X$. We want to describe a map

$$
\rho_{V}^{U}\left(\mathcal{F}^{\#}\right): \mathcal{F}^{\#}(\mathcal{U}) \rightarrow \mathcal{F}^{\#}(V)
$$

Given $f \in \mathcal{F}^{\#}(U)=\check{H}^{0}\left(U,\left.\mathcal{F}\right|_{U}\right)=\underline{\longrightarrow} \check{H}^{0}\left(\mathcal{U},\left.\mathcal{F}\right|_{U}\right)$, represent $f$ in one of the limiting groups, i.e. choose $f_{\mathcal{U}} \in \check{H}^{0}\left(\mathcal{U},\left.\mathcal{F}\right|_{U}\right)$ whose image in the direct limit is $f$. Here, $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ is some open cover of $U$. This $f_{\mathcal{U}}$ can be written as

$$
f_{\mathcal{U}}=\left(f_{i}\right) \in \prod_{i \in I} \mathcal{F}\left(U_{i}\right)
$$

Then define $\mathcal{V}$ to be the following open cover of $V$.

$$
\mathcal{V}=\left\{V_{i}=V \cap U_{i}\right\}_{i \in I}
$$

This covers $V$ because $V \subset U$. Then define $g_{i}=\left.f_{i}\right|_{V} \in \mathcal{F}\left(V_{i}\right)$, so

$$
\left(g_{i}\right) \in \check{H}^{0}\left(\mathcal{V},\left.\mathcal{F}\right|_{V}\right)
$$

Finally, we can take the image of $\left(g_{i}\right)$ in the direct limit $\underset{\longrightarrow}{\lim } \check{H}^{0}\left(\mathcal{V},\left.\mathcal{F}\right|_{V}\right)=\check{H}^{0}\left(V,\left.\mathcal{F}\right|_{V}\right)=$ $\mathcal{F}^{\#}(V)$. This image is what we call $\rho_{V}^{U}\left(\mathcal{F}^{\#}\right)(f)$.

$$
\rho_{V}^{U}\left(\mathcal{F}^{\#}\right): \mathcal{F}^{\#}(\mathcal{U}) \rightarrow \mathcal{F}^{\#}(V) \quad f \mapsto \text { image of }\left(\left.f_{i}\right|_{V}\right) \text { in direct limit }
$$

It is not entirely clear that this is well defined, independent of the choice of $f_{\mathcal{U}}$, but we omit these details.

Lemma 6.59. The previous two definitions make $\mathcal{F}^{\#}$ into a presheaf on $X$.
This is the setup to our new approach to sheafification. The general strategy is as follows: take a presheaf $\mathcal{F}$, and form the presheaf $\mathcal{F}^{\#}$. We will show that $\mathcal{F}^{\#}$ satisfies the separation axiom. Then repeat the process, forming $\left(\mathcal{F}^{\#}\right)^{\#}$, which we will denote $\mathcal{F}^{+}$. We will show that $\left(\mathcal{F}^{\#}\right)^{\#}$ satisfies both sheaf axioms.

$$
\mathcal{F} \rightsquigarrow \mathcal{F}^{\#} \rightsquigarrow\left(\mathcal{F}^{\#}\right)^{\#}=\mathcal{F}^{+}
$$

Definition 6.60. We will construct a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^{\#}$. Given an arbitrary open subset $U \subset X$, we need to define a map $\mathcal{F}(U) \rightarrow \mathcal{F}^{\#}(U)=\check{H}^{0}\left(U,\left.\mathcal{F}\right|_{U}\right)$. Consider the trivial open cover of $U$, given by $\mathcal{U}=\{U\}$. For this cover, we have

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F})=\mathcal{F}(U)
$$

so we have the "identity map"

$$
\mathcal{F}(U) \rightarrow \check{H}^{0}(\mathcal{U}, \mathcal{F})=\mathcal{F}(U) \quad s \mapsto s
$$

Then we compose this with the natural map $\check{H}^{0}(\mathcal{U}, \mathcal{F}) \rightarrow \underset{\longrightarrow}{\lim } \check{H}^{0}(\mathcal{U}, \mathcal{F})=\mathcal{F}^{\#}(U)$. This gives our needed map

$$
\mathcal{F}(U) \rightarrow \mathcal{F}^{\#}(U)
$$

It is clear that these are compatible with restriction maps for $\mathcal{F}, \mathcal{F}^{\#}$, so we get a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^{\#}$.

Theorem 6.61. Let $X$ be a topological space and $\mathcal{F}$ a presheaf of abelian groups on $X$.

1. $\mathcal{F}^{\#}$ is a separated presheaf on $X$.
2. If $\mathcal{F}$ is separated, then $\mathcal{F}^{\#}$ is a sheaf on $X$. (Hence $\left(\mathcal{F}^{\#}\right)^{\#}$ is a sheaf on $X$ for any presheaf $\mathcal{F}$.)
3. The map defined above $\mathcal{F} \rightarrow \mathcal{F}^{\#}$ induces an isomorphism on stalks. (Hence repeating the construction $\mathcal{F} \rightarrow \mathcal{F}^{\#} \rightarrow\left(\mathcal{F}^{\#}\right)^{\#}$ induces an isomorphism on stalks as well.)
4. $\left(\mathcal{F}^{\#}\right)^{\#}$ satisfies the universal property of sheafification for $\mathcal{F}$.

Since the proof is so long, we break it up into four separate propositions.
Proposition 6.62. Let $X$ be a topological space and $\mathcal{F}$ a presheaf of abelian groups on $X$. Then $\mathcal{F}^{\#}$ is a separated presheaf on $X$.

Warning: The following proof is involves a lot of confusing notation, and requires a thorough understanding of the following concrete description of a direct limit of abelian groups.

$$
\mathcal{F}^{\#}(U)=\underset{\underset{\mathcal{U}}{ }}{\lim } \check{H}^{0}(\mathcal{U}, \mathcal{F})=\left(\bigsqcup_{\mathcal{U}} \check{H}^{0}(\mathcal{U}, \mathcal{F})\right) / \sim
$$

The equivalence is described as follows. Given two open covers $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ and $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ of $U$, and $\left(s_{i}\right) \in \check{H}^{0}(\mathcal{U}, \mathcal{F})$ and $\left(t_{j}\right) \in \check{H}^{0}(\mathcal{V}, \mathcal{F})$, they represent the same element of the direct limit if there is another open cover $\mathcal{W}=\left\{W_{k}\right\}_{k \in K}$ and refinement maps $\tau: K \rightarrow I, \sigma$ : $K \rightarrow J$ such that the maps induced by $\tau, \sigma$ on the zeroth Cech cohomology map $\left(s_{i}\right),\left(t_{j}\right)$ respectively to the same element of $\check{H}^{0}(\mathcal{W}, \mathcal{F})$.


Proof. Let $X, \mathcal{F}$ be as in the statement of the theorem. For $V \subset U$ open subsets of $X$, we denote the restriction map for $\mathcal{F}$ by $\rho_{V}^{U}(\mathcal{F})$ or just $\rho_{V}^{U}$, and the restriction map for $\mathcal{F}^{\#}$ by $\rho_{V}^{U}\left(\mathcal{F}^{\#}\right)$.

We want to show that $\mathcal{F}^{\#}$ is separated, which is to say, for any $U \subset X$, if two sections of $\mathcal{F}^{\#}$ over $U$ are the same everywhere locally (on some cover of $U$ ), then they are the same globally (equal in $\mathcal{F}^{\#}(U)$ ). So let $U \subset X$ be any open subset, and let $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ be an open cover of $U$. Another way to phrase the property that $\mathcal{F}^{\#}$ be separated is that the following map is injective.

$$
\prod_{j \in J} \rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right): \mathcal{F}^{\#}(U) \rightarrow \prod_{j \in J} \mathcal{F}^{\#}\left(V_{j}\right)
$$

To show it is injective, we start with two arbitrary elements $s, t \in \mathcal{F}^{\#}(U)$ with equal images under the above map. That is, for $j \in J$,

$$
\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(s)=\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(t) \quad \text { in } \mathcal{F}^{\#}\left(V_{j}\right)
$$

We want to show that $s=t$. In broad strokes, we just follow the definitions where they lead, unraveling concrete descriptions of direct limits and maps between them. We will construct cover $\mathcal{W}$ which is a much finer cover than $\mathcal{V}$, and somehow things will work out.

As noted before the proof, $\mathcal{F}^{\#}(U)$ is a direct limit over covers of $U$ of the groups $\check{H}^{0}(\mathcal{U}, \mathcal{F})$, so we may choose representative elements for each of $s, t$ in some cover-specific Čech cohomology group. That is, there exist open covers $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}, \widetilde{\mathcal{U}}=\left\{\widetilde{U}_{i}\right\}_{i \in \tilde{I}}$ of $U$ and elements

$$
\begin{aligned}
& s_{\mathcal{U}}=\left(s_{i}\right) \in \prod_{i \in I} \mathcal{F}\left(U_{i}\right)=\check{H}^{0}(\mathcal{U}, \mathcal{F}) \\
& t_{\tilde{\mathcal{U}}}=\left(t_{i}\right) \in \prod_{i \in \widetilde{I}} \mathcal{F}\left(\widetilde{U}_{i}\right)=\check{H}^{0}(\widetilde{\mathcal{U}}, \mathcal{F})
\end{aligned}
$$

such that under the natural maps to the direct limit, $\left(s_{i}\right)$ gets mapped to $s$ and $\left(t_{i}\right)$ gets mapped to $t$.

$$
\begin{array}{rlrl}
\check{H}^{0}(\mathcal{U}, \mathcal{F}) & \rightarrow \mathcal{F}^{\#}(U) & & \left(s_{i}\right) \mapsto s \\
\check{H}^{0}(\widetilde{\mathcal{U}}, \mathcal{F}) \rightarrow \mathcal{F}^{\#}(U) & & \left(t_{i}\right) \mapsto t
\end{array}
$$

Now we attempt to unravel what the equality $\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(s)=\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(t)$ implies. In order to do this, we recall the description of the map $\rho_{V}^{U}\left(\mathcal{F}^{\#}\right)$. Having found $s_{\mathcal{U}}, t_{\tilde{\mathcal{U}}}$ representing $s$ and $t$, we consider the open covers of $V_{j}$ given by

$$
\mathcal{V}_{j}=\left\{V_{j} \cap U_{i}\right\}_{i \in I} \quad \widetilde{\mathcal{V}}_{j}=\left\{V_{j} \cap \widetilde{U}_{i}\right\}_{i \in \tilde{I}}
$$

then consider the elements

$$
\begin{aligned}
& \left.s_{i}\right|_{V_{j}}=\rho_{V_{V} \cap U_{i}}^{U_{i}}\left(s_{i}\right) \in \mathcal{F}\left(V_{j} \cap U_{i}\right) \\
& \left.t_{i}\right|_{V_{j}}=\rho_{V_{j} \cap \widetilde{U}_{i}}^{\widetilde{U}_{i}}\left(t_{i}\right) \in \mathcal{F}\left(V_{j} \cap \widetilde{U}_{i}\right)
\end{aligned}
$$

So we have

$$
\begin{aligned}
& \left(\left.s_{i}\right|_{V_{j}}\right) \in \prod_{i \in I} \mathcal{F}\left(V_{j} \cap U_{i}\right)=\check{H}^{0}\left(\mathcal{V}_{j},\left.\mathcal{F}\right|_{V_{j}}\right) \\
& \left(\left.t_{i}\right|_{V_{j}}\right) \in \prod_{i \in \tilde{I}} \mathcal{F}\left(V_{j} \cap \widetilde{U}_{i}\right)=\check{H}^{0}\left(\widetilde{\mathcal{V}}_{j},\left.\mathcal{F}\right|_{V_{j}}\right)
\end{aligned}
$$

By definition, $\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(s)$ is the image of $\left(s_{i} \mid V_{j}\right)$ in the direct limit, and $\rho_{V_{j}}^{U}\left(\mathcal{F}^{\#}\right)(t)$ is the image of $\left(\left.t_{i}\right|_{V_{j}}\right)$ in the direct limit. Following the discussion before the proof, the fact that $\left(\left.s_{i}\right|_{V_{j}}\right)$ and $\left(\left.t_{i}\right|_{V_{j}}\right)$ represent the same equivalence class in the direct limit means that there is a cover $\mathcal{W}_{j}=\left\{W_{j k}\right\}_{k \in K_{j}}$ of $V_{j}$ and refinement maps

$$
\tau_{j}: K_{j} \rightarrow I \quad \sigma_{j}: K_{j} \rightarrow \widetilde{I}
$$

such that under the induced maps $\check{\tau}_{j}^{0}, \check{\sigma}_{j}^{0}$, the elements $\left(\left.s_{i}\right|_{V_{j}}\right)$ and $\left(\left.t_{i}\right|_{V_{j}}\right)$ are mapped to the same element of $\check{H}^{0}\left(\mathcal{W}_{j},\left.\mathcal{F}\right|_{V_{j}}\right)$.


Note that for all $j$, and for $k \in K_{j}, W_{j k} \subset V_{j} \cap U_{\tau_{j}(k)}$ and $W_{j k} \subset V_{j} \cap \widetilde{U}_{\sigma_{j}(k)}$ (this is just what it means that $\tau_{j}, \sigma_{j}$ are refinements). Now consider the open cover $\mathcal{W}=\left\{W_{j k}\right\}_{j \in J, k \in K_{j}}$. Since $\mathcal{V}=\left\{V_{j}\right\}_{j \in J}$ is an open cover of $U$ and $\mathcal{W}_{j}=\left\{W_{j k}\right\}_{k \in K_{j}}$ is an open cover of $V_{j}, \mathcal{W}$ is an open cover of $U$.

Setting aside $\mathcal{W}$ for the moment, let us describe what we actually need to show in order to conclude $s=t$. For $s$ to equal $t$ in $\mathcal{F}^{\#}(U)=\underset{\longrightarrow}{\lim } \check{H}^{0}\left(\mathcal{U},\left.\mathcal{F}\right|_{U}\right)$, it suffices to find an open cover $\widetilde{\mathcal{W}}=\left\{W_{\ell}\right\}_{\ell \in L}$ of $U$ and refinement maps $\tau: L \rightarrow I, \sigma: L \rightarrow \widetilde{I}$ such that the images of $\left(s_{i}\right),\left(t_{i}\right)$ in $\check{H}^{0}(\widetilde{\mathcal{W}}, \mathcal{F})$ are equal. That is, in the following picture, we need $\check{\tau}^{0}\left(\left(s_{i}\right)\right)=\check{\sigma}^{0}\left(\left(t_{i}\right)\right)$.


But this is exactly the data we already have, as long as we piece it together carefully. Define

$$
L=\left\{(j, k) \in J \times \bigcup_{j \in J} K_{j}: k \in K_{j}\right\}
$$

and let

$$
\widetilde{\mathcal{W}}=\mathcal{W}=\left\{W_{j k}\right\}_{j \in J, k \in K_{j}}=\left\{W_{j k}\right\}_{(j, k) \in L}
$$

then define

$$
\begin{array}{ll}
\tau: L \rightarrow I & (j, k) \mapsto \tau_{j}(k) \\
\sigma: L \rightarrow \widetilde{I} & \\
(j, k) \mapsto \sigma_{j}(k)
\end{array}
$$

Then we claim that $\check{\tau}^{0}\left(\left(s_{i}\right)\right)=\check{\sigma}^{0}\left(\left(t_{i}\right)\right)$, which will finish the proof of (1). To prove these are equal, it suffices to consider the $(j, k)$ th component in $\check{H}^{0}(\mathcal{W}, \mathcal{F})=\prod_{(j, k) \in L} \mathcal{F}\left(W_{j k}\right)$. Note from earlier that $W_{j k} \subset V_{j} \cap U_{\tau_{j}(k)} \cap \widetilde{U}_{\sigma_{j}(k)}$, so the restriction maps we are about to write down make sense. We already know that for $j \in J$ and $k \in K_{j}$, the following elements are equal.

$$
\begin{aligned}
& \check{\tau}_{j}^{0}\left(\left(\left.s_{i}\right|_{V_{j}}\right)\right)_{k}=\left.\left(\left.s_{\tau_{j}(k)}\right|_{V_{j}}\right)\right|_{W_{j k}}=\left.\rho_{V_{j}(k)}^{U_{\tau_{\tau_{j}}(k)}\left(s_{\tau_{j}(k)}\right)}\right|_{W_{j k}}=\rho_{W_{j k} \cap U_{\tau_{j}(k)}}^{V_{j} \cap \rho_{V_{j} \cap U_{\tau_{j}(k)}}^{U_{\tau_{j}(k)}}\left(s_{\tau_{j}(k)}\right)=\rho_{W_{j k}}^{U_{\tau_{j}(k)}}\left(s_{\tau_{j}(k)}\right)} \\
& \check{\sigma}_{j}^{0}\left(\left(\left.t_{i}\right|_{V_{j}}\right)\right)_{k}=\left.\left(t_{\sigma_{j}(k)} \mid V_{j}\right)\right|_{W_{j k}}=\left.\rho_{V_{j} \cap U_{\sigma_{j}(k)}}^{U_{\sigma_{j}(k)}}\left(t_{\sigma_{j}(k)}\right)\right|_{W_{j k}}=\rho_{W_{j k} \cap U_{\sigma_{j}(k)}}^{V_{V_{j}}} \rho_{V_{j} \cap U_{\sigma_{j}(k)}}^{U_{\sigma_{j}(k)}}\left(t_{\sigma_{j}(k)}\right)=\rho_{W_{j k}}^{U_{\sigma_{j}(k)}}\left(t_{\sigma_{j}(k)}\right)
\end{aligned}
$$

Thus for $(j, k) \in L$, we have

$$
\begin{aligned}
\check{\tau}^{0}\left(\left(s_{i}\right)\right)_{(j, k)} & =\left.s_{\tau(j, k)}\right|_{W_{j k}}=\left.s_{\tau_{j}(k)}\right|_{W_{j k}}=\rho_{W_{j k}}^{U_{\tau_{j}(k)}}\left(s_{\tau_{j}(k)}\right) \\
& =\rho_{W_{j k}}^{U_{\sigma_{j}(k)}}\left(t_{\sigma_{j}(k)}\right)=\left.t_{\sigma_{j}(k)}\right|_{W_{j k}}=\left.t_{\sigma(j, k)}\right|_{W_{j k}}=\check{\sigma}^{0}\left(\left(t_{i}\right)\right)_{(j, k)}
\end{aligned}
$$

Thus $\check{\tau}^{0}\left(\left(s_{i}\right)\right)=\check{\sigma}^{0}\left(\left(t_{i}\right)\right)$ in $\check{H}^{0}(\mathcal{W}, \mathcal{F})$, hence $s=t$ in the direct limit. Hence the map we originally considered is injective, so $\mathcal{F}^{\#}$ is a separated presheaf.

We continue the proof of Theorem 6.61, proving the second part using a lemma.
Lemma 6.63. Let $\mathcal{F}$ be a separated presheaf of abelian groups on a space $X$. Then for any open subset $U \subset X$ and any open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $U$, the canonical map associated with the direct limit

$$
\check{H}^{0}\left(\mathcal{U},\left.\mathcal{F}\right|_{U}\right) \rightarrow \check{H}^{0}\left(U,\left.\mathcal{F}\right|_{U}\right)
$$

is injective.

## Proof. Omitted.

Proposition 6.64. Let $\mathcal{F}$ be a separated presheaf of abelian groups on $X$. Then $\mathcal{F}^{\#}$ is a sheaf.

Proof. Fom the previous proposition, we know $\mathcal{F}^{\#}$ is separated, so we just need to verify that it satisfies gluing. Let $U \subset X$ be an open set, and let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $U$. Suppose we are given elements $s_{i} \in \mathcal{F}^{\#}\left(U_{i}\right)$ for $i \in I$ which agree on double intersections. That is, for $i_{1}, i_{2} \in I$, we have

$$
\rho_{U_{i_{1}} \cap U_{i_{2}}}^{U_{i_{1}}}\left(\mathcal{F}^{\#}\right)\left(s_{i_{1}}\right)=\left.s_{i_{1}}\right|_{U_{i_{1} \cap U_{i_{2}}}}=\rho_{U_{i_{1} \cap U_{i_{2}}}}^{U_{i_{2}}}\left(\mathcal{F}^{\#}\right)\left(s_{i_{2}}\right)=\left.s_{2_{1}}\right|_{U_{i_{1} \cap U_{i_{2}}}}
$$

We need to find $s \in \mathcal{F}^{\#}(U)$ such that $\left.s\right|_{U_{i}}=s_{i}$ for all $i \in I$. Using the concrete description of the direct limit, each $s_{i} \in \mathcal{F}^{\#}\left(U_{i}\right)=\lim _{\mathcal{T}} \check{H}^{0}\left(\mathcal{T},\left.\mathcal{F}\right|_{U_{i}}\right){ }^{13}$ is represented by some $s_{i, \mathcal{V}_{i}} \in$ $\check{H}^{0}\left(\mathcal{V}_{i},\left.\mathcal{F}\right|_{U_{i}}\right)$ where $\mathcal{V}_{i}=\left\{V_{i j}\right\}_{j \in J_{i}}$ is an open cover of $U_{i}$.

$$
s_{i, \mathcal{V}_{i}}=\left(s_{i j}\right)_{j \in J_{i}} \in \check{H}^{0}\left(\mathcal{V}_{i},\left.\mathcal{F}\right|_{U_{i}}\right)=\prod_{j \in J_{i}} \mathcal{F}\left(V_{i j}\right)
$$

Now fix $i_{1}, i_{2} \in I$, and consider the cover

$$
\mathcal{W}=\left\{V_{i_{1} j_{1}} \cap V_{i_{2} j_{2}}\right\}_{\substack{j_{1} \in J_{i_{1}} \\ j_{2} \in J_{i_{2}}}}
$$

[^11]of $U_{i_{1}} \cap U_{i_{2}}$. Then consider the elements
\[

$$
\begin{aligned}
& \widetilde{s}_{i_{1}}=\left(\left.s_{i_{1} j_{1}}\right|_{V_{i_{1} j_{1}} \cap V_{i_{2} j_{2}}}\right)_{\substack{j_{1} \in J_{i_{1}} \\
j_{2} \in J_{i_{2}}}} \in \check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{U_{i_{1}} \cap U_{i_{2}}}\right)=\prod_{\substack{j_{1} \in J_{i_{1}} \\
j_{2} \in J_{i_{2}}}} \mathcal{F}\left(V_{i_{1} j_{2}} \cap V_{i_{2} j_{2}}\right) \\
& \widetilde{s}_{i_{2}}=\left(\left.s_{i_{2} j_{2}}\right|_{V_{i_{1} j_{1}} \cap V_{i_{2} j_{2}}}\right)_{\substack{j_{1} \in J_{i_{1}} \\
j_{2} \in J_{i_{2}}}} \in \check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{U_{i_{1} \cap U_{i_{2}}}}\right)=\prod_{\substack{j_{1} \in J_{i_{1}} \\
j_{2} \in J_{i_{2}}}} \mathcal{F}\left(V_{i_{1} j_{2}} \cap V_{i_{2} j_{2}}\right)
\end{aligned}
$$
\]

We have the canonical map of the direct limit

$$
\check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{U_{i_{1}} \cap U_{i_{2}}}\right) \rightarrow \xrightarrow[\longrightarrow]{\lim }\left(\mathcal{T},\left.\mathcal{F}\right|_{U_{i_{1}} \cap U_{i_{2}}}\right)=\check{H}^{0}\left(U_{i_{1}} \cap U_{i_{2}},\left.\mathcal{F}\right|_{U_{i_{1}} \cap U_{i_{2}}}\right)=\mathcal{F}^{\#}\left(U_{i_{1}} \cap U_{i_{2}}\right)
$$

where the direct limit ranges over open covers $\mathcal{T}$ of $U_{i_{1}} \cap U_{i_{2}}$. Under this map,

$$
\begin{aligned}
& \widetilde{s}_{i_{1}} \mapsto \rho_{U_{i_{1}} \cap U_{i_{2}}}^{U_{i_{1}}}\left(\mathcal{F}^{\#}\right)\left(s_{i_{1}}\right) \\
& \widetilde{s}_{i_{2}} \mapsto \rho_{U_{i_{1}} \cap U_{i_{2}}}^{U_{i_{2}}}\left(\mathcal{F}^{\#}\right)\left(s_{i_{2}}\right)
\end{aligned}
$$

by definition of the restriction maps for $\mathcal{F}^{\#}$. But by assumption, the two elements on the RHS above are equal, so $\widetilde{s}_{i_{1}}, \widetilde{s}_{i_{2}}$ get mapped to the same class in the direct limit. Then by Lemma 6.63, $\widetilde{s}_{i_{1}}=\widetilde{s}_{i_{2}}$ as elements of $\check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{U_{i_{1} \cap U_{i_{2}}}}\right)$, meaning that

$$
\left.s_{i_{1} j_{1}}\right|_{V_{i_{1} j_{1} \cap} \cap V_{i_{2} j_{2}}}=\left.s_{i_{2} j_{2}}\right|_{V_{i_{1} j_{1} \cap} \cap V_{i_{2} j_{2}}}
$$

for $j_{1} \in J_{i_{1}}, j_{2} \in J_{i_{2}}$. Now consider the open cover

$$
\mathcal{V}=\left\{V_{i j}\right\}_{\substack{i \in J_{i} \\ j \in J_{i}}}
$$

of $U$. Since $s_{i_{1} j_{1}}=s_{i_{2} j_{2}}$ on the intersection $V_{i_{1} i_{2}} \cap V_{i_{2} j_{2}}$, we have a well defined element

$$
\widetilde{s}=\left(s_{i j}\right)_{\substack{i \in I \\ j \in J_{i}}} \in \check{H}^{0}\left(\mathcal{V},\left.\mathcal{F}\right|_{U}\right)=\prod_{\substack{i \in I \\ j \in J_{i}}} \mathcal{F}\left(V_{i j}\right)
$$

Finally, we define $s$ to be the image of $\widetilde{s}$ under the canonical map to the direct limit.

$$
\check{H}^{0}\left(\mathcal{V},\left.\mathcal{F}\right|_{U}\right) \rightarrow \underset{\mathcal{\mathcal { T }}}{\lim } \check{H}^{0}\left(\mathcal{T},\left.\mathcal{F}\right|_{U}\right)=\check{H}^{0}\left(U,\left.\mathcal{F}\right|_{U}\right)=\mathcal{F}^{\#}(U)
$$

We claim that $s$ is the required glued global section, that is, $\left.s\right|_{U_{i}}=s_{i}$ in $\mathcal{F}^{\#}\left(U_{i}\right)$. We know that $s_{i}$ is represented by $\left(s_{i j}\right)_{i \in J_{i}} \in \check{H}^{0}\left(\mathcal{V},\left.\mathcal{F}\right|_{U_{i}}\right)$ using the open cover $\mathcal{V}_{i}$ of $U_{i}$.

On the other hand, $s$ is represented by $\left(s_{i j}\right)_{i \in I, j \in J_{i}} \in \check{H}^{0}\left(\mathcal{V}_{i},\left.\mathcal{F}\right|_{U}\right) \in \check{H}^{0}\left(\widetilde{\mathcal{V}},\left.\mathcal{F}\right|_{U_{i}}\right)$ using the open cover $\mathcal{V}$ of $U$, so $\left.s\right|_{U_{i}}$ is represented by $\left(\left.s_{i j}\right|_{U_{i}}\right)_{i \in I, j \in J_{i}}$, using the open cover $\widetilde{\mathcal{V}}_{i}=$ $\left\{U_{i} \cap V_{i j}\right\}$ of $U_{i}$. But this is silly, since $V_{i j} \subset U_{i}$, so $\widetilde{\mathcal{V}}_{i}=\mathcal{V}$ is the same cover. So clearly $s_{i}$ and $\left.s\right|_{U_{i}}$ are represented by the same thing, so they are equal in the direct limit.

This finishes the proof of part (2) of Theorem 6.61. Now we discuss part (3). I deliberately use the word "discuss" instead of "prove," because the "proof" given below is basically hand waving and nonsense.

Proposition 6.65. Let $\mathcal{F}$ be a presheaf of abelian groups on a space $X$. The morphism of sheaves $\mathcal{F} \rightarrow \mathcal{F}^{\#}$ of Definition 6.60 induces an isomorphism on stalks.

Proof. We have the morphism $\phi: \mathcal{F} \rightarrow \mathcal{F}^{\#}$ which is described on an open subset $U \subset X$ by composing the identity map $\mathcal{F}(U) \rightarrow \breve{H}^{0}\left(\mathcal{U}_{0}, \mathcal{F}\right)$ (where $\mathcal{U}_{0}=\{U\}$ is the trivial cover of $U$ ) with the canonical map to the direct limit $\mathcal{F}^{\#}(U)$. We need to show that the induced map on stalks $\phi_{x}$ is an isomorphism for all $x \in X$.

Let $x \in X$, let $U$ be an open subset containing $x$, and let $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $U$. As $\phi$ is a morphism of presheaves, we have the following commutative diagram.


We recall the description of the map $\phi_{x}$. Given an element of the stalk $\bar{s} \in \mathcal{F}_{x}$, we can choose a neighborhood $V$ of $x$ and a section $s \in \mathcal{F}\left(V_{s}\right)$ such that $\rho_{x}^{V_{s}}(s)=\bar{s}$. Then by commutativity of the above diagram,

$$
\phi_{x}(\bar{s})=\phi_{x} \circ \rho_{x}^{V_{s}}(\mathcal{F})(s)=\rho_{x}^{V_{s}}\left(\mathcal{F}^{\#}\right) \circ \phi_{V_{s}}(s)
$$

Now consider an element $\bar{t} \in \mathcal{F}_{x}^{\#}$. Then represent $\bar{t}$ by a section $t \in \mathcal{F}^{\#}\left(V_{t}\right)$, where $V_{t}$ is some neighborhood of $x$. Then since $\mathcal{F}^{\#}\left(V_{t}\right)$ is a direct limit also, represent $t$ by some $\widetilde{t} \in \check{H}^{0}\left(\mathcal{W}, V_{t}\right)$, where $\mathcal{W}=\left\{W_{j}\right\}_{j \in J}$ is an open cover of $V_{t}$.

$$
\tilde{t}=\left(t_{j}\right)_{j \in J} \in \prod_{j \in J} \mathcal{F}\left(W_{j}\right)=\check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{V_{t}}\right)
$$

That is, $\tilde{t}$ maps to $t$ in the canonical map to the direct limit $\mathcal{F}^{\#}\left(V_{t}\right)$. Fix $j_{x} \in J$ to be any index such that $x \in W_{j_{0}}$, which exists since $\mathcal{W}$ is an open cover of $V_{t}$ which contains $x$. Now consider the map

$$
\psi_{\mathcal{W}}: \check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{V_{t}}\right) \rightarrow \mathcal{F}_{x} \quad\left(u_{j}\right)_{j \in J} \mapsto \rho_{x}^{U}(\mathcal{F})\left(u_{j_{x}}\right)
$$

We now make several outrageous claims, which complete the proof if true.

1. The class of $\psi_{\mathcal{W}}\left(\left(u_{j}\right)\right)$ in $\mathcal{F}_{x}$ does not depend on the choice of index $j_{x}$.
2. As $\mathcal{W}$ ranges over open covers of $V_{t}$, the family of maps $\psi_{\mathcal{W}}$ are compatible in such a way that they pass to a map on the direct limit

$$
\psi_{t}=\underset{\underset{\mathcal{W}}{l}}{\lim } \psi_{\mathcal{W}}: \xrightarrow[\longrightarrow]{\lim } \check{H}^{0}\left(\mathcal{W},\left.\mathcal{F}\right|_{V_{t}}\right)=\mathcal{F}^{\#}\left(V_{t}\right) \rightarrow \mathcal{F}_{x}
$$

3. As $V_{t}$ ranges over neighborhoods of $x$ on which is is possible to represent $\bar{t}$, the family of maps $\psi_{t}$ are compatible in such a way that they pass to a map on the direct limit

$$
\psi=\underset{V_{t}}{\lim } \psi_{t}: \xrightarrow{\lim } \mathcal{F}^{\#}\left(V_{t}\right)=\mathcal{F}_{x}^{\#} \rightarrow \mathcal{F}_{x}
$$

4. The map $\psi: \mathcal{F}_{x}^{\#} \rightarrow \mathcal{F}_{x}$ is inverse to $\phi_{x}$.

I'm far too lazy to work these out, so you get to do it on your own.
Proposition 6.66. Let $\mathcal{F}$ be a presheaf of abelian groups on a space $X$, and let $\mathcal{F}^{+}=\left(\mathcal{F}^{\#}\right)^{\#}$ be the associated sheaf, and $\theta: \mathcal{F} \rightarrow \mathcal{F}^{+}$the composition of two maps following Definition 6.60. Then $\left(\mathcal{F}^{+}, \theta\right)$ satisfy the universal property of sheafification, which is to say, given any sheaf $\mathcal{G}$ of abelian groups on $X$ and a morphism of presheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$, there exists a unique factorization through $\theta$, that is, there exists a unique morphism $\psi: \mathcal{F}^{+} \rightarrow \mathcal{G}$ making the following diagram commute.


Proof. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves, with $\mathcal{G}$ a sheaf. By functoriality of Čech cohomology (Proposition 6.48), we have a morphism $\phi^{\#}: \mathcal{F}^{\#} \rightarrow \mathcal{G}^{\#}$ of presheaves. It fits into the commutative below, where $\alpha: \mathcal{F} \rightarrow \mathcal{F}^{\#}$ and $\beta: \mathcal{G} \rightarrow \mathcal{G}^{\#}$ are the canonical maps of Definition 6.60


Following the discussion in Example 6.35, since $\mathcal{G}$ is a sheaf, the canonical map $\mathcal{G}(U) \rightarrow$ $\mathcal{G}^{\#}(U)$ is an isomorphism, so we obtain a morphism $\beta_{U}^{-1} \phi_{U}^{\#}: \mathcal{F}^{\#}(U) \rightarrow \mathcal{G}(U)$, giving a morphism of presheaves $\beta^{-1} \phi^{\#}: \mathcal{F}^{\#} \rightarrow \mathcal{G}$. Iterating this process, we obtain a morphism of sheaves $\psi: \mathcal{F}^{+} \rightarrow \mathcal{G}$ which is precisely the needed factorization.

All that remains is to establish unqiueness of $\psi$. This is essentially a formal consequence of the fact that $\theta$ induces an isomorphism on stalks. Since $\theta$ induces isomorphisms on stalks, if $\psi, \psi^{\prime}$ are two morphisms $\mathcal{F}^{+} \rightarrow \mathcal{G}$ which compose with $\theta$ to give the same morphism, they are the same on stalks of $\mathcal{F}^{+}$, but then they are the same morphism by Corollary 4.3.

This completes the proof of Theorem 6.61, which we restate for convenience.

Theorem 6.67. Let $X$ be a topological space and $\mathcal{F}$ a presheaf of abelian groups on $X$.

1. $\mathcal{F}^{\#}$ is a separated presheaf on $X$.
2. If $\mathcal{F}$ is separated, then $\mathcal{F}^{\#}$ is a sheaf on $X$. (Hence $\left(\mathcal{F}^{\#}\right)^{\#}$ is a sheaf on $X$ for any presheaf $\mathcal{F}$.)
3. The map defined above $\mathcal{F} \rightarrow \mathcal{F}^{\#}$ induces an isomorphism on stalks. (Hence repeating the construction $\mathcal{F} \rightarrow \mathcal{F}^{\#} \rightarrow\left(\mathcal{F}^{\#}\right)^{\#}$ induces an isomorphism on stalks as well.)
4. $\left(\mathcal{F}^{\#}\right)^{\#}$ satisfies the universal property of sheafification for $\mathcal{F}$.

## 7 Representable functors and Yoneda's lemma

This section is material covered in a guest presentation for the class by another student. The material in this section is not critical to anything after it in these notes, so it may be reasonably skipped over.

In this section, we build up to the statement of Yoneda's lemma with some definitions and examples using algebras over a field. We skip any proof of the lemma. Then we give some application to developing the category of affine group schemes and their connection to the category of Hopf algebras.

Definition 7.1. A category is locally small if between any two objects $X, Y$, the morphisms $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ form a set.

Definition 7.2. Let $\mathcal{C}$ be a locally small category, and let $X$ be an object in $\mathcal{C}$. The associated covariant functor to $X$ is

$$
h_{X}: \mathcal{C} \rightarrow \text { Set } \quad Y \mapsto \operatorname{Hom}_{\mathcal{C}}(X, Y)
$$

This describes the functor $h_{X}$ on objects. Given a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(Y, Z), h_{X}$ acts on $f$ as follows.

$$
h_{X}: \operatorname{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \operatorname{Hom}_{\text {Set }}\left(\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, Z)\right) \quad f \mapsto(\phi \mapsto f \circ \phi)
$$

More generally, if the homomorphism sets of $\mathcal{C}$ are objects in some category $\mathcal{D}$ (say the category of groups), then an object $X$ of $\mathcal{C}$ has an associated functor $h_{X}: \mathcal{C} \rightarrow \mathcal{D}$.

Definition 7.3. A functor $\mathcal{C} \rightarrow$ Set is representable if it is naturally isomorphic to a functor $h_{X}$ for some object $X$ in $\mathcal{C}$. For such a functor, the object $X$ is called the representing object.

More generally, if $\mathcal{D}$ is a category and hom sets $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ are objects in the category $\mathcal{D}$, then a functor $\mathcal{C} \rightarrow \mathcal{D}$ is representable if it is naturally isomorphic to some $h_{X}$ as a functor $\mathcal{D} \rightarrow \mathcal{D}$.

In this section, all of our examples will focus on the category $\mathcal{C}=\operatorname{Alg}_{k}$, the category of commutative unital associative $k$-algebras, where $k$ is a fixed field. In all of the following examples, we are somewhat lazy in justifying things. We justify the fact that each functor is representable only in that we describe how the functor on objects gives isomorphisms, but we do not fully describe the naturality of these isomorphisms.

Example 7.4. The trivial functor

$$
\operatorname{Alg}_{k} \rightarrow \text { Set } \quad A \mapsto\{*\}
$$

is representable, with representing object $k$, viewed as an algebra over itself. This is because for any $k$-algebra $A$, there is a unique $k$-algebra map $k \rightarrow A$ which is determined by sending $1 \in k$ to $1 \in A$.

$$
h_{k}: \operatorname{Alg}_{k} \rightarrow \operatorname{Set} A \mapsto \operatorname{Hom}_{k}(k, A) \cong\{*\}
$$

Example 7.5. The forgetful functor

$$
\operatorname{Alg}_{k} \rightarrow \text { Set } \quad A \mapsto A
$$

is representable, with representing object $k[x]$, the polynomial ring in one variable. This is because for any $k$-algebra $A$, a $k$-algebra map $k[x] \rightarrow A$ is determined by the image of $x$, which can be any element of $A$.

$$
h_{k[x]}(A)=\operatorname{Hom}_{k}(k[x], A) \cong A \quad \phi \leftrightarrow \phi(x)
$$

Example 7.6. The functor

$$
\operatorname{Alg}_{k} \rightarrow \text { Set } \quad A \mapsto A^{\times}
$$

which takes an algebra to its group of multiplicative units is also representable, with representing $k$-algebra

$$
k[x, y] /(x y-1) \cong k\left[x, x^{-1}\right]
$$

This is because a $k$-algebra map $k\left[x, x^{-1}\right] \rightarrow A$ is determined by the image of $x$, along with the fact that $x$ must be mapped to a unit.

Example 7.7. Let $h_{A}, h_{B}$ be representable functors $\mathrm{Alg}_{k} \rightarrow$ Set, with representing $k$ algebras $A, B$ respectively. Then the functor

$$
h_{A} \times h_{B}: \operatorname{Alg}_{k} \rightarrow \text { Set } \quad C \mapsto h_{A}(C) \times h_{B}(C)
$$

is representable, with representing $k$-algebra $A \otimes_{k} B$. (On the right side, $h_{A}(C) \times h_{B}(C)$ is just the cartesian product of sets, which is the categorical product in that categoy.) This is because of the well-known natural isomorphism

$$
\operatorname{Hom}_{k}\left(A \otimes_{k} B, C\right) \cong \operatorname{Hom}_{k}(A, C) \times \operatorname{Hom}_{k}(B, C)
$$

By natural isomorphism, we mean that there is a natural isomorphism of bifunctors

$$
\operatorname{Hom}_{k}\left(-\otimes_{k}-, C\right) \cong \operatorname{Hom}_{k}(-, C) \times \operatorname{Hom}_{k}(-, C)
$$

Definition 7.8. Given a locally small category $\mathcal{C}$, let $\check{\mathcal{C}}$ be the category of covariant functors $\mathcal{C} \rightarrow$ Set, with morphisms given by natural transformations. We may then regard $h$ as a (contravariant) functor

$$
h: \mathcal{C} \rightarrow \check{\mathcal{C}} \quad X \mapsto h_{X}
$$

Alternatively, we may view $h$ as a covariant functor to the opposite category $\check{\mathcal{C}}^{\text {opp }}$.

$$
h: \mathcal{C} \rightarrow \check{\mathcal{C}}^{\text {opp }} \quad X \mapsto h_{X}
$$

Definition 7.9. Let $\mathcal{C}, \mathcal{D}$ be locally small categories and $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ a functor. For every pair of objects $X, Y$ in $\mathcal{C}, \mathcal{F}$ induces a function

$$
\mathcal{F}_{X, Y}: \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(X, Y)
$$

The functor $\mathcal{F}$ is full if for every $X, Y, \mathcal{F}_{X, Y}$ is surjective. The functor $\mathcal{F}$ is faithful if for every $X, Y, \mathcal{F}_{X, Y}$ is injective. If it is both, we say $\mathcal{F}$ is fully faithful.

Definition 7.10. A functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective if for any object $Z$ of $\mathcal{D}$, there exists an object $X$ of $\mathcal{C}$ such that $\mathcal{F}(X) \cong Z$. That is to say, not every object of $\mathcal{D}$ is in the "image" of $\mathcal{F}$, but every isomorphism class of objects of $\mathcal{D}$ is in the "image."

Remark 7.11. This is a distraction at this point, but it is an important fact that a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Lemma 7.12 (Yoneda lemma). The functor $h: \mathcal{C} \rightarrow \check{\mathcal{C}}$ is fully faithful. That is, for any objects $X, Y$ of $\mathcal{C}, h$ induces an isomorphism

$$
\operatorname{Hom}_{\mathcal{C}}(X, Y) \cong \operatorname{Hom}_{\mathcal{C}}\left(h_{Y}, h_{X}\right)
$$

More generally, for any object $\mathcal{F}$ of $\check{\mathcal{C}}, h$ induces an isomorphism

$$
\mathcal{F}(Y) \cong \operatorname{Hom}_{\check{\mathcal{C}}}\left(h_{Y}, \mathcal{F}\right)
$$

Proof. The proof isn't actually that hard. You basically just need to keep track of everything and define the logical maps to give inverses to the maps induced by $h$. Despite this, the proof is not that illuminating so we skip it.

Remark 7.13. The philosophy behind Yoneda's lemma is that it tells us a lot about representable functors and morphisms between them. In general, it is very difficult to get a handle on all of the possible natural transformations between two functors. However, what Yoneda's lemma says is that if we have two representable functors, then we can understand all of the natural transformations between them by understanding the morphisms between their representating objects, which is usually much more attainable.

### 7.1 Application of Yoneda - affine group schemes and Hopf algebras

Definition 7.14. Fix a field $k$, and let Gp be the category of groups. An affine group scheme over $k$ is a representable functor $\mathrm{Alg}_{k} \rightarrow \mathrm{Gp}$.

Example 7.15. Let $h_{A}: \operatorname{Alg}_{k} \rightarrow$ Gp be an affine group scheme over $k$, with representing $k$-algebra $A$. We have the usual structure maps associated with $A$ :

$$
\begin{array}{rll}
A \otimes_{k} A \rightarrow A & a \otimes b \mapsto a b & \text { (bilinear) multiplication } \\
A \rightarrow A & a \mapsto-a & \text { additive inversion } \\
k \rightarrow A & \lambda \mapsto \lambda 1_{A} & \text { multiplicative identity }
\end{array}
$$

We want to see how the fact that $A$ represents a functor $\operatorname{Alg}_{k} \rightarrow \mathrm{Gp}$ also induces a lot of additional structure on $A$, which is in some sense "dual" to the above algebra structure. To save some space, in the diagrams to follow we denote $h_{A}$ merely by $h$. For any $k$-algebra $B$, we know that $h B=h_{A}(B)$ is a group, so it has a multiplication map

$$
m_{B}: h B \times h B \rightarrow h B \quad(x, y) \mapsto x y
$$

Also, given a morphism $\phi: B \rightarrow C$ of $k$-algebras, the fact that $h$ is a functor (in particular, $h \phi$ is a group homomorphism) means that $m_{B}, m_{C}$ are compatible in the sense that the following diagram commutes.


Similary, $h B$ has an inversion map

$$
i_{B}: h B \rightarrow h B \quad x \mapsto x^{-1}
$$

which fits into a commutative diagram


Also, $h B$ has a special element (the identity), which we can represent categorically as a map from the trivial group $\{e\}$ to $h B$.

$$
\epsilon_{B}:\{e\} \rightarrow h B
$$

fitting into a commutative diagram


Each of these diagrams respectively says that $m, i, \epsilon$ are natural transformations.

$$
\begin{aligned}
m & : h \times h \rightarrow h \\
i & : h \rightarrow h \\
\epsilon & : \text { Trivial functor } \rightarrow h
\end{aligned}
$$

Based on our examples from the previous section, the functor $h \times h$ and the trivial functor are representable, in particular, $h \times h$ is represented by $A \otimes_{k} A$, and the trivial functor is represented by $k$ (viewed as a $k$-algebra). Hence by Yoneda's lemma, these natural transformations correspond to $k$-algebra homomorphisms

$$
\begin{aligned}
m^{*} & : A
\end{aligned} \rightarrow A \otimes_{k} A, ~ \begin{aligned}
i^{*} & : A
\end{aligned} \rightarrow A
$$

We have not written them out, but one can write out various properties for these maps, which are formally induced by various commutative diagrams associated with $h_{A}$ being a functor. For all the usual properties of $A$ (such as having a unit, associativity, etc.) there is a "dual" property of the maps above, with names like counit, coassociativity, etc. The technical word to summarize all of this is that the maps $m^{*}, i^{*}, \epsilon^{*}$ give $A$ the structure of a coalgebra, which the interested reader can read more about.

In fact, such an algebra $A$ representing a functor $\mathrm{Alg}_{k} \rightarrow \mathrm{Gp}$ is not merely an algebra and coalgebra simultaneously. It is a bialgebra, which just means that a few technical compatibilities are imposed on the relationship between the algebra and coalgebra structures. Actually, there is even more structure on $A$ - it has the properties of being a Hopf algebra, which is a bialgebra with even more structure.

Remark 7.16. In the previous example, we demonstrated that the $k$-algebra representing an affine group scheme over $k$ (a representable functor $\mathrm{Alg}_{k} \rightarrow \mathrm{Gp}$ ) has a lot more structure imposed upon it, to the point where $A$ is given the structure of a Hopf algebra. This gives a functor from the category of affine group $k$-schemes to the category of Hopf $k$-algebras.

If one is careful and has the appropriate definitions in hand, one can trace back the construction. Basically, one observes that given a Hopf $k$-algebra $A$, one can take the associated functor $h_{A}: \operatorname{Alg}_{k} \rightarrow$ Set, $B \mapsto \operatorname{Hom}_{k}(A, B)$ and use the various commutative diagrams codifying the Hopf algebra properties to see that the set $h_{A}(B)$ has a reasonable group structure which makes $h_{A}$ into a functor $\mathrm{Alg}_{k} \rightarrow \mathrm{Gp}$. That is, a Hopf $k$-algebra defines an affine group scheme, via the functor $h$ of the Yoneda lemma.

Finally, as one would hope in the best of all possible worlds, the two constructions above of functors we have described are quais-inverses, meaning that they give an equivalence of categories.

$$
\{\text { affine group } k \text {-schemes }\} \leftrightarrow\{\text { Hopf } k \text {-algebras }\}
$$

We state this in the following theorem, without much in the way of justification other than this discussion.

Theorem 7.17. Fix a field $k$. The category of affine group schemes over $k$ is equivalent to the category of Hopf algebras over $k$.

## 8 Functors between categories of sheaves

So far, we have dealt with the category $\operatorname{Sh}(X)$ (or $\operatorname{PSh}(X)$ ), the category of sheaves (or presheaves) of abelian groups on a fixed topological space $X$. Now, we will consider how a continuous map $f: X \rightarrow Y$ of topological spaces induces various functors between $\operatorname{Sh}(X)$ and $\operatorname{Sh}(Y)$. We will discuss several such functors, and various adjunction properties between them. We give a quick list as a preview. Fix a map $f: X \rightarrow Y$.

1. Direct image functor $f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$, also sometimes called pushforward.
2. Inverse image functor $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$, also sometimes called pullback.
3. Extension by zero - when $f$ is injective and the image is closed or open in $Y$, we have $f_{!}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$. This is read as " $f$ lower shriek."
4. Exceptional image functor $f^{!}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$, read as " $f$ upper shriek."

As mentioned, perhaps the most important property of these functors is various adjunctions, for example, $\left(f^{-1}, f_{*}\right)$ are an adjoint pair.

### 8.1 Direct image functor

Definition 8.1 (Direct image functor). Let $f: X \rightarrow Y$ be a continuous map of topological spaces. We will define the direct image functor $f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$. First, we define it on objects. We can even just define it on presheaves. Let $\mathcal{F}$ be a presheaf of abelian groups on $X$. For $V \subset Y$, define a presheaf $f_{*} \mathcal{F}$ on $Y$ by

$$
\left(f_{*} \mathcal{F}\right)(V)=\mathcal{F}\left(f^{-1}(V)\right)
$$

The restriction maps for $f_{*} \mathcal{F}$ are defined as follows. For $V_{1} \subset V_{2} \subset Y$ open sets, we have $f^{-1}\left(V_{1}\right) \subset f^{-1}\left(V_{2}\right) \subset X$ open subsets, so we may define the restriction map $\rho_{V_{1}}^{V_{2}}\left(f_{*} \mathcal{F}\right)$ by commutativity of the following diagram.


There are some quick things to check to verify that $f_{*} \mathcal{F}$ as defined is a presheaf, but these basically come from the analogous properties of $\mathcal{F}$. Similarly, one can verify that if $\mathcal{F}$ is a sheaf, then $f_{*} \mathcal{F}$ is a sheaf. This completes our description of the functor $f_{*}$ acting on objects of $\operatorname{Sh}(X)$.

Now we discuss how $f_{*}$ acts on morphisms. Let $\phi: \mathcal{F}_{1} \rightarrow \mathcal{F}_{2}$ be a morphism of sheaves (of abelian groups) on $X$. So for every $U \subset X$ open, we have a map $\phi_{U}: \mathcal{F}_{1}(U) \rightarrow \mathcal{F}_{2}(U)$ fitting
into an appropriate commutative square. To define a morphism $f_{*} \phi=\phi_{*}: f_{*} \mathcal{F}_{1} \rightarrow f_{*} \mathcal{F}_{2}$, it suffices to define

$$
\left(\phi_{*}\right)_{V}:\left(f_{*} \mathcal{F}_{1}\right)(V) \rightarrow\left(f_{*} \mathcal{F}_{2}\right)(V)
$$

for each $V \subset Y$ open. We define $\left(\phi_{*}\right)_{V}$ by commutativity of the following diagram.


It is immediate to verify that this makes $\phi_{*}$ into a morphism of sheaves. After some additional mild verification, this definition makes $f_{*}$ into a covariant functor, $\operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$. This is called the direct image functor associated to $f$, or sometimes called the pushforward of $f$.

Example 8.2 (Direct image functor associated to inclusion of a point). Let $X$ be a topological space and $x \in X$. Consider the inclusion map $\iota:\{x\} \hookrightarrow X$, which is continuous. Let $S$ be an abelian group, and let $\mathcal{F}$ be the constant sheaf on $\{x\}$ with value group $S$. That is,

$$
\mathcal{F}(U)= \begin{cases}S & U=\{x\} \\ \{0\} & U=\emptyset\end{cases}
$$

What is the pushforward of $\mathcal{F}$ under $\iota$ ? We compute, since there are only two simple cases. Let $V \subset X$ be open.

$$
\begin{aligned}
\left(\iota_{*} \mathcal{F}\right)(V) & =\mathcal{F}\left(\iota^{-1}(V)\right) \\
& = \begin{cases}\mathcal{F}(\{x\}) & x \in V \\
\mathcal{F}(\emptyset) & x \notin V\end{cases}
\end{aligned} \quad=\left\{\begin{array}{ll}
S & x \in V \\
\{0\} & x \notin V
\end{array}\right] .
$$

Hence $\iota_{*} \mathcal{F}$ is the skyskraper sheaf on $X$ concentrated at $x$ (with value group $S$ ).
Example 8.3 (Pushforward of locally constant sheaf need not be locally constant). In this example, we will see that for the right function $f$, the pushforward of a locally constant sheaf need not be locally constant. This will also serve as an example of how two sheaves may have the same stalks at each point, and not be isomorphic as sheaves.

Let $X, Y$ both be the unit circle $S^{1}=\{z \in \mathbb{C}:|z|=1\} \subset \mathbb{C}$. Consider the two sheeted covering

$$
f: X \rightarrow Y \quad z \mapsto z^{2}
$$

Let $S$ be a nontrivial abelian group, and let $\mathcal{F}$ be the locally constant sheaf on $X$ with value group $S$. That is, for $U \subset X, \mathcal{F}(U)$ is the group of locally constant functions $U \rightarrow S$. Hence if $U$ has $n$ connected components, then

$$
\mathcal{F}(U) \cong S \times \cdots \times S
$$

with an $n$-fold product. Given an arbitrary open subset $U \subset Y$, we do not attempt to describe $\left(f_{*} \mathcal{F}\right)(U)$, since this is a bit complicated. Instead, we will just attempt to describe the stalks of the pushforward sheaf $f_{*} \mathcal{F}$. Given $y \in Y$, there is a sufficiently small neighborhood $V$ of $y$ such that $f^{-1}(V)$ has exactly two connected components. Hence

$$
\left(f_{*} \mathcal{F}\right)\left(f^{-1}(V)\right) \cong S \times S
$$

Also, any smaller neighborhood of $y$ will still have a preimage with two connected components. So taking the direct limit over shrinking neighborhoods of $y$, we see that the stalk $\left(f_{*} \mathcal{F}\right)_{y}$ is two copies of $S$.

$$
\left(f_{*} \mathcal{F}\right)_{y}=\lim _{y \in V} \mathcal{F}\left(f^{-1}(V)\right) \cong S \times S
$$

Now, on the other hand, consider the locally constant sheaf $\mathcal{G}$ on $Y$ with value group $S \times S$. From previous examples (long ago), we know that the stalk of $\mathcal{G}$ at any point is $S \times S$.

$$
\mathcal{G}_{y} \cong S \times S
$$

So for any point $y \in Y$, the stalks $\mathcal{G}_{y}$ and $\left(f_{*} \mathcal{F}\right)_{y}$ are isomorphic. Yet we claim that these sheaves are not isomorphic. This is immediate from considering global sections. The global sections of $\mathcal{G}$ are

$$
\mathcal{G}(Y) \cong S \times S
$$

But the global sections of $f_{*} \mathcal{F}$ are just one copy of $S$, since $f^{-1}(Y)=X$ is connected.

$$
f_{*} \mathcal{F}(Y) \cong S \not \approx \mathcal{G}(Y)
$$

Thus $f_{*} \mathcal{F}, \mathcal{G}$ are not isomorphic sheaves. If $f_{*} \mathcal{F}$ was a locally constant sheaf, it would have to be $\mathcal{G}$, since we know what the stalks of $f_{*} \mathcal{F}$ are, but this is not the case. Hence $f_{*} \mathcal{F}$ is not a locally constant sheaf. This provides our example of a pushforward of a locally constant sheaf which is not locally constant.

As mentioned at the beginning, this example also gives a subtle example of how stalks do not entirely determine the sheaf. From a previous result, we know that if a morphism of sheaves induces isomorphisms on every stalk, then it is an isomorphism of sheaves. In this sense, stalks "determine" the sheaf.

However, in the above example, we saw two sheaves $f_{*} \mathcal{F}$ and $\mathcal{G}$ which do have the same stalks everywhere, yet are not the same. This is because the isomorphisms between stalks are not induced by a morphism of sheaves. In fact, they cannot be, since then the previous result would force them to be isomorphic as sheaves, which they are not.

Proposition 8.4 (Direct image functor is left exact). Let $X, Y$ be topological spaces.

1. For any continuous map $f: X \rightarrow Y$, the direct image functor $f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$ is left exact.
2. If $\iota: X \hookrightarrow Y$ is injective with closed image, then the direct image functor $\iota_{*}$ is exact.

Proof. (1) Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of sheaves (of abelian groups) on $X$. We need to show that the sequence

$$
0 \rightarrow f_{*} \mathcal{F} \rightarrow f_{*} \mathcal{G} \rightarrow f_{*} \mathcal{H}
$$

is exact. Following section 4.2, we know that for any $U \subset X$, the sequence

$$
0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)
$$

is a short exact sequence of abelian groups. Hence for $V \subset Y$ open, we get that the sequence

$$
0 \rightarrow \mathcal{F}\left(f^{-1}(V)\right) \rightarrow \mathcal{G}\left(f^{-1}(V)\right) \rightarrow \mathcal{H}\left(f^{-1}(V)\right)
$$

is exact, but this is the same as the sequence

$$
0 \rightarrow\left(f_{*} \mathcal{F}\right)(V) \rightarrow\left(f_{*} \mathcal{G}\right)(V) \rightarrow\left(f_{*} \mathcal{H}\right)(V)
$$

So $0 \rightarrow f_{*} \mathcal{F} \rightarrow f_{*} \mathcal{G} \rightarrow f_{*} \mathcal{H}$ is exact as a sequence of presheaves, hence exact as a sequence of sheaves.
(2) Let $\iota: X \hookrightarrow Y$ be injective with closed image. As $\iota$ is injective, we identify $X$ with $\iota(X)$. Let $\mathcal{F}$ be a sheaf on $X$. First, we describe the stalks of $\iota_{*} \mathcal{F}$. For $y \in Y$ if $y \notin X$, since $X$ is closed, there exists $V \subset Y$ with $y \in V$ and $V \cap X=\emptyset$. Then

$$
\left(\iota_{*} \mathcal{F}\right)(V)=\mathcal{F}\left(\iota^{-1}(V)\right)=\mathcal{F}(\emptyset)=0
$$

where 0 represents the trivial group. Hence taking the direct limit over shrinking neighborhoods of $y$, we see that the stalk at $y$ is zero, if $y \notin X$. On the other hand, if $y \in X$, then the stalk of $\iota_{*} \mathcal{F}$ is just the stalk of $\mathcal{F}$. Putting this together,

$$
\left(\iota_{*} \mathcal{F}\right)_{y}= \begin{cases}\mathcal{F}_{y} & y \in X \\ 0 & y \notin X\end{cases}
$$

Now we return to exactness. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of sheaves on $X$. Then the sequence on stalks is exact for any $x \in X$.

$$
0 \rightarrow \mathcal{F}_{x} \rightarrow \mathcal{G}_{x} \rightarrow \mathcal{H}_{x} \rightarrow 0
$$

We want to show that $0 \rightarrow \iota_{*} \mathcal{F} \rightarrow \iota_{*} \mathcal{G} \rightarrow \iota_{*} \mathcal{H} \rightarrow 0$ is exact. For $y \in Y$, consider its sequence of stalks.

$$
0 \rightarrow\left(\iota_{*} \mathcal{F}\right)_{y} \rightarrow\left(\iota_{*} \mathcal{G}\right)_{y} \rightarrow\left(\iota_{*} \mathcal{H}\right)_{y} \rightarrow 0
$$

If $y \in X$, then this is exact since it is identical to the previous sequence, and if $y \notin X$, then all terms in this are zero, so it is trivially exact. Hence $\iota_{*}$ is exact.

Definition 8.5. Let $\mathcal{F}$ be a sheaf on $X$, and let $f: X \rightarrow Y$ be a continuous map. We want to describe an induced map

$$
\check{H}^{n}\left(Y, f_{*} \mathcal{F}\right) \rightarrow \check{H}^{n}(X, \mathcal{F})
$$

First, let $\mathcal{V}=\left\{V_{i}\right\}_{i \in I}$ be an open cover of $Y$, and let $\mathcal{U}=\left\{U_{i}=f^{-1}\left(V_{i}\right)\right\}_{i \in I}$ be the induced open cover of $X$. Note that for $i_{0}, \cdots, i_{n} \in I$, we have

$$
U_{i_{0} \cdots i_{n}}=f^{-1}\left(V_{i_{0} \cdots i_{n}}\right)
$$

using basic properties of preimages, hence

$$
\left(f_{*} \mathcal{F}\right)\left(V_{i_{0} \cdots i_{n}}\right)=\mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)
$$

So the corresponding groups of Čech $n$-cochains are equal.

$$
\check{C}^{n}(\mathcal{U}, \mathcal{F})=\check{C}^{n}\left(\mathcal{V}, f_{*} \mathcal{F}\right)
$$

They also have the same Čech differentials, since these are given by "the same" combinatorial formula. Hence we obtain an isomorphism on cover-specific Čech cohomology groups.

$$
\check{H}^{n}(\mathcal{U}, \mathcal{F}) \cong \check{H}^{n}\left(\mathcal{V}, f_{*} \mathcal{F}\right)
$$

Then composing with the canonical map to the direct limit, we obtain a map

$$
\check{H}^{n}\left(\mathcal{V}, f_{*} \mathcal{F}\right) \xrightarrow{\cong} \check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{n}(X, \mathcal{F})
$$

Modulo some tedious verification, these maps are compatible with refinements of covers, hence pass a map on the direct limit.

$$
\check{H}^{n}\left(Y, f_{*} \mathcal{F}\right) \rightarrow \check{H}^{n}(X, \mathcal{F})
$$

Note that despite the many isomorphisms on cover-specific cohomology groups, this map is in general not an isomorphism. The main reason for this is that every open cover of $Y$ gives an open cover of $X$ by taking preimages, but not every open cover of $X$ arises in this way. Furthermore, two different open covers of $Y$ may give the same open cover of $X$. Hence the direct limits on each side are generally over very different indexing sets.

However, if $f$ is a homeomorphism, then $f$ does give a bijection between open covers of $X$ and $Y$ in this way, hence the induced map above on Čech cohomology is an isomorphism. In the next definition and remark, we give a slightly weaker condition on $f$ for this to happen.

Definition 8.6. Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Then $f$ induces a map

$$
\tilde{f}:\{\text { open covers of } Y\} \rightarrow\{\text { open covers of } X\} \quad\left\{V_{i}\right\}_{i \in I} \mapsto\left\{f^{-1}\left(V_{i}\right)\right\}_{i \in I}
$$

We say the map $f$ is a cover isomorphism if the induced map $\tilde{f}$ is a bijection.

Example 8.7. If $f$ is a homeomorphism, then $f$ is a cover isomorphism.
Example 8.8. Let $X, Y$ be nonempty sets, each with the trivial topology (the only open sets are the whole set and the empty set). Let $f: X \rightarrow Y$ be any set map. Then $f$ is continuous, and $f$ is a cover isomorphism.

Remark 8.9. It is mildly tempting to think that a cover isomorphism must have some structure approximating a homeomorphism, but the previous somewhat silly example demonstrates that a cover isomorphism need not be open, surjective, or injective.

Proposition 8.10. If $f: X \rightarrow Y$ is a cover isomorphism, and $\mathcal{F}$ is a sheaf on $X$, then the induced map $\check{H}^{n}\left(Y, f_{*} \mathcal{F}\right) \rightarrow \check{H}^{n}(X, \mathcal{F})$ is an isomorphism.

Proof. Since $f$ gives a bijection between open covers, in a way which is compatible with refinements as previously asserted (without proof, sorry), then the induced map $\widetilde{f}$ gives an isomorphism of directed systems (of open covers with refinements). This gives rise to an isomorphism of directed systems between abelian groups, $H^{n}(\mathcal{U}, * \mathcal{F})$ with $\mathcal{U}$ ranging over open covers of $Y$, and $\check{H}^{n}(\mathcal{V}, \mathcal{F})$ with $\mathcal{V}$ ranging over open covers of $X$. Passing to the direct limit, this induces an isomorphism on the direct limit, which is the claimed isomorphism.

### 8.2 Inverse image functor

Unfortunately, the inverse image functor is much less simple to describe than the direct image functor. However, since we have built up enough understanding of abstract direct limits and sheafification, it is not so bad.

The general intuition is as follows. Let $f: X \rightarrow Y$ be continuous. We want a functor $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$. Recall that for a sheaf $\mathcal{F}$ on $X$, we define

$$
\left(f_{*} \mathcal{F}\right)(V)=\mathcal{F}\left(f^{-1}(V)\right) \quad V \subset Y \text { open }
$$

Given a sheaf $\mathcal{G}$ on $Y$, the logical analogous definition would be

$$
\left(f^{-1} \mathcal{G}\right)(U)=\mathcal{G}(f(U)) \quad U \subset X \text { open }
$$

However, there is an issue with this, which is that $f(U)$ may or may not be an open subset of $Y$. If $f$ is an open map, this definition works, and we will actually show that our slightly more complicated definition agrees with this in that particular case.

However, if $f(U)$ is not open, we need to somehow "approximate" $f(U)$ by open subsets of $Y$. The natural way to do this is using a direct limit over all open subsets of $Y$ containin $f(U)$. So instead of $f(U)$, we use a direct limit.

Definition 8.11 (Inverse image functor on objects). Let $f: X \rightarrow Y$ e a continuous map of topological spaces. We will define the inverse image functor $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$, just on objects at this time. Let $\mathcal{G}$ be a sheaf (of abelian groups) on $Y$. First, we need to define
an auxiliary presheaf, which is not going to be our final definition of $f^{-1} \mathcal{G}$. For $U \subset X$ open, define

$$
\mathcal{F}(U)=\lim _{\substack{f(U) \subset V \\ V \text { open }}} \mathcal{G}(V)
$$

The direct limit is taken over all $V \subset Y$ which are open and contain $f(U)$, the partial ordering on such subsets is by reverse inclusion, with maps being restriction maps. We want to make $\mathcal{F}$ a presheaf, so we need to define restriction maps for $\mathcal{F}$. Given $U_{1} \subset U_{2} \subset X$, it is clear that

$$
\left\{V \subset Y \text { open, } f\left(U_{1}\right) \subset V\right\} \subset\left\{V \subset Y \text { open, } f\left(U_{2}\right) \subset V\right\}
$$

so one directed system is actually just a subset of the other. So we obtain a map

Some verification is required, but these restriction maps make $\mathcal{F}$ into a presheaf (of abelian groups) on $Y$. However, now we run into a second issue - we have no reason to expect that $\mathcal{F}$ as defined is a sheaf. In order to fix that, we "simply" sheafify it. Let $\mathcal{F}^{+}$be the sheafification of $\mathcal{F}$. Then we define the inverse image of $\mathcal{G}$ to be

$$
f^{-1} \mathcal{G}:=\mathcal{F}^{+}
$$

This defines part of a functor $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$, called the inverse image functor or sometime pullback. We will define it on morphisms a bit later.

Example 8.12 (Inverse image for open map). Let $f: X \rightarrow Y$ be an open map. Then for $U \subset X, f(U) \subset Y$ is open. So for any sheaf $\mathcal{G}$ on $Y$, the associated sheaf $\mathcal{F}$ as described in the previous definition is

$$
\mathcal{F}(U)=\underset{f(U) \subset V}{\lim _{f}} \mathcal{G}(V)=\mathcal{G}(f(U))
$$

In this case, $\mathcal{F}$ is a sheaf, by transferring properties of $\mathcal{G}$, so the sheafification is just $\mathcal{F}$. Hence

$$
f^{-1} \mathcal{G}=\mathcal{F}^{+}=\mathcal{F}
$$

Example 8.13 (Inverse image for inclusion of a point). Let $X$ be a space and $x \in X$. Let $\iota:\{x\} \hookrightarrow X$ be the inclusion. Let $\mathcal{G}$ be a sheaf on $X$. Then the auxiliary sheaf $\mathcal{F}$ is defined by

$$
\mathcal{F}(\{x\})=\lim _{x \in V} \mathcal{G}(V)=\mathcal{G}_{x}
$$

Hence the pullback $\iota^{-1} \mathcal{G}$ is the constant sheaf on $\{x\}$ with value group equal to the stalk $\mathcal{G}_{x}$.
Definition 8.14 (Inverse image functor on morphisms). Now we describe the inverse image functor on morphisms. Let $f: X \rightarrow Y$ be a continuous map, and let $\phi: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ be a morphism of sheaves on $Y$. So for $V \subset Y$ open, we have a map (of abelian groups)

$$
\phi_{V}: \mathcal{G}_{1}(V) \rightarrow \mathcal{G}_{2}(V)
$$

which is compatible with restriction maps. Let $\mathcal{F}_{1}, \mathcal{F}_{2}$ be the auxiliary presheaves on $X$, defined by

$$
\begin{aligned}
& \mathcal{F}_{1}(U)=\underset{f(U) \subset V}{\lim _{f}} \mathcal{G}_{1}(V) \\
& \mathcal{F}_{2}(U)=\underset{f(U) \subset V}{\lim _{\vec{~}}} \mathcal{G}_{2}(V)
\end{aligned}
$$

Since the maps $\phi_{V}$ are compatible with restrictions, which are the maps of the above direct limits, using the maps $\phi_{V}$ we obtain a map $\phi_{U}$ as below.

$$
\phi_{U}=\underset{f(U) \subset V}{\lim _{f}} \phi_{V}: \mathcal{F}_{1}(U)=\underset{f(U) \subset V}{\lim _{\rightarrow}} \mathcal{G}_{1}(V) \rightarrow \mathcal{F}_{2}(U)=\underset{f(U) \subset V}{\lim _{\vec{C}}} \mathcal{G}_{2}(V)
$$

Modulo some verification, these maps $\phi_{U}$ give us a morphism of presheaves $\widetilde{f^{-1} \phi}: \mathcal{F}_{1} \rightarrow \mathcal{F}_{2}$. Let $\theta_{1}: \mathcal{F}_{1} \rightarrow \mathcal{F}_{1}^{+}, \theta_{2}: \mathcal{F}_{2} \rightarrow \mathcal{F}_{2}^{+}$be the respective sheafifications. Using the universal property of sheafification, we obtain a morphism of sheaves

$$
f^{-1} \phi: f^{-1} \mathcal{G}_{1}=\mathcal{F}_{1}^{+} \rightarrow f^{-1} \mathcal{G}_{2}=\mathcal{F}_{2}^{+}
$$

After some mild verification, one checks that this makes $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$ into a covariant functor.

Remark 8.15. Occasionally the functor $f^{-1}$ is also denoted $f^{*}$, but this is not technically correct. Properly speaking, the notation $f^{*}$ should only be used when $f: X \rightarrow Y$ is a morphism of schemes, in which case $\operatorname{Sh}(X), \operatorname{Sh}(Y)$ usually refer not to sheaves of abelian groups, but to sheaves of $\mathcal{O}_{X^{-}}, \mathcal{O}_{Y^{-}}$modules respectively. However, the definition/construction of $f^{*}$ (in that situation) is very similar to our description of $f^{-1}$ above.

Proposition 8.16 (Stalks of the inverse image sheaf). Let $f: X \rightarrow Y$ be a continuous map, and $\mathcal{G}$ be a sheaf (of abelian groups) on $Y$. Let $x \in X$. There is a natural isomorphism between stalks

$$
\left(f^{-1} \mathcal{G}\right)_{x} \cong \mathcal{G}_{f(x)}
$$

Here, the word "natural" means that if $\phi: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ is a morphism of sheaves on $Y$, then the following diagram commutes for any $x \in X$.


Proof. We will just describe the isomorphism, the naturality is a mild verification/diagram chase after the definitions are laid out. Let $\mathcal{F}$ be the auxiliary sheaf associated to $\mathcal{G}$ and $f$. Since $\mathcal{F}, \mathcal{F}^{+}=f^{-1} \mathcal{G}$ have isomorphic stalks, it suffices to describe an isomorphism

$$
\mathcal{F}_{x} \cong \mathcal{G}_{f(x)}
$$

for $x \in X$. Let $x \in X$, and let $U \subset X$ be an open neighborhood of $x$. If we have an open subset $V \subset Y$ with $f(U) \subset V$, then $f(x)=y \in V$. This gives a map

Modulo some verification, this passes to the direct limit over all $U \subset X$ containing $x$, giving a map

$$
\mathcal{F}_{x}=\lim _{x \subset U} \mathcal{F}(U)=\underset{x \subset U}{\lim } \underset{f(U) \subset V}{\lim } \rightarrow \mathcal{G}_{f(x)}
$$

We leave the verification that this map is an isomorphism to the reader.
Corollary 8.17 (Inverse image functor is exact). Let $f: X \rightarrow Y$ be a continuous map. The inverse image functor $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$ is exact.
Proof. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of sheaves (of abelian groups) on $Y$. Then for $y \in Y$, the sequence on stalks is exact.

$$
0 \rightarrow \mathcal{F}_{y} \rightarrow \mathcal{G}_{y} \rightarrow \mathcal{H}_{y} \rightarrow 0
$$

Hence for $x \in X$, using naturality of the isomorphism $(\mathcal{F})_{f(x)} \cong\left(f^{-1} \mathcal{F}\right)_{x}$ of the previous proposition, we obtain a commutative diagram as below.


Hence the lower sequence is also exact, so $0 \rightarrow f^{-1} \mathcal{F} \rightarrow f^{-1} \mathcal{G} \rightarrow f^{-1} \mathcal{H} \rightarrow 0$ is a short exact sequence of sheaves. Hence $f^{-1}$ is exact.

Now we give some description/discussion of the inverse image functor involving the étale space description of sheafification. We start by recalling the definition of fiber produt in the category of topological spaces.

Definition 8.18. Let $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ be continuous maps. The fiber product or pullback of $X$ and $Y$ (or more properly, of $f$ and $g$ ) with respect to $Z$ is the object $X \times_{Z} Y$ which is universal in the following diagram.


Concretely, we may describe it as

$$
X \times{ }_{Z} Y=\{(x, y) \in X \times Y: f(x)=g(y)\}
$$

The topology is given by the subspace topology from the product $X \times Y$. The maps to $X$ and $Y$ in the diagram above are just the projections to each component, respectively.

Definition 8.19. Let $f: X^{\prime} \rightarrow X$ be a continuous map and let $\pi: E \rightarrow X$ be a local homeomorphism. Recall that in this situation, we call $E$ the total space, and $X$ the base space. Let $E^{\prime}=X^{\prime} \times_{X} E$ be the fiber product, with projection maps $\pi^{\prime}: E^{\prime} \rightarrow X^{\prime}$ and $f^{\prime}: E^{\prime} \rightarrow E$.


In this situation, we say that $\pi^{\prime}$ is obtained by base change from $\pi$ via $f$.
Remark 8.20. Let $f: X^{\prime} \rightarrow X, \pi: E \rightarrow X$, and $\pi^{\prime}: E^{\prime} \rightarrow X^{\prime}$ be as above. For any $x^{\prime} \in X^{\prime}$, there is a bijection (just of sets)

$$
\left(\pi^{\prime}\right)^{-1}\left(x^{\prime}\right) \cong \pi^{-1}\left(f\left(x^{\prime}\right)\right)
$$

This follows from a simple diagram chase, which we now describe. Given $e^{\prime}=\left(x^{\prime}, e\right) \in$ $\left(\pi^{\prime}\right)^{-1}\left(x^{\prime}\right) \subset E^{\prime}$, we get $f^{\prime}\left(e^{\prime}\right)=e \in \pi^{-1}\left(f\left(x^{\prime}\right)\right.$. Conversely, given $e \in \pi^{-1}\left(f\left(x^{\prime}\right)\right)$, the element $\left(x^{\prime}, e\right) \in X^{\prime} \times E$ satisfies $f\left(x^{\prime}\right)=\pi(e)$, so we get $e^{\prime}=\left(x^{\prime}, e\right) \in\left(\pi^{\prime}\right)^{-1}\left(x^{\prime}\right)$. These processes are clearly inverse, so this gives the bijection above.

Lemma 8.21 (Local homeomorphisms preserved by base change). Let $f: X^{\prime} \rightarrow X$ be a continuous map, and $\pi: E \rightarrow X$ be a local homeomorphism. Form the fiber product $E^{\prime}=X^{\prime} \times_{X} E$, with projections $\pi^{\prime}: E^{\prime} \rightarrow X^{\prime}$ and $f^{\prime}: E^{\prime \prime} \rightarrow E$. Then $\pi^{\prime}$ is a local homeomorphism.

Proof. Let $e^{\prime}=\left(x^{\prime}, e\right) \in E^{\prime}=X^{\prime} \times_{X} E$. Note that $\pi(e)=f\left(x^{\prime}\right)$. We need to find an open neighborhood $U \subset E^{\prime}$ of $e^{\prime}$ such that $\left.\pi^{\prime}\right|_{U}: U \rightarrow \pi^{\prime}(U)$ is a homeomorphism. Since $\pi$ is a local homeomorphism, there is an open neighborhood $V \subset E$ of $e$ such that $\left.\pi\right|_{V}: V \rightarrow \pi(V)$ is a homeomorphism.

Then let $W=f^{-1}(\pi(V)) \subset X^{\prime}$. Since $\pi$ is an open map, $\pi(V)$ is open in $X$, so by continuity of $f, W$ is open in $X^{\prime}$. Also, since $f\left(x^{\prime}\right)=\pi(e), x^{\prime} \in W$. Then by the definition of the product topology on $X^{\prime} \times E, W \times V$ is an open neighborhood of $\left(x^{\prime}, e\right)$ in $X^{\prime} \times E$. Finally, let $U=(W \times V) \cap E^{\prime}$, so that $U$ is an open neighborhood of $e^{\prime}=\left(x^{\prime}, e\right)$ in $E^{\prime}$. We claim that $\left.\pi^{\prime}\right|_{U}$ is a homeomorphism, which will complete the proof.

To show that $\left.\pi^{\prime}\right|_{U}$ is a homeomorphism, we construct an inverse. First, note that the image of $\left.\pi^{\prime}\right|_{U}=W$. Let $x^{\prime} \in W=f^{-1}(\pi(V))$. Then $f\left(x^{\prime}\right) \in \pi(V)$. So define

$$
\eta: W \rightarrow U \quad x^{\prime} \mapsto\left(x^{\prime},\left(\left.\pi\right|_{V}\right)^{-1}\left(f\left(x^{\prime}\right)\right)\right)
$$

Then we verify that $\eta,\left.\pi^{\prime}\right|_{U}$ are inverse to each other. Let $u=\left(x^{\prime}, e\right) \in U$, where $x^{\prime} \in W, e \in$ $V$, and $f\left(x^{\prime}\right)=\pi(e)$. Then

$$
\left.\eta \pi^{\prime}\right|_{U}\left(x^{\prime}, e\right)=\eta\left(x^{\prime}\right)=\left(x^{\prime}, \pi^{-1}\left(f\left(x^{\prime}\right)\right)\right)=\left(x^{\prime}, \pi^{-1}(\pi(e))=\left(x^{\prime}, e\right)\right.
$$

On the other hand, it is obvious that $\left.\pi^{\prime}\right|_{U} \eta\left(x^{\prime}\right)=x^{\prime}$ for $x^{\prime} \in W$. So they are inverses. It is immediately obvious that $\eta$ is continuous, but we omit the proof of this fact. It should follow relatively quickly from the fact that $\pi$ is an open map.
Remark 8.22. The previous lemma is often phrased as saying "local homeomorphisms are preserved under base change."

Lemma 8.23. Let $\pi_{1}: E_{1} \rightarrow X, \pi_{2}: E_{2} \rightarrow X$ be local homeomorphisms, and let $\psi: E_{1} \rightarrow E_{2}$ be a map (not necessarily continuous) over $X$ which is a bijection.


Then $\psi$ is a homeomorphism.
Proof. Proof omitted.
Remark 8.24. One consequence of the previous lemma is as follows. Let $\mathcal{F}$ be a presheaf on a space $X$, with étale space $\pi_{\mathcal{F}}: E_{\mathcal{F}} \rightarrow X$. Let $\theta: \mathcal{F} \rightarrow \mathcal{F}^{+}$be the sheafification of $\mathcal{F}$, and $\pi_{\mathcal{F}+}: E_{\mathcal{F}^{+}} \rightarrow X$ the étale space of $\mathcal{F}^{+}$.

$$
E_{\mathcal{F}}=\bigsqcup_{x \in X} \mathcal{F}_{x} \quad E_{\mathcal{F}+}=\bigsqcup_{x \in X} \mathcal{F}_{x}^{+}
$$

We know that $\theta_{x}: \mathcal{F}_{x} \rightarrow \mathcal{F}_{x}^{+}$is an isomorphism for all $x$, so this gives a map

$$
\psi: E_{\mathcal{F}} \rightarrow E_{\mathcal{F}^{+}} \quad e \in \mathcal{F}_{x} \mapsto \theta_{x}(e) \in \mathcal{F}_{x}^{+}
$$

which is a bijection, and a map over $X$. Hence by the previous lemma, $\psi$ is a homeomorphism.
Remark 8.25. Let $f: X \rightarrow Y$ be continuous, and $\mathcal{G}$ a sheaf on $Y$. Recall that to define $f^{-1} \mathcal{G}$, we constructed an auxiliary presheaf $\mathcal{F}$ on $X$, and then defined $f^{-1} \mathcal{G}$ to be the sheafification of this sheaf. We also know a lot about a particular concrete description of the sheafification, using the étale space. The next proposition describes the étale space of this $\mathcal{F}$ in terms of a fiber product.

We recall the description of the étale space. Let $\mathcal{F}$ be a presheaf on a space $X$. The étale space of $\mathcal{F}$ is

$$
E_{\mathcal{F}}=\bigsqcup_{x \in X} \mathcal{F}_{x}
$$

with the obvious projection $\pi_{\mathcal{F}}: E_{\mathcal{F}} \rightarrow X$, which is an open map and a local homeomorphism. The topology is somewhat complicated to describe, but is roughly speaking generated by a basis of lifting open sets of $X$ to $E_{\mathcal{F}}$. Recall that one construction of sheafification we gave is

$$
\mathcal{F}^{+}(U)=\Gamma\left(U, \pi_{\mathcal{F}}\right)
$$

where $\Gamma\left(U, \pi_{\mathcal{F}}\right)$ is continuous sections of $\pi_{\mathcal{F}}$. Recall also that if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \cong \mathcal{F}^{+}$. Then following the remark above, there is a homeomorphism (over $X$ ) of the étale spaces $E_{\mathcal{F}} \cong E_{\mathcal{F}+}$.

Proposition 8.26. Let $f: X \rightarrow Y$ be a continuous map, and let $\mathcal{G}$ be a sheaf on $Y$. Let $\pi_{\mathcal{G}}: E_{\mathcal{G}} \rightarrow Y$, and $\pi_{f^{-1} \mathcal{G}}: E_{f^{-1} \mathcal{G}} \rightarrow X$ be the respective étale spaces of $\mathcal{G}, f^{-1} \mathcal{G}$. Form the fiber product $E_{\mathcal{G}} \times_{Y} X$ in the following diagram.


There is a homeomorphism $\psi: E_{f^{-1} \mathcal{G}} \rightarrow E_{\mathcal{G}} \times{ }_{Y} X$ over $X$, that is, the following diagram commutes.


Proof. Before embarking on the proof, we give two reductions.
(1st reduction) By Lemma 8.23 , it suffices to define a map $\psi$ which is just a bijection and a map over $X$, then it will follow from the lemma that it is a homeomorphism.
(2nd reduction) Let $\mathcal{F}$ be the auxiliary presheaf used to define $f^{-1} \mathcal{G}$, that is, $\mathcal{F}^{+}=f^{-1} \mathcal{G}$.

$$
\mathcal{F}(U)=\lim _{\substack{f(U) \subset V \\ V \text { open }}} \mathcal{G}(V)
$$

Using the Remark 8.25, we have a homeomorphism $E_{\mathcal{F}} \cong E_{f^{-1} \mathcal{G}}$ over $X$, so to prove the proposition it suffices to define a bijection $\psi: E_{\mathcal{F}} \xlongequal{\cong} E_{\mathcal{G}} \times{ }_{Y} X$ over $X$. So this is what we now do, ending our discussion of reductions.

By Proposition 8.16, for $x \in X$, we have natural isomorphisms

$$
\phi_{x}: \mathcal{F}_{x} \cong \mathcal{G}_{f(x)}
$$

Let $\pi: E_{\mathcal{F}} \rightarrow X$ be the étale space of the presheaf $\mathcal{F}$.

$$
E_{\mathcal{F}}=\bigsqcup_{x \in X} \mathcal{F}_{x} \quad E_{\mathcal{G}}=\bigsqcup_{y \in Y} \mathcal{G}_{y}
$$

Then define

$$
\Phi: E_{\mathcal{F}} \rightarrow E_{\mathcal{G}} \quad e \in \mathcal{F}_{x} \mapsto \phi_{x}(e) \in \mathcal{G}_{x} \subset E_{\mathcal{G}}
$$

Then the following diagram commutes.


We now claim that $\Phi$ is continuous. This a rather unpleasant calculation, following the definitions of the topologies on étale spaces, which we omit. Moving on, we assume $\Phi$ is continuous.

The fiber product $E_{\mathcal{G}} \times{ }_{Y} X$ is defined by a universal property, that it is universal in the following diagram.


Hence by the universal property, there is a map $\psi: E_{\mathcal{F}} \rightarrow E_{\mathcal{G}} \times_{Y} X$ making the following diagram commute.


In fact, we can describe $\psi$ a bit more concretely, using commutativity of this diagram. Let $e \in E_{\mathcal{F}}$, so $e \in \mathcal{F}_{x}$ for some $x \in X$. Let $\psi(e)=\left(e^{\prime}, x^{\prime}\right) \in E_{\mathcal{G}} \times_{Y} X$, where $e^{\prime} \in \mathcal{G}_{y} \subset E_{\mathcal{G}}$ for some $y \in Y$, and $x^{\prime} \in X$. Using the commutativity of the upper "triangle," $e^{\prime}=\Phi(e)=$ $\phi_{x}(e) \in \mathcal{G}_{f(x)}$. Using commutativity of the lower "triangle," $x^{\prime}=x$. Hence we may describe $\psi$ as

$$
\psi: E_{\mathcal{F}} \rightarrow E_{\mathcal{G}} \times_{Y} X \quad e \in \mathcal{F}_{x} \mapsto\left(\phi_{x}(e), x\right) \in \mathcal{G}_{f(x)} \times X
$$

Since $\phi_{x}$ is a bijection, $\psi$ is a bijection. Also, by Lemma 8.21 , the projection map $E_{\mathcal{G}} \times_{Y} X \rightarrow$ $X$ is a local homeomorphism, since $\pi_{\mathcal{G}}: E_{\mathcal{G}} \rightarrow Y$ is a local homeomorphism. Then by Lemma 8.23 , since $\psi$ is a bijection, it is a homeomorpism. Clearly, it is a map over $X$, as seen in the previous diagram.

Example 8.27 (Inverse image of a locally constant sheaf is a locally constant sheaf). Recall that the direct image of a locally constant sheaf need not be locally constant. However, the inverse image functor is better behaved in this respect, as the following example demonstrates.

Let $f: X \rightarrow Y$ be a continuous map. Let $S$ be an abelian group, and let $\mathcal{G}$ be the locally constant sheaf on $Y$ with value group $S$, and let $\mathcal{F}$ be the locally constant sheaf on $X$ with value group $S$. Let $\pi_{\mathcal{G}}: E_{\mathcal{G}} \rightarrow Y$ be the étale space of $\mathcal{G}$, and let $\pi_{\mathcal{F}}: E_{\mathcal{F}} \rightarrow X$ be the étale space of $\mathcal{F}$. Recall from Example 5.15 that we may describe $E_{\mathcal{G}}$ as $Y \times S$, with $\pi_{\mathcal{G}}$ being just the projection map to $Y$. Using the previous proposition, the étale space $E_{f^{-1} \mathcal{G}}$ of the inverse image sheaf is homeomorphic to the fiber product $E_{\mathcal{G}} \times{ }_{Y} X$.


Explicitly,

$$
E_{\mathcal{G}} \times_{Y}(Y \times S)=\left\{(x, y, s) \in X \times Y \times S: f(x)=\pi_{\mathcal{G}}(y, s)=y\right\} \cong X \times S
$$

That is, $E_{f^{-1} \mathcal{G}} \cong S \times X$, which is the same as the étale space $E_{\mathcal{F}}$. Then using Corollary 5.16, since this isomorphism is over $X$, it follows that $\mathcal{F} \cong f^{-1} \mathcal{G}$.

Definition 8.28. Let $X$ be a subspace of a space $Y$, and let $\iota: X \hookrightarrow Y$ be the inclusion. Let $\mathcal{G}$ be a sheaf (of abelian groups) on $Y$. The restriction of $\mathcal{G}$ to $X$, denoted $\left.\mathcal{G}\right|_{X}$, is the inverse image sheaf $\iota^{-1} \mathcal{G}$ on $X$.

We have already used this notation, but now it has some more substance behind it. In the case where $X \subset Y$ is an open subset, the restriction has a simple description as

$$
\left.\mathcal{G}\right|_{X}(U)=\mathcal{G}(U \cap X)
$$

However, if $X$ is not an open subset of $Y$, such a definition does not make sense, since for $U \subset Y$ open, the intersection $U \cap X$ may or may not be open in $Y$. Hence, in such a situation, we need to use the process of taking direct limits and sheafification, which is exactly what the inverse image functor accomplishes.

### 8.3 Adjunction between $\left(f^{-1}, f_{*}\right)$

Before discussion some generalities about adjunctions, we state the main result as motivation.
Proposition $8.29\left(\left(f^{-1}, f_{*}\right)\right.$ adjunction $\left.)\right)$. Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on $X$, and let $\mathcal{G}$ be a sheaf on $Y$. There is a natural bijection

$$
\operatorname{Hom}_{\operatorname{Sh}(X)}\left(f^{-1} \mathcal{G}, \mathcal{F}\right) \cong \operatorname{Hom}_{\mathrm{Sh}(Y)}\left(\mathcal{G}, f_{*} \mathcal{F}\right)
$$

Remark 8.30. The above proposition is more succinctly stated in saying that $f^{-1}$ is left adjoint to $f_{*}$ and $f_{*}$ is right adjoint to $f^{-1}$. In the following general definitions, we explain the precise meaning of the word "natural" in the proposition.

Definition 8.31. Let $\mathcal{C}, \mathcal{D}$ be locally small categories, and let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$ be covariant functors. We say that $(\mathcal{F}, \mathcal{G})$ form an adjoint pair or alternately that $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is right adjoint to $\mathcal{F}$ for any objects $X \in \operatorname{Ob}(\mathcal{C}), Y \in \operatorname{Ob}(\mathcal{D})$, there is a natural bijection

$$
b_{X, Y}: \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} X, Y) \cong \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{G} Y)
$$

By "natural," we mean that the various morphisms $b_{X, Y}$ are compatible with $\mathcal{C}$-morphisms and $\mathcal{D}$-morphisms in the following way. If $f: X_{1} \rightarrow X_{2}$ is a morphism in $\mathcal{C}$, and $g: Y_{1} \rightarrow Y_{2}$ is a morphism in $\mathcal{D}$, then the following diagram commutes.

$$
\begin{aligned}
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{2}, Y_{1}\right) \xrightarrow{b_{X_{2}, Y_{1}}} \operatorname{Hom}_{\mathcal{C}}\left(X_{2}, \mathcal{G} Y_{1}\right) \\
& t \mapsto g \circ \circ \circ \mathcal{F} \downarrow \downarrow \\
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{1}, Y_{2}\right) \xrightarrow{b_{x_{1}, Y_{2}}} \underset{\cong}{\cong} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, \mathcal{G} Y_{2}\right)
\end{aligned}
$$

In particular, specializing to the case where $g=\operatorname{Id}_{Y}: Y \rightarrow Y$, we get the diagram

$$
\begin{aligned}
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{2}, Y\right) \xrightarrow[\cong]{\stackrel{b_{X_{2}, Y}}{\cong}} \operatorname{Hom}_{\mathcal{C}}\left(X_{2}, \mathcal{G} Y\right) \\
& t \mapsto t \circ \mathcal{F} f \downarrow \downarrow \downarrow^{u \mapsto u \circ f} \\
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{1}, Y\right) \xrightarrow[\cong]{\stackrel{b_{X_{1}, Y}}{\cong}} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, \mathcal{G} Y\right)
\end{aligned}
$$

and similarly specializing to the case where $f=\operatorname{Id}_{X}: X \rightarrow X$, we get the diagram


Alternatively, the "naturality" condition of the isomorphisms $b_{X, Y}$ can be rephrased as saying that the isomorphisms $b_{X, Y}$ assemble to give a natural isomorphism of functors

$$
b: \operatorname{Hom}_{\mathcal{D}}(\mathcal{F}(-), \bullet) \rightarrow \operatorname{Hom}_{\mathcal{C}}(-, \mathcal{G}(\bullet))
$$

where we view both $\operatorname{Hom}_{\mathcal{D}}(\mathcal{F}(-), \bullet)$ and $\operatorname{Hom}_{\mathcal{C}}(-, \mathcal{G}(\bullet))$ as functors $\mathcal{C}^{\text {opp }} \times \mathcal{D} \rightarrow$ Set.
In some situations, directly constructing or describe a natural isomorphism as above may be somewhat difficult. Showing that two functors are adjoint is not that simple. In order to show that $\left(f^{-1}, f_{*}\right)$ form an adjoint pair, we will first develop an alternative characterization of adjointness, in terms of natural transformations called the "unit" and "counit."

Definition 8.32. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$ be an adjoint pair of functors, specifically, $\mathcal{F}$ is left adjoint to $\mathcal{G}$. Note that for $X \in \operatorname{Ob}(\mathcal{C})$, we have the isomorphism

$$
b_{X, \mathcal{F} X}: \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} X, \mathcal{F} X) \stackrel{\cong}{\rightrightarrows} \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{G} \mathcal{F} X)
$$

In particular, one element of the left side that we always have is the identity map of $\mathcal{F} X$, which we denote $\operatorname{Id}_{\mathcal{F} X}$. Then we define

$$
\eta_{X}=b_{X, \mathcal{F} X}\left(\operatorname{Id}_{\mathcal{F} X}\right): X \rightarrow \mathcal{G F} X
$$

Now consider a morphism $f: X_{1} \rightarrow X_{2}$ in $\mathcal{C}$, and the associated morphism $\mathcal{F} f: \mathcal{F} X_{1} \rightarrow \mathcal{F} X_{2}$ in $\mathcal{D}$. By naturality of $b$, we get two commutative diagrams.

$$
\begin{aligned}
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{1}, \mathcal{F} X_{1}\right) \xrightarrow{b_{X_{1}, \mathcal{F}}} \cong \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, \mathcal{G} \mathcal{F} X_{1}\right) \\
& t \mapsto \mathcal{F} f \circ \downarrow \downarrow \downarrow u \mapsto \mathcal{G} \mathcal{F} f \circ u \\
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{1}, \mathcal{F} X_{2}\right) \xrightarrow{b_{X_{1}, \mathcal{F} X_{2}}} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, \mathcal{G} \mathcal{F} X_{2}\right) \\
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{2}, \mathcal{F} X_{2}\right) \xrightarrow{b_{X_{2}, \mathcal{F} X_{2}}} \operatorname{Hom}_{\mathcal{C}}\left(X_{2}, \mathcal{G} \mathcal{F} X_{2}\right) \\
& t \mapsto t \circ \mathcal{F} \downarrow \downarrow \quad \downarrow^{u \mapsto u \circ f} \\
& \operatorname{Hom}_{\mathcal{D}}\left(\mathcal{F} X_{1}, \mathcal{F} X_{2}\right) \xrightarrow{b_{X_{1}, \mathcal{F} X_{2}}} \operatorname{Hom}_{\mathcal{C}}\left(X_{1}, \mathcal{G} \mathcal{F} X_{2}\right)
\end{aligned}
$$

In each diagram, we follow the path of the identity map starting in the top left corner. The first diagram gives

$$
\mathcal{G F} f \circ b_{X_{1} \mathcal{F} X_{1}}\left(\operatorname{Id}_{X_{1}}\right)=\mathcal{G} \mathcal{F} f \circ \eta_{X_{1}}=b_{X_{1}, \mathcal{F} X_{2}}\left(\mathcal{F} f \circ \operatorname{Id}_{X_{1}}\right)=b_{X_{1}, \mathcal{F} X_{2}}(\mathcal{F} f)
$$

The second diagram gives

$$
b_{X_{2}, \mathcal{F} X_{2}}\left(\operatorname{Id}_{X_{2}}\right) \circ f=\eta_{X_{2}} \circ f=b_{X_{1}, \mathcal{F} X_{2}}\left(\operatorname{Id}_{\mathcal{F} X_{2}} \circ \mathcal{F} f\right)=b_{X_{1}, \mathcal{F} X_{2}}(\mathcal{F} f)
$$

Combining these, we obtain $\eta_{X_{2}} \circ f=\mathcal{G F} f \circ \eta_{X_{1}}$. We depict this in the following commutative diagram.


As this commutes for an arbitrary map $f: X_{1} \rightarrow X_{2}$, this is precisely the condition that $\eta$ is a natural transformation from the identity functor on $\mathcal{C}$ to the functor $\mathcal{G F}$.

$$
\eta: \operatorname{Id}_{\mathcal{C}} \rightarrow \mathcal{G \mathcal { F }}
$$

This natural transformation $\eta$ is called the unit of the adjunction $(\mathcal{F}, \mathcal{G})$.
Definition 8.33. Similarly to the above, we define the counit associated to an adjunction $(\mathcal{F}, \mathcal{G})$. For $Y \in \operatorname{Ob}(\mathcal{D})$, we have

$$
b_{\mathcal{G} Y, Y}: \operatorname{Hom}_{\mathcal{C}}(\mathcal{F G Y}, Y) \stackrel{ }{\cong} \operatorname{Hom}_{\mathcal{C}}(\mathcal{G} Y, \mathcal{G} Y)
$$

and we define

$$
\epsilon_{Y}=b_{\mathcal{G} Y, Y}^{-1}\left(\operatorname{Id}_{\mathcal{G} Y}\right)
$$

We omit the diagram chase verification, but one can verify as with $\eta$ above that these maps assemble together to give a natural transformation

$$
\epsilon: \mathcal{F G} \rightarrow \operatorname{Id}_{\mathcal{D}}: \mathcal{F} \mathcal{G} Y \rightarrow Y
$$

As already mentioned, $\epsilon$ is called the counit of the adjunction $(\mathcal{F}, \mathcal{G})$.

All the previous work of defining the unit and counit would not be especially useful, except for the following proposition. Above, we constructed $\eta, \epsilon$ given an adjunction $(\mathcal{F}, \mathcal{G})$. The next proposition says that one can reverse-engineer the process, and actually recover the adjunction from $\eta$ and $\epsilon$, provided $\eta$ and $\epsilon$ satisfy some "triangle identities," which are expressed as commutative diagrams (how else?).

Proposition 8.34 (Adjunction equivalent to triangle identities). Given covariant functors $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$. Then $(\mathcal{F}, \mathcal{G})$ form an adjoint pair if and only if there exist natural transformations

$$
\eta: \operatorname{Id}_{\mathcal{C}} \rightarrow \mathcal{G F} \quad \epsilon: \mathcal{F} \mathcal{G} \rightarrow \operatorname{Id}_{\mathcal{D}}
$$

which make the following diagrams commute.


Remark 8.35. The two triangular commutative diagrams in the proposition are called the triangle identities.

Remark 8.36. Before embarking on at least a partial proof, let's make sure we understand what the two commutative diagrams are saying. In the first triangle identity, $\eta$ is a natural transformation $\mathrm{Id}_{\mathcal{C}} \rightarrow \mathcal{G} \mathcal{F}$, and $\mathcal{F}$ is a covariant functor $\mathcal{C} \rightarrow \mathcal{D}$. Hence $\mathcal{F} \eta$ is a natural transformation $\mathcal{F} \rightarrow \mathcal{F G \mathcal { F }}$, described as follows. For $X \in \operatorname{Ob}(\mathcal{C})$, we have $\eta_{X}: X \rightarrow \mathcal{G F} X$. Hence we have

$$
(\mathcal{F} \eta)_{X}=\mathcal{F}\left(\eta_{X}\right): \mathcal{F} X \rightarrow \mathcal{F} \mathcal{G} \mathcal{F} X
$$

which gives a natural transformation $\mathcal{F} \eta: \mathcal{F} \rightarrow \mathcal{F G \mathcal { F }}$. Similarly, given $X \in \operatorname{Ob}(\mathcal{C})$, the natural transformation $\epsilon \mathcal{F}$ is given by

$$
(\epsilon \mathcal{F})_{X}=\epsilon_{\mathcal{F} X}: \mathcal{F} \mathcal{G} \mathcal{F} X \rightarrow \mathcal{F} X
$$

So the first triangle identity is equivalent to having, for all $X \in \operatorname{Ob}(\mathcal{C})$, the following commutative diagram.


Simillarly, the second triangle identity is equivalent to having, for all $Y \in \operatorname{Ob}(\mathcal{C})$, the following commutative diagram.


Proof. (of Proposition 8.34). First, we will show that given an adjunction $(\mathcal{F}, \mathcal{G})$, the unit $\eta$ and counit $\epsilon$ satisfy the triangle identities. Actually, we will just verify the first triangle identity, since the second is similar. Let $b_{X, Y}$ be the natural isomorphisms between hom-sets associated to the adjunction. We want to show that for $X \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes.


By naturality of the isomorphisms $b_{X, Y}$, we have the following commutative diagramm, for any object $X \in \operatorname{Ob}(\mathcal{C})$.


In particular, consider the map $\epsilon_{\mathcal{F} X}$ starting in the top right corner. By construction of $\epsilon$, it gets mapped under $b_{\mathcal{G F X}, \mathcal{F} X}$ to $\operatorname{Id}_{\mathcal{G F X}}$, which then gets mapped down to $\eta_{X}$ under the vertical map. On the other hand, the preimage of $\eta_{X}$ under $b_{X, \mathcal{F} X}$ is $\operatorname{Id}_{\mathcal{F} X}$, by construction of $\eta$. Hence the vertical map on the left takes $\epsilon_{\mathcal{F} X}$ to $\operatorname{Id}_{\mathcal{F} X}$. That is,

$$
\epsilon_{\mathcal{F} X} \circ \mathcal{F}\left(\eta_{X}\right)=\operatorname{Id}_{\mathcal{F} X}
$$

This is precisely the needed commutative triangle for the triangle identity. As mentioned before, we will not do the verification for the second triangle identity, since it is similar. This completes our argument that an adjunction $(\mathcal{F}, \mathcal{G})$ induces a unit $\eta$ and counit $\epsilon$ which satisfy the triangle identities.

To complete the proof, we need to prove that given $\eta, \epsilon$ satisfying the triangle identities, it follows that $(\mathcal{F}, \mathcal{G})$ form an adjoint pair. That is, we need to construct the natural isomorphisms $b_{X, Y}$ for $X \in \operatorname{Ob}(\mathcal{C}), Y \in \operatorname{Ob}(\mathcal{D})$. We define $b_{X, Y}$ as follows.

$$
b_{X, Y}: \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} X, Y) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{G} Y) \quad \phi \mapsto G \phi \circ \eta_{X}
$$

That is, $b_{X, Y}$ is the following composition.

$$
X \xrightarrow{\eta_{X}} \mathcal{G} \mathcal{F} X \xrightarrow{\mathcal{G} \phi} \mathcal{G} Y
$$

Similarly, define

$$
\begin{aligned}
& \widetilde{b}_{X, Y}: \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{G} Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} X, Y) \quad \psi \mapsto \epsilon_{Y} \circ \mathcal{F} \psi \\
& \mathcal{F} X \xrightarrow{\mathcal{F} \psi} \mathcal{F G Y} \xrightarrow{\epsilon_{Y}} Y
\end{aligned}
$$

We then claim that $b_{X, Y}, \widetilde{b}_{X, Y}$ are inverse to each other. Consider the composition $\widetilde{b}_{X, Y} \circ b_{X, Y}$ applied to $\phi \in \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} X, Y)$.

$$
\begin{gathered}
\widetilde{b}_{X, Y} \circ b_{X, Y}(\phi)=\widetilde{b}_{X, Y}\left(G \phi \circ \eta_{X}\right)=\epsilon_{Y} \circ \mathcal{F}\left(\mathcal{G} \phi \circ \eta_{X}\right)=\epsilon_{Y} \circ \mathcal{F} \mathcal{G} \phi \circ \mathcal{F} \eta_{X} \\
\widetilde{b}_{X, Y} \circ b_{X, Y}(\phi)=\left(\mathcal{F} X \xrightarrow{\mathcal{F} \eta_{X}} \mathcal{F} \mathcal{G} \mathcal{F} X \xrightarrow{\mathcal{F} \phi} \mathcal{F} \mathcal{G} Y \xrightarrow{\epsilon_{Y}} Y\right)
\end{gathered}
$$

By naturality of $\epsilon$, we have a commutative diagram as below.


Now we can fit the above diagram into the commutative diagram below. The triangle on the left is one of the triangle identities.


Thus $\widetilde{b}_{X, Y} \circ b_{X, Y}(\phi)=\phi$. A similar diagram concatenation argument proves that the other composition $b_{X, Y} \circ \widetilde{b}_{X, Y}$ is the identity. Hence having a unit and counit $\eta, \epsilon$ induce bijections $b_{X, Y}$. We omit the proof of the naturality of the isomorphisms, since this is tedious. This completes our "proof."

With the preceeding proposition in hand, we are now prepared to prove that the direct image and inverse image functors form an adjoint pair. We will do this by constructing a unit and counit, and showing (to some degree) that they satisfy the triangle identities.

Proposition $8.37\left(\left(f^{-1}, f_{*}\right)\right.$ adjunction $)$ ). Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Then $\left(f^{-1}, f_{*}\right)$ form an adjoint pair.

Proof. Using the previous proposition, it suffices to construct natural transformations

$$
\eta: \operatorname{Id}_{\operatorname{Sh}(Y)} \rightarrow f_{*} f^{-1} \quad \epsilon: f^{-1} f_{*}: \operatorname{Id}_{\operatorname{Sh}(X)}
$$

which satisfy the triangle identities. First, we construct $\eta$. To construct a natural transformation $\operatorname{Id}_{\operatorname{Sh}(Y)} \rightarrow f_{*} f^{-1}$, we need, for a sheaf $\mathcal{G}$ on $Y$, to define

$$
\eta_{\mathcal{G}}: \mathcal{G} \rightarrow f_{*} f^{-1} \mathcal{G}
$$

Let $\mathcal{L}$ be the auxiliary presheaf on $X$ defined by

$$
\mathcal{L}(U)=\underset{f(U) \subset V}{\lim _{\vec{\longrightarrow}}} \mathcal{G}(V)
$$

Recall that $f^{-1} \mathcal{G}$ is defined as the sheafification $\mathcal{L}^{+}$of $\mathcal{L}$. Let $\theta: \mathcal{L} \rightarrow \mathcal{L}^{+}$be the canonical map of sheafification. Applying the direct image functor to $\theta$, we obtain

$$
f_{*} \theta: f_{*} \mathcal{L} \rightarrow f_{*} \mathcal{L}^{+}=f_{*} f^{-1} \mathcal{G}
$$

So to define $\eta_{\mathcal{G}}$, it suffices to define a morphism $\mathcal{G} \rightarrow f_{*} \mathcal{L}$, and then compose with $f_{*} \theta$ above. Set $\mathcal{H}=f_{*} \mathcal{L}$. Then for $V \subset Y$ open,

$$
\mathcal{H}(V)=f_{*} \mathcal{L}(V)=\mathcal{L}\left(f^{-1}(V)\right)={\underset{f\left(f^{-1}(V)\right) \subset W}{\lim _{\longrightarrow}} \mathcal{G}(W), ~(W)}
$$

Note that trivially, for $V \subset Y$, by basic properties of sets, $f\left(f^{-1}(V)\right) \subset V$. So in the direct limit above, $V$ is one possible $W$. Hence we have the canonical map to the direct limit,

$$
\mathcal{G}(V) \rightarrow \mathcal{H}(V)=\underset{W}{\lim } \mathcal{G}(W)
$$

Modulo some verification, these assemble together to give a morphism of presheaves

$$
\alpha_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{H}
$$

We then compose this with $f_{*} \theta$ to obtain

$$
\eta_{\mathcal{G}}=f_{*} \theta \circ \alpha_{\mathcal{G}}: \mathcal{G} \xrightarrow{\alpha_{\mathcal{G}}} \mathcal{H} \xrightarrow{f_{*} \theta} f_{*} f^{-1} \mathcal{G}
$$

Once again, we wave our hands and blithely assert that these $\eta_{\mathcal{G}}$ assemble to give a natural transformation

$$
\eta: \operatorname{Id}_{\mathrm{Sh}(Y)} \rightarrow f_{*} f^{-1}
$$

Now we define the counit $\epsilon$. We need to define, for a sheaf $\mathcal{F}$ on $X$, a morphism of sheaves

$$
\epsilon_{\mathcal{F}}: f^{-1} f_{*} \mathcal{F} \rightarrow \mathcal{F}
$$

For $\mathcal{F} \in \operatorname{Ob}(\operatorname{Sh}(X)), f^{-1} f_{*} \mathcal{F}$ is the sheafification of the auxiliary presheaf $\mathcal{K}$ defined by

$$
\mathcal{K}(U)=\underset{f(U) \subset V}{\lim _{f}}\left(f_{*} \mathcal{F}\right)(V)=\lim _{f(U) \subset V} \mathcal{F}\left(f^{-1}(V)\right)
$$

First, we will construct a morphism $\mathcal{K} \rightarrow \mathcal{F}$, and then apply the universal property of sheafification to get a morphism from $f^{-1} f_{*} \mathcal{F}=\mathcal{K}^{+}$. That is, we want for $U \subset X$, a morphism $\mathcal{K}(U) \rightarrow \mathcal{F}(U)$. Note that for $V$ open in $Y$ such that $f(U) \subset V$, we have $U \subset f^{-1}(V)$ by basic set theory, hence there is a restriction map (of $\mathcal{F}$ )

$$
\operatorname{res}_{U}^{f^{-1}(V)}(\mathcal{F}): \mathcal{F}\left(f^{-1}(V)\right) \rightarrow \mathcal{F}(U)
$$

Then using more handwaving, such maps are compatible with the direct limit defining $\mathcal{K}(U)$, hence the restriction maps above induce a map on the direct limit.

Using even more handwaving, these are compatible with restrictions for $\mathcal{K}, \mathcal{F}$, and so give a morphism of presheaves

$$
\mathcal{K} \rightarrow \mathcal{F}
$$

Then using the universal property of sheafification, since $\mathcal{F}$ is a sheaf, such a morphism of presheaves induces a unique morphism of sheaves

$$
\epsilon_{\mathcal{F}}: f^{-1} f_{*} \mathcal{F}=\mathcal{K}^{+} \rightarrow \mathcal{F}
$$

Using even more handwaving (we're getting really good at that in this proof) these are compatible with morphisms in such a way that they give a natural transformation

$$
\epsilon: f^{-1} f_{*} \rightarrow \operatorname{Id}_{\operatorname{Sh}(X)}
$$

This completes our construction of the unit and counit natural transformations. In order to finish the proof that $\left(f^{-1}, f_{*}\right)$ are adjoint, we would need to verify the triangle identities for $\eta, \epsilon$. The argument for this is convoluted and tediious, and not especially illustrative, so we omit it.

Lastly, we give an application of the above adjunction to a property of $f_{*}$ acting on injective objects.

Definition 8.38. Let $\mathcal{C}$ be an abelian category. An object $I \in \operatorname{Ob}(\mathcal{C})$ is injective if the (covariant) functor $\operatorname{Hom}_{\mathcal{C}}(-, I)$ is exact. Somewhat more concretely, this is equivalent to the following property: If $\phi: X \rightarrow Y$ is a monomorphism in $\mathcal{C}$ and $\psi: X \rightarrow I$ is any morphism, then there is a morphism $h: Y \rightarrow I$ making the following diagram commute.


Note that the functor $\operatorname{Hom}_{\mathcal{C}}(-, I)$ is left exact for any object in $\mathcal{C}$, it really the right exactness part that makes injective objects special.

Remark 8.39. We have not discussed abelian categories in any generality yet, but eventually we will see that $\operatorname{Sh}(X)$ is an abelian category. We are primarily interested in the above definition in the situation where $\mathcal{C}=\operatorname{Sh}(X)$. In this setting, "monomorphism" is what we have previously defined as an injective morphism of sheaves.

In the following proposition, "additive" functor just means that it respects binary products. If the abstraction of the next proposition is too daunting, just imagine that $\mathcal{F}=f^{-1}, \mathcal{G}=$ $f_{*}, \mathcal{C}=\operatorname{Sh}(Y), \mathcal{D}=\operatorname{Sh}(X)$ for some continuous map $f: X \rightarrow Y$.

Proposition 8.40. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$ be an adjoint pair of additive functors $(\mathcal{F}, \mathcal{G})$, and suppose that $\mathcal{F}$ is exact. Then $\mathcal{G}$ takes injectives to injectives. That is, if $I$ is an injective object of $\mathcal{D}$, then $\mathcal{G I}$ is an injective object of $\mathcal{C}$.

Proof. Let $I$ be an injective object of $\mathcal{D}$. Let $0 \rightarrow X \xrightarrow{\phi} Y$ be an exact sequence in $\mathcal{C}$, and let $\psi: X \rightarrow \mathcal{G} I$ be a morphism. We have the following diagram in $\mathcal{C}$, which we need to complete with a morphism $Y \rightarrow \mathcal{G} I$ to show that $\mathcal{G} I$ is injective.


Since $\mathcal{F}$ is exact, the sequence $0 \rightarrow \mathcal{F} X \xrightarrow{\mathcal{F} \phi} \mathcal{F} Y$ is exact in $\mathcal{D}$. So we have the following diagram. As $(\mathcal{F}, \mathcal{G})$ are adjoint, we have a natural isomorphisms $b_{X, I}, b_{Y, I}$ fitting into the following diagram.


Let $\widetilde{\psi}=b_{X, I}^{-1}(\psi)$. Then we have the following diagram (in $\mathcal{D}$ ), which, by injectivity of $I$ can be completed to a morphism $\widetilde{h}: \mathcal{F} Y \rightarrow I$.


Let $h=b_{Y, I}(\widetilde{h})$. We repeat the diagram as an aid to keeping track of where each morphism lives.


By commutativity of the above diagram, we have

$$
\psi=b_{X, I}(\widetilde{\psi})=b_{X, I}(\widetilde{h} \circ \mathcal{F} \phi)=b_{Y, I}(\widetilde{h}) \circ \phi=h \circ \phi
$$

That is, $h$ is the needed completion in the original diagram.


Hence $\mathcal{G I}$ is injective.
As remarked before the proof, the case that we care about for the previous proposition is for the $\left(f^{-1}, f_{*}\right)$ adjunction, where $\mathcal{F}=f^{-1}, \mathcal{G}=f_{*}, \mathcal{C}=\operatorname{Sh}(Y), \mathcal{D}=\operatorname{Sh}(X)$. We note this in the following corollary.

Corollary 8.41. Let $f: X \rightarrow Y$ be a continuous map. If $\mathcal{I}$ is an injective sheaf on $X$, then $f_{*} \mathcal{I}$ is an injective sheaf on $Y$.

### 8.4 Extension by zero

Our next topic has very little to do with the previous few sections on the adjunction between the inverse image and direct image functors. The next goal is to describe the "extension by zero" functor associated to an embedding $\iota: Z \hookrightarrow X$, where $Z \subset X$ is open or closed. We consider the closed case first, since it is quite simple to describe, and is in fact just a special case of the direct image functor.

Definition 8.42. Let $i: Z \hookrightarrow X$ be an embedding of topological spaces, so we identify $Z$ with $i(Z)$. Assume $Z$ is closed in $X$. Given a sheaf $\mathcal{G}$ on $Z$, the extension by zero of $\mathcal{G}$ is the sheaf $i_{*} \mathcal{G}$. We also denote this functor by $i_{!}$, which is read as " $i$ lower shriek."

Remark 8.43. The goal of this remark is to explain the use of the phrase "extension by zero." Let $i: Z \hookrightarrow X$ be a closed embedding of topological spaces, and $\mathcal{G}$ be a sheaf on $Z$. Let $\mathcal{F}=i_{*} \mathcal{G}$ be the extension by zero of $\mathcal{G}$. As we have previously seen, for $x \in X$, the stalk of $\mathcal{F}$ is

$$
\mathcal{F}_{x}=\left(i_{*} \mathcal{G}\right)_{x}= \begin{cases}\mathcal{G}_{x} & x \in Z \\ 0 & x \notin Z\end{cases}
$$

Also note that $\left.\mathcal{F}\right|_{Z}=i^{-1} \mathcal{F}=i^{-1} i_{*} \mathcal{G}$. In the previous section, we constructed the unit

$$
\eta: i^{-1} i_{*} \rightarrow \operatorname{Id}_{\mathrm{Sh}(Z)}
$$

which is a natural transformation, given on a sheaf $\mathcal{G}$ by

$$
\eta_{\mathcal{G}}: i^{-1} i_{*} \mathcal{G} \rightarrow \mathcal{G}
$$

which is an isomorphism on stalks at $x \in Z$ (one can trace through the construction of $\eta_{\mathcal{G}}$ and verify this). Hence $\eta$ is an isomorphism of sheave $\mathcal{F}_{Z} \cong \mathcal{G}$. Hence $\mathcal{F}=i_{*} \mathcal{G}$ is a sheaf on $X$ whose restriction to $Z$ coincides with $\mathcal{G}$, and whose stalks outside of $Z$ are zero. This explains the use of the phrase "extension by zero," since $\mathcal{F}=i_{*} \mathcal{G}$ is an "extension" of $\mathcal{G}$ in the above sense - it is "the same" as $\mathcal{G}$ on $Z$, but "extended" to all of $X$ in a way that the values outside $Z$ are zero.

The previous discussion sets our model for what extension by zero should be. The extension by zero should be a new sheaf on the total space $X$, whose stalks are zero outside of $Z$ and whose restriction to $Z$ is the original sheaf.

The next question is, what happens if the embedding $Z \hookrightarrow X$ has image which is not closed? In this situation, we may still take the direct image. However, even when the image of $Z$ is open in $X$, the next example shows that the direct image does not have the desired properties of the previous paragraph. This will motivate defining the extension by zero in a more careful way in this situation.

Example 8.44. Let $Y=\mathbb{R}^{2}$, and let $X \subset Y$ be the open unit disc,

$$
X=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}<1\right\}
$$

Let $\iota: X \hookrightarrow Y$ be the inclusion, which is an embedding with open image. Let $\mathcal{G}$ be the locally constant sheaf on $X$ with value group $S$, where $S$ is any nontrivial abelian group.

Let $\mathcal{F}=\iota_{*} \mathcal{G}$ be the direct image sheaf on $Y$. For $x \in X$, the stalk of $\mathcal{G}$ is just $S, \mathcal{G}_{x} \cong S$, and the stalk of the $\mathcal{F}$ is the same, $\mathcal{F}_{x}=\left(\iota_{*} \mathcal{G}\right)_{x} \cong S$. Also, for any $y \in Y$ along the boundary of $X$, we claim that the stalk is also $S$. To see this, note that every open neighborhood $U$ of $y$ intersects $X$, and we may always shrink such a neighborhood $U$ so that the intersection $U \cap X$ is connected, in which case

$$
\left(\iota_{*} \mathcal{G}\right)(U)=\mathcal{G}\left(\iota^{-1}(U)\right)=\mathcal{G}(U \cap X) \cong S
$$

Passing to the direct limit of the stalk at $y$, all the groups of the direct limit eventually become $S$, so $\left(\iota_{*} \mathcal{G}\right)_{y} \cong S$. Putting this together,

$$
\left(\iota_{*} \mathcal{G}\right)_{y}= \begin{cases}S & y \in \bar{X} \\ 0 & y \notin \bar{X}\end{cases}
$$

where $\bar{X}$ is the closure of $X$, the closed unit disk $\bar{X}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2} \leq 1\right\}$. So in contrast to the previous situation of a closed embedding, the direct image of $\mathcal{G}$ is not a good choice for "extension by zero," since the stalks along the boundary will usually not vanish.

Definition 8.45 (Extension by zero for open embeddings). Let $j: U \hookrightarrow X$ be an embedding of topological spaces with open image. Let $\mathcal{G}$ be a sheaf (of abelian groups) on $U$, and define $\mathcal{H}$ to be the presheaf on $X$ defined by

$$
\mathcal{H}(V)= \begin{cases}\mathcal{G}(V) & V \subset U \\ 0 & \text { else }\end{cases}
$$

for $V \subset X$ open. In general, $\mathcal{H}$ is not a sheaf, so we just take the sheafification, and define the extension by zero of $\mathcal{G}$ by $j$ to be the sheafification $\mathcal{H}^{*}$. We denote this as $j!\mathcal{G}$.

Remark 8.46. It is pretty quick to see from the definitions that the stalks of $j!\mathcal{G}$ behave as we want an extension by zero to behave.

$$
\left(j_{!} \mathcal{G}\right)_{x}= \begin{cases}\mathcal{H}_{x}=\mathcal{G}_{x} & x \in U \\ 0 & x \notin U\end{cases}
$$

It is also immediate from the definition that

$$
j^{-1} j_{!} \mathcal{G}=j^{-1} \mathcal{H}^{+} \cong \mathcal{G}
$$

Proposition 8.47. Let $j: U \hookrightarrow X$ be an open embedding and $j: Z \hookrightarrow X$ be a closed embedding. Then the functors $j_{!}, i_{!}$are both exact.

Proof. Pretty immediate from the description of stalks.
Remark 8.48. Using the definitions above, it is possible to extend the "extension by zero functor" to so-called "locally closed" ${ }^{[14}$ subspaces of $X$ in a way which makes sense. This is only really used in algebraic geometry when dealing with sheaves on varieties, since that's the only place where people care about locally closed subspaces.

Proposition 8.49. Let $X$ be a topological space, and let $\mathcal{F}$ be a sheaf (of abelian groups) on $X$. Let $Z \subset X$ be a closed subset, with fixed embedding $i: Z \hookrightarrow X$. Let $U=X \backslash Z$, with fixed embedding $j: U \hookrightarrow Z$. Then we have a short exact sequence of sheaves

$$
0 \rightarrow j_{!}\left(\left.\mathcal{F}\right|_{U}\right) \xrightarrow{\epsilon_{\mathcal{F}}} \mathcal{F} \xrightarrow{\eta_{\mathcal{F}}} i_{*}\left(\left.\mathcal{F}\right|_{Z}\right) \rightarrow 0
$$

Remark 8.50. We may alternatively write the short exact sequence above as

$$
0 \rightarrow j!j^{-1} \mathcal{F} \rightarrow \mathcal{F} \rightarrow i!i^{-1} \mathcal{F} \rightarrow 0
$$

Proof. (Proposition 8.49) We will just describe the maps of the short exact sequence, and leave verification of exactness. As $j$ is an open map, the sheaf $\left.\mathcal{F}\right|_{U}=j^{-1} \mathcal{F}$ on $U$ is describable without direct limits,

$$
\left.\mathcal{F}\right|_{U}(V)=j^{-1} \mathcal{F}(V)=\mathcal{F}(V)
$$

Then by definition, $j!\left(\left.\mathcal{F}\right|_{U}\right)=j!j^{-1} \mathcal{F}$ is the sheafification of the presheaf $\widetilde{\mathcal{F}}$, which is defined by

$$
\widetilde{\mathcal{F}}(V)= \begin{cases}\mathcal{F}(V) & V \subset U \\ 0 & \text { else }\end{cases}
$$

where 0 represents the trivial group. We have a morphism of sheaves $\phi: \widetilde{\mathcal{F}} \rightarrow \mathcal{F}$ given by

$$
\phi_{V}: \widetilde{\mathcal{F}}(V) \rightarrow \mathcal{F}(V) \quad \phi_{V}= \begin{cases}\operatorname{Id}_{\mathcal{F}(V)} & V \subset U \\ 0 & \text { else }\end{cases}
$$

[^12]where 0 represents the unique map to the trivial group. Then by the universal property of sheafification, $\phi$ extends to a morphism $\epsilon_{\mathcal{F}}$ on $\widetilde{\mathcal{F}}^{+}=j_{!} j^{-1} \mathcal{F}$.
$$
\epsilon_{\mathcal{F}}: j_{!} j^{-1} \mathcal{F} \rightarrow \mathcal{F}
$$

This is the morphism in the claimed short exact sequence. The second map $\eta_{\mathcal{F}}$ is simply the previously described unit of the $\left(i^{-1}, i_{*}\right)$ adjunction.

$$
\eta: \operatorname{Id}_{\mathrm{Sh}(X)} \rightarrow i_{*} i^{-1} \quad \eta_{\mathcal{F}}: \mathcal{F} \rightarrow i_{*} i^{-1} \mathcal{F}=i_{*}\left(\left.\mathcal{F}\right|_{Z}\right)=i_{!}\left(\left.\mathcal{F}\right|_{Z}\right)=i_{!} i^{-1} \mathcal{F}
$$

We omit verification of exactness, since it is not that hard to write down what all the stalks are for these functors.

Remark 8.51. Let $j: U \hookrightarrow X$ be an open embedding as above. It turns out that the map $\epsilon_{\mathcal{F}}: j_{!} j^{-1} \mathcal{F} \rightarrow \mathcal{F}$ is the counit of a different adjunction than the one we previously discussed. Among other things, this means that these maps give a natural transformation

$$
\epsilon: j!j^{-1} \rightarrow \operatorname{Id}_{\operatorname{Sh}(X)}
$$

There is an adjunction $\left(j!, j^{-1}\right)$. The unit of this adjunction is described by

$$
\begin{aligned}
\eta: \operatorname{Id}_{\mathrm{Sh}(U)} & \rightarrow j^{-1} j_{1} \mathcal{G} & & \\
\eta_{\mathcal{G}}: \mathcal{G} & \rightarrow j^{-1} j_{1} \mathcal{G} & & \mathcal{G} \in \mathrm{Ob}(\operatorname{Sh}(U)) \\
\left(\eta_{\mathcal{G}}\right)_{V}=\operatorname{Id}_{\mathcal{G}(V)}: \mathcal{G}(V) & \rightarrow j^{-1} j_{!} \mathcal{G}(V) & & V \subset U \text { open }
\end{aligned}
$$

We repeat this fact in the following proposition, but give no attempt at a proof.
Proposition 8.52. For an open embedding $j: U \hookrightarrow X$ of topological spaces, the functors ( $j!j^{-1}$ ) form an adjoint pair.

### 8.5 Exceptional inverse image

Above, we saw that for $j: U \hookrightarrow X$ an open embedding, $j$ ! has a right adjoint, namely $j^{-1}$. When $i: Z \hookrightarrow X$ is a closed embedding, $i_{!}=i_{*}$ also has a right adjoint. It is the goal of this section to describe this adjoint, at least in rough outline. This right adjoint will be called the "exceptional inverse image" functor, and it is more complicated to describe than any of the functors between sheaf categories that we have so far described.

Definition 8.53. Let $\mathcal{F}$ be a sheaf of abelian groups on a space $X$. Let $U \subset X$ be an open subset, and let $s \in \mathcal{F}(U)$ be a section. The support of $s$ is the set

$$
\operatorname{supp}(s)=\left\{x \in U: \rho_{x}^{U}(s) \neq 0 \text { in } \mathcal{F}_{x}\right\}
$$

Lemma 8.54. Let $\mathcal{F}, X, U$, s be as above. The set $\operatorname{supp}(s)$ is closed in $U$.

Proof. We will show that the complement of $\operatorname{supp}(s)$ is open. Let $x \in U \backslash \operatorname{supp}(s)$. By definition of $\operatorname{supp}(s), \rho_{x}^{U}(s)=0$ in $\mathcal{F}_{x}$. If something is zero in the stalk, it means there is a small open neighborhood $V$ of $x$ such that $\rho_{V}^{U}(s)=0$ in $\mathcal{F}(V)$. Then by shrinking $V$ if necessary, $V \cap \operatorname{supp}(s)=\emptyset$. Thus $U \backslash \operatorname{supp}(s)$ is open, $\operatorname{sonpp}(s)$ is closed.

Definition 8.55. Let $i: Z \hookrightarrow X$ be an embedding of a closed subspace, and let $\mathcal{F}$ be a sheaf on $X$. Define, for $U \subset X$,

$$
\mathcal{F}_{Z}(U)=\{s \in \mathcal{F}(U): \operatorname{supp}(s) \subset Z\}
$$

Note that we have an obvious injective morphism of presheaves $\mathcal{F}_{Z} \rightarrow \mathcal{F}$, given by inclusions $\mathcal{F}_{Z}(U) \hookrightarrow \mathcal{F}(U)$, so $\mathcal{F}_{Z}$ is a subsheaf of $\mathcal{F}$. The exceptional inverse image of $\mathcal{F}$ is the sheaf $i^{-1} \mathcal{F}_{Z}$. This defines a functor

$$
i^{!}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Z)
$$

which acts on objects as described above, and on objects as follows. Given a morphism of sheaves on $X, \phi: \mathcal{F} \rightarrow \mathcal{G}$ we have $\phi_{U}: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, and we note that

$$
\phi_{U}\left(\mathcal{F}_{Z}(U)\right) \subset \mathcal{G}_{Z}(U)
$$

So the maps $\left.\phi_{U}\right|_{\mathcal{F}_{Z}(U)}$ give a morphism of sheaves $\left.\phi\right|_{\mathcal{F}_{Z}}: \mathcal{F}_{Z} \rightarrow \mathcal{G}_{Z}$. Then $i^{!} \phi$ is defined to be the inverse image applied to the morphism $\left.\phi\right|_{\mathcal{F}_{Z}}$.
Proposition 8.56. Let $i: Z \hookrightarrow X$ be an embedding with closed image. The functor $i^{!}$: $\operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Z)$ is left exact.
Proof. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be a short exact sequence of sheaves on $X$. Then we claim that $0 \rightarrow \mathcal{F}_{Z} \rightarrow \mathcal{G}_{Z} \rightarrow \mathcal{H}_{Z}$ is also an exact sequence of sheaves on $X$. Since $\mathcal{F}_{Z}$ is a subsheaf of $\mathcal{F}$, for $x \in X$ we have inclusions $\left(\mathcal{F}_{Z}\right)_{x} \hookrightarrow \mathcal{F}_{x}$, making the following commutative diagram.


It is immediate from this diagram that $\left(\mathcal{F}_{Z}\right)_{x} \rightarrow\left(\mathcal{G}_{Z}\right)_{x}$ is injective. Somewhat less immediately, but just by a simple diagram chase, we get exactness at $\left(\mathcal{G}_{Z}\right)_{x}$. Then apply the inverse image functor $i^{-1}$, which is left exact, and we get an exact sequence $0 \rightarrow i^{!} \mathcal{F} \rightarrow i^{!} \mathcal{G} \rightarrow$ $i^{!} \mathcal{H}$.

Proposition 8.57. Let $i: Z \hookrightarrow X$ be an embedding with closed image. The functors $\left(i_{!}=i_{*}, i^{!}\right)$form an adjoint pair.

Proof. Omitted.
Remark 8.58. In general, if $f: X \rightarrow Y$ is any continuous map between locally compact spaces, a functor $f^{!}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$ can be defined on the level of derived categories, though not actually on the level of sheaves. This leads into a great deal of abstraction, including results like Grothendieck-Verdier duality.

### 8.6 Summary of adjunctions and exactness results

Since we have introduced so many functors associated to continuous maps, and various exactness and adjunction properties among them, we include the following table as a reference for all of this information. Let

$$
f: X \rightarrow Y \quad i: Z \hookrightarrow X \quad j: U \hookrightarrow X
$$

be continuous maps, with $i, j$ injective and $i(Z)$ closed in $X, j(U)$ open in $X$.

| Functor | Name | Exactness | Adjunctions |
| :--- | :--- | :--- | :--- |
| $f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$ | Direct image | Left exact | $\left(f^{-1}, f_{*}\right)$ |
| $f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$ | Inverse image | Exact | $\left(f^{-1}, f_{*}\right)$ |
| $i_{!}=i_{*}: \operatorname{Sh}(Z) \rightarrow \operatorname{Sh}(X)$ | Extension by zero | Exact | $\left(i_{1}, i^{!}\right),\left(i^{-1}, i_{!}\right)$ |
| $j_{!}: \operatorname{Sh}(U) \rightarrow \operatorname{Sh}(X)$ | Extension by zero | Exact | $\left(j_{!}, j^{-1}\right)$ |
| $i^{!}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Z)$ | Exceptional inverse image | Left exact | $\left(i_{!}, i^{!}\right)$ |

## 9 General homological algebra

As our next step toward defining sheaf cohomology, we need some more category theory. This section covers the required general homological algebra such as injective and projective objects, resolutions, and delta functors which set the stage for derived functors. We do not delve too deeply into this abstraction, since that would involve going all the way to derived categories. We'll cover just enough homological algebra to get to sheaf cohomology.

### 9.1 Injectives

Let $R$ be a unital, associative ring (not necessarily commutative). By an $R$-module, we mean a left $R$-module. Fix an $R$-module $Q$, and consider the contravariant functor $\operatorname{Hom}_{R}(-, Q)$. This functor is left exact, meaning that given a short exact sequence of $R$-modules,

$$
0 \rightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \rightarrow 0
$$

the resulting sequence after applying $\operatorname{Hom}_{R}(-, Q)$ is left exact, meaning that the following is exact.

$$
0 \rightarrow \operatorname{Hom}_{R}(N, Q) \xrightarrow{\phi^{\prime}} \operatorname{Hom}_{R}(M, Q) \xrightarrow{\psi^{\prime}} \operatorname{Hom}_{R}(L, Q)
$$

The maps $\phi^{\prime}, \psi^{\prime}$ are given by $f \mapsto f \circ \phi, g \mapsto g \circ \psi$, respectively. We want to understand when this functor is exact, which is to say, when it is also right exact, which is to say, when $\psi^{\prime}$ is surjective. A morphism $f: L \rightarrow Q$ is in the image of $\psi^{\prime}$ if it factors through $\psi$, which is to say, there exists $h: M \rightarrow Q$ making the following diagram commute.


Before moving on to discuss when this happens, we give a concrete example of failure.
Example 9.1. Let $R=\mathbb{Z}$, and consider the short exact sequence of abelian groups ( $\mathbb{Z}$ modules)

$$
0 \rightarrow \mathbb{Z} \xrightarrow{\psi=2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z} / 2 \mathbb{Z} \rightarrow 0
$$

Let $Q=\mathbb{Z} / 2 \mathbb{Z}$, and consider the map $f=\bmod 2: \mathbb{Z} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$. Then if $h: \mathbb{Z} \rightarrow \mathbb{Z} / 2 \mathbb{Z}$ is any map, $h \circ \psi=0$, which is not $f$.


Proposition 9.2. Let $Q$ be an $R$-module. The following are equivalent.

1. $\operatorname{Hom}_{R}(-, Q)$ is exact.
2. $\operatorname{Hom}_{R}(-, Q)$ is right exact.
3. Given any injective morphism $\psi: L \rightarrow M$ and any morphism $f: L \rightarrow Q$, there exists $h: M \rightarrow Q$ such that $h \circ \psi=f$.

4. For any ideal $I \subset R$ with inclusion $\psi: I \hookrightarrow R$, and any morphism $f: I \rightarrow Q$, there exists $h: R \rightarrow Q$ such that $h \circ \psi=f$.

5. For any module $M$, if $Q$ is a submodule of $M$, then $Q$ is a direct summand of $M$.
6. Every short exact sequence $0 \rightarrow Q \rightarrow M \rightarrow N \rightarrow 0$ splits.

Proof. The equivalence of (1), (2), (3), (5), (6) is not hard, done in other places, e.g. Lang's graduate algebra textbook. The equivalence of (4) with the rest is commonly known as Baer's criterion, and the proof is somewhat complicated, involving Zorn's lemma. This can also be found in Lang, or other places, such as the online Stacks Project.

Definition 9.3. An $R$-module $Q$ satsifying the above equivalent conditions is called injective.

Definition 9.4. Let $R$ be a ring and $M$ an $R$-module. For $x \in R$, consider the map

$$
x: M \rightarrow M \quad m \mapsto x m
$$

The module $M$ is divisible if for every $x \neq 0$, the map above is surjective. This is usually stated as saying that $x M=M$ for every $x \neq 0$.

Example 9.5. $\mathbb{Q}$ viewed as a $\mathbb{Z}$-module is divisible. $\mathbb{Z}$ is not divisible as a $\mathbb{Z}$-module.
Example 9.6. A quotient of a divisible module is divisible. For example, $\mathbb{Q} / \mathbb{Z}$ is a divisible $\mathbb{Z}$-module.

Proposition 9.7. Let $R$ be a PID. An $R$-module is injective if and only if it is divisible.

Proof. We use Baer's criterion. Let $Q$ be an $R$-module, and let $I \subset R$ be an ideal, so $I=(r)$ for some $x \in R$. An $R$-module homomorphism $f: I \rightarrow Q$ is determined by $f(r)=q$, and an $R$-module homomorphism $h: R \rightarrow Q$ is determined by $h(1)=q^{\prime} . Q$ is injective if and only if there is a map $h: R \rightarrow Q$ such that $h(r)=q$.


If $Q$ is divisible, then there exists $q^{\prime} \in Q$ such that $r q^{\prime}=q$, hence we may define $h(1)=q^{\prime}$, and then $h(r)=r h(1)=r q^{\prime}=q$. Hence $Q$ is injective.

Conversely, if $Q$ is injective, then for any $r \in R$, consider the above diagram with $I=(r)$, and for $q \in Q$, consider the map $f: I \rightarrow Q$ given by $f(r)=q$. Since $Q$ is injective, there exists $h$ as above, in particular, there exists $q^{\prime}=h(1) \in Q$ with $r q^{\prime}=r h(1)=h(r)=q$. Hence $Q$ is divisible.

Remark 9.8. Let $R$ be a Noetherian ring. Then a direct sum of injective modules is injective. In fact, by a somewhat high-powered result, $R$ is Noetherian if and only if every countable direct sum of injective modules is injective.

Definition 9.9. A category $\mathcal{C}$ has enough injectives if every object in $\mathcal{C}$ is a subobject of an injective object.

Proposition 9.10. The category of abelian groups has enough injectives.
Proof. Let $M$ be an abelian group, and let $A$ be a set of generators for $M$. Let $F$ be the free abelian group on $A$, so we have a surjection

$$
F \rightarrow M
$$

Let $K \subset F$ be the kernel of this map, so by the first isomorphism theorem, $M \cong F / K$. Let $Q$ be the free $\mathbb{Q}$-module on $A$. Note that $Q$ is an injective $\mathbb{Z}$-module, as it is a countable direct sum of injective objects (namely copies of $\mathbb{Q}$ ). Then $K \subset F \subset Q$. Then

$$
M \cong F / K \subset Q / K
$$

Since $Q / K$ is a quotient of a divisible group, it is divisible. So $M$ is a subobject of the injective module $Q / K$.

Remark 9.11. For any (unital, associative) ring $R$, the category of $R$-modules has enough injectives. For a topological space $X$, the category $\operatorname{Sh}(X)$ of abelian groups on $X$ has enough injectives.

Proposition 9.12. Let $\mathcal{C}$ be an abelian category with enough injectives. Then every object $X$ has an injective resolution.

$$
0 \rightarrow X \rightarrow I_{0} \rightarrow I_{1} \rightarrow I_{2} \rightarrow \cdots
$$

Proof. Omitted.
Remark 9.13. Injective resolutions are not unique. However, they are unique up to chain homotopy, which is to say, any two injective resolutions of an object $X$ are chain homotopic.

Remark 9.14. Suppose $0 \rightarrow M^{\prime} \rightarrow M \rightarrow M^{\prime \prime} \rightarrow 0$ is a short exact sequence in a category with enough injectives. Then there are simultaneous resolutions of $M^{\prime}, M, M^{\prime \prime}$ which form together to make a short exact sequence of chain complexes.


### 9.2 Projectives

In this section, we omit nearly all of the proofs, since they are not materially different from the dual statements in the previous section on injectives.

Definition 9.15. An $R$-module $P$ is projective if it satsifies the following equivalent conditions.

1. Given a morphisms $g: M \rightarrow M^{\prime \prime}$ and $f: P \rightarrow M^{\prime \prime}$ with $g$ surjective, there exists a morphism $h: P \rightarrow M$ such that $g h=f$.

2. Every short exact sequence $0 \rightarrow M \rightarrow M^{\prime \prime} \rightarrow P \rightarrow 0$ is split.
3. $P$ is a direct summand of a free $R$-module.
4. The (covariant) functor $\operatorname{Hom}_{R}(P,-)$ is exact.
5. The functor $\operatorname{Hom}_{R}(P,-)$ is right exact.

Remark 9.16. By condition (3) above, a free $R$-module is projective. The converse is not true in general. However, if $R$ is a PID or a local ring, then the converse does hold, that is, over a PID or local ring, projective is equivalent to free.

Lemma 9.17. Every $R$-module is a quotient of a projective module. (More generally, every module is a quotient of a free module.)

Remark 9.18. The category of $R$-modules has enough projectives. That is, every object is a quotient of a projective object.

Definition 9.19. A projective resolution of an $R$-module $M$ is a long exact sequence

$$
\cdots \rightarrow P_{2} \rightarrow P_{1} \rightarrow M \rightarrow 0
$$

with each $P_{i}$ projective.
Proposition 9.20. If a category $\mathcal{C}$ has enough projectives, then every object has a projective resolution.

Remark 9.21. In the category of $R$-modules, every object even has a free resolution.
Lemma 9.22. Let $f: M \rightarrow N$ be a morphism of $R$-modules. Given any two projective resolutions of $M, N$ respectively, $f$ extends to a chain map between the projective resolutions.


This extension is not unique, bu it is unique up to chain homotopy, which is to say, any two such extensions are chain homotopic.

Corollary 9.23. Any two projective resolutions of a fixed module $M$ are homotopy equivalent.

Proof. Apply the previous lemma to Id : $M \rightarrow M$.
The statement of the next lemma is not intended to be precise or help someone understand if they have not seen it before. For that, the confused reader should consult other sources.

Lemma 9.24. A short exact sequence of chain complexes induces a long exact sequence on homology.

Proposition 9.25. Let $R$ be an integral domain and not a field. Suppose $M$ is an $R$-module which is projective and injective. Then $M=0$.

Proof. Let $R$ be an integral domain, and suppose $M$ is an $R$-module which is projective an injective. We will show that if $M \neq 0$, then $R$ is a field.

As a first step, we show that any nonzero morphism $M \rightarrow R$ is surjective. Consider a nonzero morphism $f: M \rightarrow R$ of $R$-modules, so there exists $m \in M$ with $f(m)=b \neq 0$. Since $M$ is injective, it is divisible, so there exists $m^{\prime} \in M$ such that $b m^{\prime}=m$. Then

$$
b f\left(m^{\prime}\right)=f\left(b m^{\prime}\right)=f(m)=b
$$

hence $f\left(m^{\prime}\right)=1$, so $f$ is surjective. Hence every nonzero morphism $M \rightarrow R$ is surjective.
As $M$ is projective, it is a direct summand of a free $R$-module $F, F \cong M \oplus Q$. Write $F$ as

$$
F \cong \bigoplus_{i} F_{i}
$$

where $F_{i} \cong R$, and let $\pi_{i}: F \rightarrow F_{i}=R$ be projection onto the $i$ th component. Let $x \in R$ be nonzero, and consider the composition

$$
M \hookrightarrow F \xrightarrow{\pi_{i}} R \xrightarrow{x} R
$$

where $x$ denotes left multiplication by $x$. Recall that we want to show $R$ is a field, so it suffices to show that $x$ is a unit. Since $R$ is an integral domain, the map $x: R \rightarrow R$ is nonzero, and clearly the other maps are nonzero, so we have a nonzero map $M \rightarrow R$. By the previous discussion, any such map is surjective. In particular, $x: R \rightarrow R$ is surjective, so there exists $y \in R$ such that $x y=1$, which is to say, $x$ is a unit, and $R$ is a field.

### 9.3 Abelian categories

We'll build up to the definition of an abelian category. We'll define additive categories, then describe the additional condition for an additive category to be abelian. First, we recall some definitions.

Definition 9.26. Let $\mathcal{C}$ be a category.

1. A morphism $f: A \rightarrow B$ in $\mathcal{C}$ is a monomorphism if for any object $C$ and morphisms $g, h: C \rightarrow A, f g=f h \Longrightarrow g=h$. That is, $f$ has a left cancellative property.

$$
C \underset{h}{\stackrel{g}{\Longrightarrow}} A \xrightarrow{f} B
$$

2. A morphism $f: A \rightarrow B$ in $\mathcal{C}$ is an epimorphism if for any object $C$ and morphisms $g, h: C \rightarrow A, g f=h f \Longrightarrow g=h$. That is, $f$ has a right cancellative property.

$$
A \xrightarrow{f} B \xrightarrow[h]{\stackrel{g}{\rightrightarrows}} C
$$

Example 9.27. In the category of abelian groups, monomorphism is equivalent to injective, and epimorphism is equivalent to surjective.

Definition 9.28. Let $\mathcal{C}$ be a category and $B$ and object in $\mathcal{C}$.

1. A subobject of $B$ is a pair $(A, i)$ where $A$ is an object and $i: A \rightarrow B$ is a monomorphism.
2. A quotient of $B$ is a pair $(C, p)$ where $C$ is an object and $p: B \rightarrow C$ is an epimorphism.

Definition 9.29. Let $f, g: A \rightarrow B$ be morphisms in a category $\mathcal{C}$. The equalizer of $f$ and $g$, if it exists, is denoted eq $(f, g)$, and is the limit of the diagram

$$
A \underset{h}{\stackrel{g}{\rightrightarrows}} B
$$

The coequalizer, if it exists, is the colimit of the above diagram. It is denoted $\operatorname{coeq}(f, g)$.
Remark 9.30. Since it has been a while since we discussed limits and colimits, let us write concretely what the limit and colimit of the simple diagram above are. The limit, if it exists, is a pair $(C, h)$ where $h: C \rightarrow A$ is a morphism such that $f h=g h$, and such that $C$ is universal with this property.


That $C$ is universal in this diagram means specifically that if $h^{\prime}: C^{\prime} \rightarrow A$ is another morphism such that $f h^{\prime}=g h^{\prime}$,

then $h^{\prime}$ factors through $h$, meaning there exists a unique morphism $k: C^{\prime} \rightarrow C$ such that $h^{\prime}=h k$.


Remark 9.31. Similar to the above, we describe more concretely the coequalizer as the colimit of such a diagram $f, g: A \rightarrow B$. It is a pair $(C, h)$ where $h: B \rightarrow C$ is a morphism such that $h f=h g$, and such that $C$ is univeral among such objects.


That is, if $h^{\prime}: B \rightarrow C$ is another such morphism, there exists a unique morphism $k: C \rightarrow C^{\prime}$ such that the following diagram commutes.


Lemma 9.32. Let $g, f: A \rightarrow B$ be morphisms in a category $\mathcal{C}$.

1. If $\mathrm{eq}(f, g)=(C, h)$ exists, then $h$ is a monomorphism.
2. If $\operatorname{coeq}(f, g)=(D, k)$ exists, then $k$ is an epimorphism.

Proof. We just do the proof of (1), since (2) is similar. Let $(C, h)$ be the equalizer of $f$ and $g$. Let $\alpha, \beta: D \rightarrow C$ be morphisms such that $h \alpha=h \beta$.

$$
D \underset{\beta}{\stackrel{\alpha}{\rightrightarrows}} C \xrightarrow{h} A \underset{g}{\stackrel{f}{\rightrightarrows}} D
$$

We need to show that $\alpha=\beta$. Since $h \alpha=h \beta$ and $f h=g h$, we get

$$
f h \alpha=f h \beta \quad g h \alpha=g h \beta
$$

So $h \alpha, h \beta$ are both maps $D \rightarrow A$ such that composing with $f$ or $g$ gives the same morphism. By the universal property of $(C, h)$ then, there is a unique map $k: D \rightarrow C$ such that $k h=h \alpha=h \beta$. In particular, we can choose $k=\alpha$ or $k=\beta$, and then by uniqueness $\alpha=\beta$.

Definition 9.33. A category $\mathcal{C}$ is additive if

1. For any two objects $A, B, \operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, and the composition of morphisms is bilinear. That is, if we have morphisms

$$
A \xrightarrow{f} B \underset{g_{2}}{\stackrel{g_{1}}{\leftrightarrows}} C \xrightarrow{h} D
$$

then

$$
h \circ\left(g_{1}+g_{2}\right) \circ f=h \circ g_{1} \circ f=h \circ g_{2} \circ f
$$

2. Binary products and coproducts exist in $\mathcal{C}$. (Then by induction, all finite products and coproducts exist.)
3. $\mathcal{C}$ has a zero object (an obejct 0 which is initial and terminal).

Definition 9.34. A functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ between additive categories is additive if for any objects $A, B$ in $\mathcal{C}$, the induced map

$$
\operatorname{Hom}_{\mathcal{C}}(A, B) \rightarrow \operatorname{Hom}_{\mathcal{D}}(\mathcal{F} A, \mathcal{F} B)
$$

is a homomorphism of abelian groups.
Example 9.35. For any commutative ring $R$, the category of $R$-modules is additive. The full subcategory of finitely generated $R$-modules is also additive.

Definition 9.36. Let $\mathcal{C}$ be an additive category and $f: A \rightarrow B$ be a morphism. Let 0 denote the zero element of $\operatorname{Hom}_{\mathcal{C}}(A, B)$.

1. The kernel of $f$, if it exists, is the equalizer of $f$ and 0 .
2. The cokernel of $f$, if it exists, is the coequalizer of $f$ and 0 .

$$
A \underset{0}{\stackrel{f}{\Longrightarrow}} B
$$

It is important to note that the kernel of $f$ is not just an object, but is a pair $(K, i)$ where $i: K \rightarrow A$ is a morphism. Similarly, the cokernel is a pair $(C, p)$ where $p: B \rightarrow C$ is a morphism. Despite this warning, we will often refer to the object $K$ as the kernel of $f$, and the morphism $i: K \rightarrow A$ as the "canonical map associated with the kernel." Ditto for the cokernel.

Remark 9.37. Let us be a bit more concrete about the universal properties of the kernel and cokernel of a morphism, assuming they exist. The kernel of $f$ is a pair $(K, i)$ where $i: K \rightarrow A$ with $f i=0$, and if $g: E \rightarrow A$ is any morphism such that $f g=0$, then $g$ factors through $i$ via a unique morphism $h: E \rightarrow K$.


Dually, the cokernel of $f$ is a pair $(C, p)$ where $p: B \rightarrow C$ with $p f=0$, and if $g: B \rightarrow E$ is any morphism such that $g f=0$, then $g$ factors through $p$ via a unique morphism $h: C \rightarrow E$.


Remark 9.38. Let $f: A \rightarrow B$ be a morphism such that the kernel $i: K \rightarrow A$ and cokernel $p: B \rightarrow C$ exist. Then

1. $(K, i)$ is a subobject of $A$, and $i$ is a monomorphism.
2. $(C, p)$ is a quotient of $B$, and $p$ is an epimorphism.

Definition 9.39. Let $f: A \rightarrow B$ be a morphism in an additive category $\mathcal{C}$, and assume the kernel and cokernel of $f$ exist. Let $i: K \rightarrow A$ and $p: B \rightarrow C$ be the kernel and cokernel, respectively.

1. The image of $f$ is the kernel of of $p: B \rightarrow C$, if it exists. That is, $\operatorname{im} f=\operatorname{ker}$ coker $f$.
2. The coimage of $f$ is the cokernel of $i: K \rightarrow A$, if it exists. That is, $\operatorname{coim} f=$ coker ker $f$.

Remark 9.40. By definition, $\operatorname{im} f$ is a subobjects of $B$, and $\operatorname{coim} f$ is a quotient of $A$.
Example 9.41. In the category of $R$-modules, the categorical kernel, cokernel, and image all coincide with the usual concrete set-theoretic descriptions. Every morphism $f$ fits into an exact sequence

$$
0 \rightarrow \operatorname{ker} f \hookrightarrow A \xrightarrow{f} B \rightarrow B / \operatorname{im} f \rightarrow 0
$$

We also have a short exact sequence

$$
0 \rightarrow \operatorname{im} f \rightarrow B \rightarrow B / \operatorname{im} f \rightarrow 0
$$

Which is to say, $\operatorname{im} f$ is the kernel of the cokernel of $f$, that is, $\operatorname{im} f \cong \operatorname{coim} f$. This is just the content of the first isomorphism theorem. This explains why the notion of coimage does not come up much in commutative algebra, since it just coincides with the image.

Remark 9.42. Although the image and coimage coincide for $R$-modules, they do not coincide in general for additive categories. We will give a concrete example later.

Recall that we abuse language slightly in referring to $\operatorname{ker} f$ as an object, rather than as a pair. But we do it anyway, because who can stop us?

Lemma 9.43. Let $f: A \rightarrow B$ be a morphism in an additive category $\mathcal{C}$.

1. If $\operatorname{ker} f$ exists, then $f$ is a monomorphism if and only if $\operatorname{ker} f$ is the zero object.
2. If coker $f$ exists, then $f$ is an epimorphism if and only if coker $f$ is the zero object.

Proof. Use some universal properties. Been there, done that.
Lemma 9.44. Let $f: A \rightarrow B$ be a morphism in an additive category $\mathcal{C}$, and assume $\operatorname{ker} f, \operatorname{coker} f, \operatorname{im} f, \operatorname{coim} f$ exist. Let $s: A \rightarrow \operatorname{coim} f$ and $t: \operatorname{im} f \rightarrow B$ be the canonical maps. Then there is a unique morphism

$$
u: \operatorname{coim} f \rightarrow \operatorname{im} f
$$

such that $t \circ u \circ s=f$.


Proof. Let $i: \operatorname{ker} f \rightarrow A$ and $j: B \rightarrow$ coker $f$ be the canonical maps associated with the kernel and cokernel respectively. By definition,

$$
\operatorname{im} f=\operatorname{ker} \operatorname{coker} f=\operatorname{ker} j \quad \operatorname{coim} f=\operatorname{coker} \operatorname{ker} f=\operatorname{coker} i
$$

By definition of the cokernel, $j \circ f=0$. Then by the universal property of the kernel, $f$ factors through ker $j=\operatorname{im} f$. That is, there exists a unique morphism $f^{\prime}: A \rightarrow \operatorname{im} f=\operatorname{ker} j$ making the following diagram commute.


Then $t \circ f^{\prime} \circ i=f \circ i=0$ by definition of the kernel. So $t \circ\left(f^{\prime} \circ i\right)=t \circ\left(f^{\prime} \circ 0\right)$.

$$
\operatorname{ker} f \xrightarrow[f^{\prime} \circ 0]{\stackrel{f^{\prime} \circ i}{\longrightarrow}} \operatorname{im} f \xrightarrow{t} B
$$

By Lemma 9.32, $t$ is a monomorphism, so we can cancel it to get $f^{\prime} \circ i=0$. Then by the universal property of the cokernel, $f^{\prime}$ factors (uniquely) through coim $f=$ coker $i$. That is, there exists a unique morphism $u: \operatorname{coim} f=\operatorname{coker} i \rightarrow \operatorname{im} f=\operatorname{ker} j$ making the following diagram commute.


Thus $u$ satisfies

$$
t \circ u \circ s=t \circ f^{\prime}=f
$$

Uniqueness of $u$ is clear from the uniqueness of all the maps constructed in the proof.
Remark 9.45. The morphism $u: \operatorname{coim} f \rightarrow \operatorname{im} f$ is called the canonical map associated to $f$.

Definition 9.46. Let $f: A \rightarrow B$ be a morphism in an additive category $\mathcal{C}$. The morphism $f$ is strict if the canonical map $u: \operatorname{coim} f \rightarrow \operatorname{im} f$ is an isomorphism (and all of the required kernels and cokernels exist, of course).

Example 9.47. As previously discussed, the fact that every morphism in the category of $R$-modules is strict is essentially the content of the first isomorphism theorem.

Definition 9.48. An abelian category $\mathcal{A}$ is an additive category such that

1. Every morphism has a kernel and cokernel.
2. Every morphism is strict.

Remark 9.49. The previous example of $R$-mod serves as a good philosophical framework to understand abelian categories - they are categories in which a sort of generalized first isomorphism theorem holds.

Example 9.50. The category of $R$-modules is abelian.
Example 9.51. Let $R$ be a non-Noetherian ring. The subcategory of finitely generated $R$-modules is additive but NOT abelian. In particular, kernels may fail to exist.

As a concrete and simple example, take an ideal $I$ of $R$ which is not finitely generated (such $I$ exists precisely because $R$ is non-Noetherian). Then consider the quotient map

$$
f: R \rightarrow R / I
$$

As an $R$-module, $R$ is finitely generated (it is generated by 1 , in fact), and so is $R / I$ (it is generated by the class of 1 ). However, the kernel (in the category of $R$-modules) is $I$, and the kernel in this subcategory would also have to be $I$, except that $I$ is not an object in this category, since $I$ is not finitely generated as an $R$-module.

Example 9.52. As previously promised, we now give a concrete example of a homomorphism in an additive category for which the image and coimage are distinct objects. In particular, this gives an example of an additive category in which kernels and cokernels exist, but that not every morphism is strict.

As our category, take $\mathcal{C}$ to be the category of Hausdorff topological abelian groups. Both $\mathbb{R}$ and $\mathbb{Q}$, using the additive structures and the standard topologies (subspace topology from $\mathbb{R}$ on $\mathbb{Q}$ ), are objects in this category. Consider the inclusion map

$$
f: \mathbb{Q} \hookrightarrow \mathbb{R}
$$

which is a continuous group homomorphism, and so a morphism in $\mathcal{C}$. It is relatively clear that the kernel of $f$ is the pair $(0,0)$, where $0: 0 \hookrightarrow \mathbb{Q}$ is the trivial morphism. Slightly less obviously, the cokernel of $f$ is also the pair $(0,0)$, where $0: \mathbb{R} \rightarrow 0$ is the trivial morphism. We verify the universal property. Suppose $g: \mathbb{R} \rightarrow E$ is a homomorphism of topological groups such that $g f=0$, which is to say, $g$ vanishes on $\mathbb{Q}$.


Since $g$ is continuous, and vanishes on the dense subset $\mathbb{Q}, g=0$. So there is a unique map $0: 0 \rightarrow E$ making the diagram commute. So $(0,0)$ is in fact the cokernel of $f$. From there is is immediate to calculate the image and coimage.

$$
\begin{aligned}
& \operatorname{im} f=\operatorname{ker} \text { coker } f=\operatorname{ker}(\mathbb{R} \rightarrow 0)=\mathbb{R} \\
& \operatorname{coim} f=\operatorname{coker} \operatorname{ker} f=\operatorname{coker}(0 \rightarrow \mathbb{Q})=\mathbb{Q}
\end{aligned}
$$

So the image and coimage are not isomorphic. In particular, the canonical map coim $f \rightarrow$ $\operatorname{im} f$ cannot be an isomorphism.

We end this section with a statement (and no proof) of a classic theorem about abelian categories which is simultaneously very important philosophically, but not very practically useful.

Theorem 9.53 (Freyd-Mitchell embedding theorem). Let $\mathcal{A}$ be a small abelian category. There exists a ring $R$ and an exact, fully faithful functor $\mathcal{A} \rightarrow \mathrm{R}$-mod. That is, $\mathcal{A}$ is equivalent to a full abelian subcategory of $\mathrm{R}-\mathrm{mod}$ for some ring $R$.

Remark 9.54. The previous theorem philosophically and logically justifies doing "diagram chases" in an arbitrary abelian category, even though properly speaking the objects of a general abelian category do not have "elements."

However, it is easy to overestimate the importance and usefulness of the theorem, since some aspects of the structure of $\mathcal{A}$ may not be preserved. In particular, a fully faithful exact additive functor necessarily preserves finite limits and colimits, but may not preserve infinite limits and colimits. Hence not necessarily all aspects of the structure of $\mathcal{A}$ are captured by this "embedding."

### 9.4 Homology in abelian categories

Definition 9.55. Let $\mathcal{A}$ be an abelian category. A cochain complex in $\mathcal{A}$ is a diagram

$$
\cdots \rightarrow A^{i-1} \xrightarrow{d_{A}^{i-1}} A^{i} \xrightarrow{d_{A}^{i}} A^{i+1} \rightarrow \cdots
$$

such that $d_{A}^{i} \circ d_{A}^{i-1}=0$ for every $i \in \mathbb{Z}$. We use the notation $A^{\bullet}=\left(A^{i}, d_{A}^{i}\right)$ to refer to the whole cochain complex.

Definition 9.56. Suppose

$$
A \xrightarrow{f} B \xrightarrow{g} C
$$

are morphisms in an abelian category $\mathcal{A}$ such that $g f=0$. As $f$ is strict, we can write $f$ as a composition

$$
A \xrightarrow{\ell} \operatorname{im} f \cong \operatorname{coim} f \xrightarrow{k} B
$$

where we identify $\operatorname{im} f$ with coim $f$ with the canonical isomorphism. Note that in the above, $\ell$ is an epimorphism, and $k$ is a monomorphism. Since $0=g f=g k \ell=0 \ell$, since $\ell$ is an epimorphism, this implies $g k=0$.

Using the universal property of the kernel, we then get a unique morphism $t: \operatorname{im} f \rightarrow \operatorname{ker} g$ such that $u t=k$, where $u$ is the canonical morphism associated with ker $g$. Since both $k, u$ are monomorphisms, $t$ is also.

The homology or cohomology ${ }^{15}$ of the sequence $A \rightarrow B \rightarrow C$ is the cokernel of the monomoprhism $t$, by which we really mean the object associated with the cokernel of $t$. We say that $A \rightarrow B \rightarrow C$ is exact at $B$ if the homology is the zero object, or equivalently, if $t$ is an epimorphism, or equivalently, if $t$ is an isomorphism.

Definition 9.57. A sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ (or longer) is exact if it exact at each term.

Definition 9.58. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ be an additive functor between abelian categories. $\mathcal{F}$ is left exact if for every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C$ in $\mathcal{C}$, the sequence

$$
0 \rightarrow \mathcal{F} A \rightarrow \mathcal{F} B \rightarrow \mathcal{F} C
$$

is exact. Similarly, one defines right exact and exact functors, as you would expect.
Definition 9.59. Let $A^{\bullet}=\left(A^{i}, d_{A}^{i}\right), B^{\bullet}=\left(B^{i}, d_{B}^{i}\right)$ be cochain complexes. A morphism of cochain complexes $f: A^{\bullet} \rightarrow B^{\bullet}$ is a family of morphisms $f_{i}: A^{i} \rightarrow B^{i}$ which commute with the differentials.

$$
d_{B}^{i} \circ f_{i}=f_{i+1} \circ d_{A}^{i}
$$

That is, a certain large commutative diagram which looks like a ladder is commutative. Similarly following the definitions in the category R-mod, we define chain homotopy between morphisms of cochain complexes.

Given an abelian category $\mathcal{A}$, cochain complexes with entries in $\mathcal{A}$ form a category of their own, $\operatorname{Kom}(\mathcal{A})$, with morphisms given by morphisms of cochain complexes. $\operatorname{Kom}(\mathcal{A})$ is also an abelian category, although we omit the proof.

Definition 9.60. A short exact sequence of cochain complexes is a sequence $A^{\bullet} \rightarrow$ $B^{\bullet} \rightarrow C^{\bullet}$ such that each sequence $A^{i} \rightarrow B^{i} \rightarrow C^{i}$ is exact.

[^13]Remark 9.61. Properly speaking, the previous definition is the "wrong" definition, in the sense that one should just define it using the previous definition of exactness in the abelian category $\operatorname{Kom}(\mathcal{A})$. However, it turns out that the previous definition is then a theorem, so whatever.

Definition 9.62. Let $A^{\bullet}=\left(A^{i}, d_{A}^{i}\right)_{i \in \mathbb{Z}}$ be a cochain complex in an abelian category $\mathcal{A}$. Define

$$
Z^{i}\left(A^{\bullet}\right)=\operatorname{ker} d_{A}^{i} \quad B^{i}\left(A^{\bullet}\right)=\operatorname{coker} d_{A}^{i-1}
$$

with canonical monomorphism $Z^{i}\left(A^{\bullet}\right) \rightarrow A^{i}$ and canonical epimorphism $A^{i-1} \rightarrow B^{i}\left(A^{\bullet}\right)$. Following our previous definition of cohomology at a single term, there is a canonically induced morphism

$$
t^{i}: B^{i}\left(A^{\bullet}\right) \rightarrow Z^{i}\left(A^{\bullet}\right)
$$

We define the $i$ th cohomology of $A^{i}$ to be the (object associated with) the cokernel of $t^{i}$.

$$
H^{i}\left(A^{\bullet}\right)=\operatorname{coker} t^{i}
$$

Remark 9.63. The previous definition of cohomology of cochain complexes is the appropriate generalization of cohomology of chain complexes of $R$-modules to a general abelian category. It has all the properties we would expect, which we do not spend time to justify.

1. A morphism of chain complexes $A^{\bullet} \rightarrow B^{\bullet}$ induces morphisms on each cohomology object $H^{i}\left(A^{\bullet}\right) \rightarrow H^{i}\left(B^{\bullet}\right)$. In fact, given an abelian category $\mathcal{A}$, we can think of each $H^{i}(-)$ as a (covariant, additive) functor $\operatorname{Kom}(A) \rightarrow \mathcal{A}$.
2. If two morphisms of chain complexes $A^{\bullet} \rightarrow B^{\bullet}$ are chain homotopic, then they induce the same morphisms on homology. (The cohomology functors "factor" through the "homotopy category of chain complexes.")
3. A short exact sequence of chain complexes induces a long exact sequence on cohomology, in the same manner as in the category of $R$-modules.

Definition 9.64. Let $\mathcal{C}$ be a category. An object $I$ in $\mathcal{C}$ is injective if for every monomor$\operatorname{phism} f: X \rightarrow Y$ and every morphism $g: X \rightarrow I$, there exists a morphism $h: Y \rightarrow I$ such that $h f=g$.


Remark 9.65. If $\mathcal{C}$ is a locally small category (which includes all of the categories we care about in these notes), then an object $I$ is injective in $\mathcal{C}$ if and only if the induced map

$$
\operatorname{Hom}_{\mathcal{C}}(Y, I) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, I) \quad \phi \mapsto \phi \circ f
$$

is surjective. (This only makes sense when $\mathcal{C}$ is locally small, since otherwise we would have to talk about a "surjective function" between things which are not sets, which is scary, and also nonsense.)

If $\mathcal{C}$ is furthermore an additive category, then the previous condition is equivalent to exactness (or just right exactness) of the functor $\operatorname{Hom}_{\mathcal{C}}(-, I)$ (which is a functor from $\mathcal{C}$ to abelian groups, and is always left exact).

Definition 9.66. A category $\mathcal{C}$ has enough injectives if every object $X$ is a subobject of an injective object. Equivalently, for every object $X$, there is a monomorphism $X \rightarrow I$ where $I$ is an injective object.

Remark 9.67. Injective resolutions are defined in an abelian category as you would expect, following the definition in R-mod. If an abelian category has enough injectives, then every object has an injective resolution. The proof is essentially the same as in R-mod, just more categorical. As in R-mod, a morphism between objects extends to a chain map between any two injective resolutions, and this extension is unique up to chain homotopy.

Proposition 9.68. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$ be an adjoint pair of additive functors $(\mathcal{F}, \mathcal{G})$, and suppose that $\mathcal{F}$ is exact. Then $\mathcal{G}$ takes injectives to injectives. That is, if $I$ is an injective object of $\mathcal{D}$, then $\mathcal{G I}$ is an injective object of $\mathcal{C}$.

Proof. This is just Proposition 8.40 repeated.
Now we get to the big result, which is the point of building up all of the previous formalism about abelian categories.

Theorem 9.69. Let $X$ be a topological space. The category $\operatorname{Sh}(X)$ of sheaves of abelian groups on $X$ is an abelian category with enough injectives.

Proof. We need to prove the following list of things.

1. $\operatorname{Sh}(X)$ is additive.
2. $\operatorname{Sh}(X)$ has kernels.
3. $\operatorname{Sh}(X)$ has cokernels.
4. Every morphism in $\operatorname{Sh}(X)$ is strict.
5. $\operatorname{Sh}(X)$ has enough injectives.
(1) To prove $\operatorname{Sh}(X)$ is additive, we need to describe the abelian group structure on a hom set, describe the zero object, and show that finite products and coproducts exist.

Let $\mathcal{F}, \mathcal{G}$ be sheaves on $X$, and $\phi, \psi: \mathcal{F} \rightarrow \mathcal{G}$ be morphisms. The sum $\phi+\psi$ is defined on sections on an open set $U \subset X$ by

$$
(\phi+\psi)_{U}=\phi_{U}+\psi_{U}: \mathcal{F}(U) \rightarrow \mathcal{G}(U)
$$

This is compatible with restriction maps in the appropriate way to make $\phi+\psi$ a morphism of sheaves. It is clear that this makes $\operatorname{Hom}_{\operatorname{Sh}(X)}(\mathcal{F}, \mathcal{G})$ into an abelian group, in a way which is compatible with composition of morphisms.

The zero object in $\operatorname{Sh}(X)$ is the constant sheaf given by $0(U)=\{*\}$.
Finite products and coproducts coincide in the category of abelian groups, they are usually denoted $\times$ or $\oplus$. Given two sheaves $\mathcal{F}, \mathcal{G}$ on $X$, the product sheaf is given by $\mathcal{F} \times \mathcal{G}$, which is defined on an open set $U \subset X$ by

$$
(\mathcal{F} \times \mathcal{G})(U)=\mathcal{F}(U) \times \mathcal{G}(U)
$$

There is some checking to do, but this is a product and coproduct in the category of sheaves on $X$. This completes the proof that $\operatorname{Sh}(X)$ is additive.
(2) Kernels in $\operatorname{Sh}(X)$ are not that hard to describe, just use the kernel presheaf which we have already defined. More precisely, given a morphism of sheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$, let $\mathcal{K}=\operatorname{ker} \phi$ be the presheaf defined by

$$
\mathcal{K}(U)=\operatorname{ker}\left(\phi_{U}\right)
$$

We showed before that this is a sheaf. We will give a rough justifcation that this is the kernel. Let $i: \mathcal{K} \rightarrow \mathcal{F}$ be the expected morphism. Any morphism $\alpha: \mathcal{H} \rightarrow \mathcal{F}$ which satisfies $\phi \alpha=0$ is going to be zero on any open subset, so $\alpha_{U}$ will factor (uniquely) through the inclusion $i_{U}: \mathcal{K}(U) \hookrightarrow \mathcal{F}(U)$, so $\alpha$ factors through $i$ (uniquely).
(3) Now we show that $\operatorname{Sh}(X)$ has cokernels. This is more complicated than kernels, because the obvious choice of object for the cokernel is merely a presheaf, and not in general a sheaf, so we have to resort to sheafification to rectify the situation.

Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morhpism of sheaves, and let $\mathcal{C}$ be the cokernel presheaf, defined by

$$
\mathcal{C}(U)=\operatorname{coker}\left(\phi_{U}\right)=\mathcal{G}(U) / \operatorname{im} \phi_{U}
$$

Let $j: \mathcal{G} \rightarrow \mathcal{C}$ be the obvious morphism, where $j_{U}: \mathcal{G}(U) \rightarrow \mathcal{C}(U)$ is the quotient map. As $\mathcal{C}$ is not generally a sehaf, let $\theta: \mathcal{C} \rightarrow \mathcal{C}^{+}$be the sheafification of $\mathcal{C}$. Then define $c: \mathcal{G} \rightarrow \mathcal{G}$ by $c=\theta \circ j$. We claim that $\left(\mathcal{C}^{+}, c\right)$ satisfies the universal property of the cokernel (for $\phi$ ) in the category of sheaves on $X$.

Suppose $\psi: \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of sheaves such that $\psi \phi=0$. We need a morphism $\mathcal{H} \rightarrow \mathcal{C}^{+}$making the following diagram commute, and to show that such a morphism is unique.


Note that $j$ is an epimorphism, and that $\theta$ induces isomorphisms on stalks at each $x \in X$, $\theta_{x}: \mathcal{C}_{x} \cong \mathcal{C}_{x}^{+}$, which makes $c: \mathcal{G} \rightarrow \mathcal{C}^{+}$an epimorphism.

For every open subset $U \subset X$, we have $\psi_{U}: \mathcal{G}(U) \rightarrow \mathcal{H}(U)$, and $\psi_{U} \circ \phi_{U}=0$, so by the universal property of cokernels for abelian groups, we get a unique induced morphism $\mathcal{C}(U) \rightarrow \mathcal{H}(U)$. Modulo some small verification, this gives a morphism of presheaves $\mathcal{C} \rightarrow \mathcal{H}$.

Then using the universal property of sheafification, since $\mathcal{H}$ is a sheaf, the map above factors through $\mathcal{C}^{+}$, giving the desired map $\widetilde{\psi}: \mathcal{C}^{+} \rightarrow \mathcal{H}$, where $\widetilde{\psi}=\psi \circ c$. This is the desired map in the previous commutative diagram. Uniqueness of $\widetilde{\psi}$ comes (after some details) from the fact that $c$ is an epimorphism, so there is a right cancellation property. This completes our proof that $\left(\mathcal{C}^{+}, c\right)$ is the cokernel of $\phi$ in $\operatorname{Sh}(X)$.
(4) We will just sketch the proof that every morphism in $\operatorname{Sh}(X)$ is strict. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves, and let $u: \operatorname{coim} \phi \rightarrow \operatorname{im} \phi$ be the induced natural map. Consider the induced map on sections, $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$, and $u_{x}:(\operatorname{coim} \phi)_{x} \rightarrow(\operatorname{im} \phi)_{x}$. One can show that $u_{x}$ is the natural map from the coimage to the image of $\phi_{x}$; that is, $(\operatorname{coim} \phi)_{x}=\operatorname{coim}\left(\phi_{x}\right)$ and $(\operatorname{im} \phi)_{x}=\operatorname{im}\left(\phi_{x}\right)$, and $u_{x}$ is the associated natural map. By the first isomorphism theorem (aka morphisms of abelian groups are strict), $u_{x}$ is an isomorphism of abelian groups. Hence $u$ is an isomorphism of sheaves.
(5) Let $\mathcal{F}$ be a sheaf on $X$. For $x \in X$, the stalk $\mathcal{F}_{x}$ is an abelian group. We know that the category of abelian groups has enough injectives, so let

$$
i_{x}: \mathcal{F}_{x} \hookrightarrow I_{x}
$$

be an embedding of $\mathcal{F}_{x}$ into an injective (divisible) abelian group $I_{x}$. We want to translate this local data into global data, that is, into a monomorphism of sheaves $\mathcal{F} \rightarrow \mathcal{I}$ where $\mathcal{I}$ is some injective sheaf on $X$, whose stalk at $x$ is the groups $I_{x}$.

We can view $i_{x}$ as a morphism of constant sheaves on the space $\{x\}$. In fact, the category of (constant) sheaves on $\{x\}$ is equivalent to the category of abelian groups. That is, both abelian groups $\mathcal{F}_{x}, I_{x}$ determine constant sheaves on $\{x\}$, meaning that the respective constant sheaves take the values $\mathcal{F}_{x}, I_{x}$ respectively on the set $\{x\}$. Note that $I_{x}$ is an injective sheaf on $\{x\}$, using the equivalence.

Now consider the inclusion

$$
j_{x}:\{x\} \hookrightarrow X
$$

We can then consider the pushforward of $I_{x}$ under $j_{x}$, and the pullback of $\mathcal{F}$ under $j_{x}$. The pushforward is $\left(j_{x}\right)_{*} I_{x}$, which is a sheaf on $X$. Recall that $\left(j_{x}^{-1},\left(j_{x}\right)_{*}\right)$ form an adjoint pair, so by Proposition $8.40,\left(j_{x}\right)_{*} I_{x}$ is an injective sheaf on $X$.

The pullback $j_{x}^{-1} \mathcal{F}$ is a sheaf on $\{x\}$, defined as the sheafification of some direct limit, but since the space $\{x\}$ is so small, this is easy to describe - it is just the constant sheaf with value group $\mathcal{F}_{x}$. That is,

$$
j_{x}^{-1}(\mathcal{F})(\{x\})=\mathcal{F}_{x}
$$

So identifying (the constant sheaf on $\{x\}) \mathcal{F}_{x}$ with $j_{x}^{-1} \mathcal{F}$, we can now view $i_{x}$ as a morphism (of sheaves on $\{x\}$ )

$$
i_{x}: j_{x}^{-1} \mathcal{F} \rightarrow I_{x}
$$

Recall that $\left(j_{x}^{-1},\left(j_{x}\right)_{*}\right)$ form an adjoint pair (Proposition 8.29). That is, we have an isomorphism

$$
\operatorname{Hom}_{\operatorname{Sh}(\{x\})}\left(j_{x}^{-1} \mathcal{F}, I_{x}\right) \cong \operatorname{Hom}_{\operatorname{Sh}(X)}\left(\mathcal{F},\left(j_{x}\right)_{*} I_{x}\right)
$$

Let $i_{x}^{\prime}: \mathcal{F} t=\left(j_{x}\right)_{*} I_{x}$ be the the morphism of sheaves on $X$ corresponding to $i_{x}$ under the above isomorphism. Define

$$
\mathcal{I}=\prod_{x \in X}\left(j_{x}\right)_{*} I_{x}
$$

Since each $\left(j_{x}\right)_{*} I_{x}$ is an injective sheaf, $\mathcal{I}$ is an injective sheaf. Taking the product of maps $i_{x}^{\prime}: \mathcal{F} \rightarrow\left(j_{x}\right)_{*} I_{x}$, we obtain a morphism of sheaves

$$
i=\prod_{x \in X} i_{x}^{\prime}: \mathcal{F} \rightarrow \mathcal{I}=\prod_{x \in X}\left(j_{x}\right)_{*} I_{x}
$$

All that remains to show is that $i$ is a monomorphism in the category $\operatorname{Sh}(X)$, or equivalently, that the morphism induced on stalks is injective (in the category of abelian groups). However, it is obvious that the morphism induced on stalks is the original map $i_{x}: \mathcal{F}_{x} \rightarrow I_{x}$, so this is proven.

### 9.5 Sheaves of $\mathcal{O}$-modules

This section is a quick aside on a generalization of the category of sheaves, which is commonly used in algebraic geometry.

Definition 9.70. A ringed space is a pair $(X, \mathcal{O})$ where $X$ is a topological space and $\mathcal{O}$ is a sheaf of (usually commutative) rings on $X$.

Definition 9.71. Let $(X, \mathcal{O})$ be a ringed spaced. A sheaf of $\mathcal{O}$-modules, or just an $\mathcal{O}$ module, is a sheaf $M$ of abelian groups on $X$, such that for each open subset $U \subset X$, there is a map

$$
\mu_{U}: \mathcal{O}(U) \times M(U) \rightarrow M(U)
$$

making $M(U)$ into an $\mathcal{O}(U)$-module, and in a way so that the maps $\mu_{U}$ are compatible with the restriction maps for $\mathcal{O}$ and $M$ in the "expected way." ${ }^{16}$ We omit describing this in detail, for details consult other sources such as Hartshorne's book on algebraic geometry.

Definition 9.72. Given a ringed space $(X, \mathcal{O})$, using the definition above we get a category of (sheaves of) $\mathcal{O}$-modules on $X$.

Remark 9.73. The category $\operatorname{Sh}(X)$ is just a special case of the category of $\mathcal{O}$-modules on $X$, where $\mathcal{O}$ is the locally constant sheaf with value group $\mathbb{Z}$.

Theorem 9.74. The category of $\mathcal{O}$-modules on a ringed space $X$ is an abelian category with enough injectives.

Proof. The proof is essentially a repeat of our proof that $\operatorname{Sh}(X)$ is abelian with enough injectives. The only real difference is in the injectives aspect, which is a bit more complicated.

[^14]
## $9.6 \quad \delta$-functors

Definition 9.75. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A cohomological $\delta$-functor from $\mathcal{A}$ to $\mathcal{B}$ is a collection of functors $T^{i}: \mathcal{A} \rightarrow \mathcal{B}$ for $i \in \mathbb{Z}_{\geq 0}$ together with morphisms $\delta^{i}$ so that for every short exact sequence

$$
0 \rightarrow A^{\prime} \rightarrow A \rightarrow A^{\prime \prime} \rightarrow 0
$$

in $\mathcal{A}$, we have a morphism in $\mathcal{B}, \delta^{i}: T^{i}\left(A^{\prime \prime}\right) \rightarrow T^{i}\left(A^{\prime}\right)$, such that there is a long exact sequence (in $\mathcal{B}$ )

$$
0 \rightarrow T^{0}\left(A^{\prime}\right) \rightarrow T^{0}(A) \rightarrow T^{0}\left(A^{\prime \prime}\right) \xrightarrow{\delta^{0}} T^{1}\left(A^{\prime}\right) \rightarrow \cdots
$$

Furthermore, the process of taking short exact sequences to long exact sequences must be functorial, meaning that a morphism of short exact sequences induces a morphism of long exact sequences. The content of the previous statement is just that given a morphism of short exact sequences

the following diagrams commute.


This condition for the cohomological $\delta$-functor is called naturality.
Definition 9.76. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A cohomological $\delta$-functor $\left(T^{i}, \delta^{i}\right)$ is called universal if given any other $\delta$-functor $\left(\left(T^{i}\right)^{\prime}\right)$ and natural transformation $\mathcal{F}^{0}: T^{0} \rightarrow\left(T^{0}\right)^{\prime}$, there is a unique sequence of natural transformations $\mathcal{F}^{i}: T^{i} \rightarrow\left(T^{i}\right)^{\prime}$ for $i \geq 0$ starting with $\mathcal{F}^{0}$, such that the $\mathcal{F}^{i}$ commute with the $\delta^{i}$ maps.

Remark 9.77. Philosophically speaking, a universal $\delta$-functor is determined up to isomorphism by the zeroth term (the zeroth functor $T^{0}$ ).

Definition 9.78. An additive functor $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories is effaceable if for each object $A$ of $\mathcal{A}$, there exists an object $M$ in $\mathcal{A}$ such that $\mathcal{F}(M)=0$ and a monomorphism $u: A \rightarrow M$.
Example 9.79. Given a ring $R$ and an injective $R$-module $A$, the functor $\operatorname{Ext}_{R}^{n}(A,-)^{17}$ vanishes on injective objects, and every $R$-module admits a monomorphism to an injective object $I$, so $\operatorname{Ext}_{R}^{n}(A,-)$ is effaceable.
Theorem 9.80 (Grothendieck universality theorem). Let $T=\left(T^{i}\right)$ be a cohomological $\delta$ functor between abelian categories $\mathcal{A}, \mathcal{B}$. If $T^{i}$ is effaceable for each $i \geq 1$, then $T$ is universal.

[^15]
### 9.7 Right derived functors

Definition 9.81. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact covariant additive functor. For an object $A \in \operatorname{Ob}(\mathcal{A})$, choose an injective resolution.

$$
I^{\bullet} \quad 0 \rightarrow A \rightarrow I^{0} \rightarrow I^{1} \rightarrow \cdots
$$

Applying $\mathcal{F}$ and dropping the $A$ term, we obtain a chain complex, which is in general not exact.

$$
\mathcal{F} I^{\bullet} \quad 0 \rightarrow \mathcal{F} I^{0} \rightarrow \mathcal{F} I^{1} \rightarrow \cdots
$$

We define the $i$ th right derived functor of $\mathcal{F}$ to be the functor $R^{i} \mathcal{F}$ given by $R^{i} \mathcal{F}(A)=$ $H^{i}\left(\mathcal{F} I^{\bullet}\right)$.

Theorem 9.82. Let $\mathcal{F}, \mathcal{A}, \mathcal{B}$ be as in the previous definition.

1. For each $A \in \operatorname{Ob}(\mathcal{A})$, the object $R^{i} \mathcal{F}(A)$ is well defined up to isomorphism (independent of the injective resolution of $A$ ), and $R^{i} \mathcal{F}$ is an additive functor $\mathcal{A} \rightarrow \mathcal{B}$.
2. There is a natural isomorphism of functors $\mathcal{F} \cong R^{0} \mathcal{F}$.
3. The collection $\left(R^{i} \mathcal{F}\right)_{i \geq 0}$ is a cohomological $\delta$-functor from $\mathcal{A}$ to $\mathcal{B}$.
4. If $I$ is injective, $R^{i} \mathcal{F}(I)=0$ for $i \geq 1$. That is, $R^{i} \mathcal{F}$ is effacable, so the family $\left(R^{i} \mathcal{F}\right)$ is a universal $\delta$-functor.

Proof. (1) Let $A, A^{\prime}$ be objects in $\mathcal{A}$, and take injective resolutions $I^{\bullet}(A), I^{\bullet}\left(A^{\prime}\right)$. A morphism $f: A \rightarrow A^{\prime}$ extends to a morphism of chain complexes $f: I^{\bullet}(A) \rightarrow I^{\bullet}\left(A^{\prime}\right)$, uniquely up to homotopy.

Consider two resolutions of $A, I_{1}^{\bullet}(A), I_{2}^{\bullet}(A)$. Extend the identity map Id : $A \rightarrow A$ to a morphism of complexes $f_{1}: I_{1}^{\bullet}(A) \rightarrow I_{2}^{\bullet}(A)$ and also to a morphism of complexes $f_{2}: I_{2}^{\bullet}(A) \rightarrow I_{1}^{\bullet}(A)$. Then by the uniqueness up to homotopy, $f_{1} \circ f_{2}$ and $f_{2} \circ f_{1}$ are each chain homotopic to the respective identity chain maps on $I_{1}^{\bullet}(A), I_{2}^{\bullet}(A)$. In particular, $f_{1}, f_{2}$ must induce isomorphisms on homology.

Now apply $\mathcal{F}$ to the whole situation (to some gigantic diagram which I'm too lazy to type up). Then $\mathcal{F} f_{1}, \mathcal{F} f_{2}$ also induce isomorphisms on homology. This proves that $R^{i} \mathcal{F}(A)$ is well defined regardless of the choice of injective resolution.

From the procedure above, it is relatively clear how to induce a map $R^{i} \mathcal{F}(A) \rightarrow R^{i} \mathcal{F}(B)$ form a morphism $A \rightarrow B$, using the same methods. This makes $R^{i} \mathcal{F}$ a functor. It is clearly additive since a direct sum of objects has an injective resolution given by taking the direct sum at each term of respective injective resolutions (noting that the direct sum of injective objects is injective).
(2) Since $\mathcal{F}$ is left exact,

$$
0 \rightarrow \mathcal{F} A \rightarrow \mathcal{F} I^{0} \rightarrow \mathcal{F} I^{1}
$$

is exact. So $R^{0} \mathcal{F}(A)$, which is by definition the kernel of $\mathcal{F} I^{0} \rightarrow \mathcal{F} I^{1}$, is isomorphic to $A$. It takes some more checking, but this is in fact a natural isomorphism of functors $R^{0} \mathcal{F} \cong \mathcal{F}$.
(3) Given a short exact sequence $0 \rightarrow A^{\prime} \rightarrow A \rightarrow A^{\prime \prime} \rightarrow 0$ in $\mathcal{A}$, we need to product a long exact sequence involving the right derived functors of $\mathcal{F}$. Using the horseshoe lemma, we can choose compatible injective resolutions of $A^{\prime}, A, A^{\prime \prime}$ making a short exact sequence of complexes,

$$
0 \rightarrow I^{\bullet}\left(A^{\prime}\right) \rightarrow I^{\bullet}(A) \rightarrow I^{\bullet}\left(A^{\prime \prime}\right) \rightarrow 0
$$

For each $i \geq 0$, the object $I^{i}(A)$ is injective, so the sequence $0 \rightarrow I^{i}\left(A^{\prime}\right) \rightarrow I^{i}(A) \rightarrow I^{i}\left(A^{\prime \prime}\right) \rightarrow$ 0 is split. So applying $\mathcal{F}$ to this, it is still exact. As long as we drop the first row (of the original sequence $0 \rightarrow A^{\prime} \rightarrow A \rightarrow A^{\prime \prime} \rightarrow 0$ after applying $\mathcal{F}$, we get a short exact sequence of chain complexes

$$
0 \rightarrow \mathcal{F} I^{\bullet}\left(A^{\prime}\right) \rightarrow \mathcal{F} I^{\bullet}(A) \rightarrow \mathcal{F} I^{\bullet}\left(A^{\prime \prime}\right) \rightarrow 0
$$

We need to drop the $A$ terms because that sequence does not necessarily remain exact after applying $\mathcal{F}$, since $\mathcal{F}$ is only left exact. But once we do that, we have this short exact sequence of chain complexes, and then by a standard lemma in homological algebra, this gives rise to a long exact sequence on homology.

$$
0 \rightarrow R^{0} \mathcal{F}\left(A^{\prime}\right) \rightarrow R^{0} \mathcal{F}(A) \rightarrow R^{0} \mathcal{F}\left(A^{\prime \prime}\right) \rightarrow R^{1} \mathcal{F}\left(A^{\prime}\right) \rightarrow \cdots
$$

which makes the family $\left(R^{i} \mathcal{F}\right)$ into a cohomological $\delta$-functor. We have omitted many details, such as why the naturality condition holds.
(4) If $I$ is injective, we have the somewhat trivial resolution

$$
0 \rightarrow I \rightarrow I \rightarrow 0
$$

of $I$, applying $\mathcal{F}$ and dropping the first term, we get the chain complex $0 \rightarrow \mathcal{F} I \rightarrow 0$, whose only homology is in degree zero. Hence $R^{0} \mathcal{F}(I)=\mathcal{F}(I)$, and $R^{i} \mathcal{F}(I)=0$ for $i \geq 1$. Hence $R^{i} \mathcal{F}$ is effaceable for $i \geq 1$.
(5) By Grothendieck's theorem, since $R^{i} \mathcal{F}$ is effaceable for each $i \geq 1$, the $\delta$-functor ( $R^{i} \mathcal{F}$ ) is universal.

Definition 9.83. Let $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact additive covariant functor between abelian categories, such that $\mathcal{A}$ has enough injectives. An object $J$ of $\mathcal{A}$ is called $\mathcal{F}$-acyclic if $R^{i} \mathcal{F}(J)=0$ for $i \geq 1$.

Example 9.84. Injective objects (of $\mathcal{A}$ ) are always acyclic for any left exact functor, but in general the class of acyclics may be larger than that. The advantage of dealing with acyclic objects is that we can compute the derived functors using acyclic resolutions instead of just injective resolutions, as the next result shows.

Proposition 9.85. Let $\mathcal{F}, \mathcal{A}, \mathcal{B}$ be as above, and $A \in \operatorname{Ob}(\mathcal{A})$. Suppose

$$
J^{\bullet} \quad 0 \rightarrow A \rightarrow J^{0} \xrightarrow{f^{0}} J^{1} \xrightarrow{f^{1}} \cdots
$$

is a resolution of $A$ by $\mathcal{F}$-acyclic ojects. Then we may compute $R^{i} \mathcal{F}(A)$ as the ith cohomology of the chain complex

$$
\mathcal{F} J^{\bullet} \quad 0 \rightarrow \mathcal{F} J^{0} \rightarrow \mathcal{F} J^{1} \rightarrow \cdots
$$

In other words, $R^{i} \mathcal{F}(A) \cong H^{i}\left(\mathcal{F} J^{\bullet}\right)$ for all $i \geq 0$.

Proof. First, note that since $\mathcal{F}$ is left exact, $R^{0} \mathcal{F}(A)=\mathcal{F}(A) \cong H^{0}\left(\mathcal{F} J^{\bullet}\right)$, so the case $i=0$ is proved.

For $i \geq 0$, let $K^{i}=\operatorname{ker} f^{i} \cong \operatorname{im} f^{i-1}$, viewed as a subobject of $J^{i}$. In particular, $K^{0} \cong A$. For each $i$, we get a short exact sequence

$$
0 \rightarrow K^{i} \xrightarrow{e^{i}} J^{i} \xrightarrow{g^{i}} K^{i+1} \rightarrow 0
$$

where $e^{i}$ is the canonical monomorphism (inclusion) and $g^{i}$ is the canonical epimorphism (quotient), and $f^{i}=e^{i+1} \circ g^{i}$. Note that $H^{i}\left(\mathcal{F} J^{\bullet}\right) \cong K^{i} / \operatorname{im} \mathcal{F} f^{i}{ }^{18}$ Then using the fact that $R \mathcal{F}^{i}$ is a $\delta$-functor, we get an induced long exact sequence.

$$
0 \rightarrow \mathcal{F} K^{i} \rightarrow \mathcal{F} J^{i} \rightarrow \mathcal{F} K^{i+1} \rightarrow R^{1} \mathcal{F}\left(K^{i}\right) \rightarrow R^{1} \mathcal{F}\left(J^{i}\right) \rightarrow R^{1} \mathcal{F}\left(K^{i+1}\right) \rightarrow \cdots
$$

Since $J$ is $\mathcal{F}$-acyclic, $R^{k} \mathcal{F}\left(J^{i}\right)=0$ for $k \geq 1$, and all $i$. So in the above long exact sequence, every 3rd term vanishes starting with $R^{1} \mathcal{F}\left(J^{i}\right)$, and we get isomorphisms

$$
R^{j} \mathcal{F}\left(K^{i+1}\right) \cong R^{j+1} \mathcal{F}\left(K^{i}\right) \quad \forall j \geq 1
$$

Remeber that we want to show $R^{i} \mathcal{F}(A) \cong H^{i}\left(\mathcal{F} J^{\bullet}\right)$. For $i \geq 1$, we have (using the isomorphisms above),

$$
R^{i+1} \mathcal{F}(A) \cong R^{i+1} \mathcal{F}\left(K^{0}\right) \cong R^{i} \mathcal{F}\left(K^{1}\right) \cong R^{i-1} \mathcal{F}\left(K^{2}\right) \cong \cdots \cong R^{2} \mathcal{F}\left(K^{i+1}\right) \cong R^{1} \mathcal{F}\left(K^{i}\right)
$$

So to complete the proof, it suffices to prove $R^{1} \mathcal{F}\left(K^{i}\right) \cong H^{i+1}\left(\mathcal{F} J^{\bullet}\right)$ for $i \geq 0$. Returning to the long exact sequence from before, $R^{1} \mathcal{F}\left(K^{i}\right)$ is the cokernel of $\mathcal{F} g^{i}: \mathcal{F} J^{i} \rightarrow \mathcal{F} K^{i+1}$, since the $R^{1} \mathcal{F}\left(J^{i}\right)$ vanishes. That is, the following sequence is exact.

$$
\begin{equation*}
0 \rightarrow \mathcal{F} K^{i} \xrightarrow{\mathcal{F} e^{i}} \mathcal{F} J^{i} \xrightarrow{\mathcal{F} g^{i}} \mathcal{F} K^{i+1} \xrightarrow{\delta} R^{1} \mathcal{F}\left(K^{i}\right) \rightarrow 0 \tag{9.1}
\end{equation*}
$$

By considering the sequence $0 \rightarrow K^{i+1} \xrightarrow{e^{i+1}} J^{i+1} \rightarrow K^{i+2} \rightarrow 0$ and applying $\mathcal{F}$, by left exactness of $\mathcal{F}$, the following is also exact.

$$
0 \rightarrow \mathcal{F} K^{i+1} \xrightarrow{\mathcal{F} e^{i+1}} \mathcal{F} J^{i+1}
$$

In particular, $\mathcal{F} e^{i+1}$ is a monomorphism. Recall that $f^{i}=e^{i+1} \circ g^{i}$, so $\mathcal{F} f^{i}=\left(\mathcal{F} e^{i+1}\right) \circ\left(\mathcal{F} g^{i}\right)$. Since $\mathcal{F} e^{i+1}$ is a monomorphism,

$$
\operatorname{im} \mathcal{F} g^{i} \cong \operatorname{im} \mathcal{F} f^{i}
$$

So using equation 9.1 again, we see

$$
R^{1} \mathcal{F}\left(K^{i}\right)=\operatorname{coker} \mathcal{F} g^{i} \cong \mathcal{F} K^{i+1} / \operatorname{im} \mathcal{F} g^{i} \cong \mathcal{F} K^{i+1} / \operatorname{im} \mathcal{F} f^{i}=H^{i+1}\left(\mathcal{F} J^{\bullet}\right)
$$

which completes the proof.

[^16]We end this chapter with a result which we will need shortly in the next chapter to establish some key properties of sheaf cohomology.

Definition 9.86. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A^{\bullet}, B^{\bullet}$ be cochain complexes with objects in $\mathcal{A}$. Let $f^{\bullet}: A^{\bullet} \rightarrow B^{\bullet}$ be a morphism of cochain complexes, with induced maps on homology $\widetilde{f}^{n}: H^{n}\left(A^{\bullet}\right) \rightarrow H^{n}\left(B^{\bullet}\right)$. The morphism $f$ is a quasi-isomorphism if each $\tilde{f}$ is an isomorphism. If there exists a quasi-isomorphism between chain complexes $A^{\bullet}, B^{\bullet}$, we say they are quasi-isomorphic as chain complexes.

Theorem 9.87. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $M^{\bullet}$ be a cochain complex in $\mathcal{A}$, such that $M^{n}=0$ for all $n<0$. Then there is a cochain complex $I^{\bullet}$ and a quasi-isomorphism $\phi: M^{\bullet} \rightarrow I^{\bullet}$ such that each $\phi^{n}$ is a monomorphism.

## 10 Sheaf cohomology

Finally, we have reached the culmination of this course, and we can finally define sheaf cohomology. Since we have built up so much theory regarding sheaves, abelian categories, and derived functors, the definition is relatively quick to give.

We then turn to an "application" of sheaf cohomology, which is that both de Rham cohomology and singular cohomology of smooth manifolds can be realized as a particular instance of sheaf cohomology. In fact, both are the same instance of sheaf cohomology, so this gives a proof that de Rham and singular cohomology agree for smooth manifolds.

### 10.1 Defining sheaf cohomology

Definition 10.1. Let $X$ be a topological space, and $\operatorname{Sh}(X)$ be the category of sheaves of abelian groups on $X$. Recall that $\operatorname{Sh}(X)$ is an abelian category with enough injectives. The global sections functor is

$$
\Gamma(X,-): \operatorname{Sh}(X) \rightarrow \mathrm{AbGp}
$$

which is described on objects (sheaves $\mathcal{F}$ ) by

$$
\mathcal{F} \mapsto \Gamma(X, \mathcal{F})=\mathcal{F}(X)
$$

Given a morphism of sheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$, the global sections functor takes $\phi$ to the corresponding map on global sections, $\phi_{X}: \mathcal{F}(X) \rightarrow \mathcal{G}(X)$.

Remark 10.2. Previously, we proved that $\Gamma(X,-)$ is a left exact functor. Even slightly more generally, if $U \subset X$ is any open subset, and $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is a short exact sequence of sheaves, then

$$
0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)
$$

is an exact sequence of abelian groups. Taking $U=X$ is precisely the statement that $\Gamma(X,-)$ is left exact.

Definition 10.3. The $i$ th sheaf cohomology functor, which we denote $H^{i}(X,-)$ is the $i$ th right derived functor of $\Gamma(X,-)$. That is, $H^{i}(X,-)=R^{i} \Gamma(X,-)$. This is also sometimes called the $i$ th cohomology group of $X$ with coefficients in $\mathcal{F}$.

Remark 10.4. All of the machinery and generality we have developed about right derived functors applies to the sheaf cohomology functors. For example, $H^{0}(X, \mathcal{F}) \cong \mathcal{F}(X)$, and a short exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ of sheaves gives rise to a long exact sequence of abelian groups

$$
0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{H}(X) \rightarrow H^{1}(X, \mathcal{F}) \rightarrow H^{1}(X, \mathcal{G}) \rightarrow H^{1}(X, \mathcal{H}) \rightarrow H^{2}(X, \mathcal{F}) \rightarrow \cdots
$$

Remark 10.5. Sheaf cohomology is generally quite hard to compute explicitly, directly from the definition. The usual strategies are either to identify it with something more combinatorially computable like Čech cohomology, or to give a geometric interpretation of a particular cohomology group with something like the Picard group.

Recall that we showed previously how a continuous map $f: X \rightarrow Y$ induces a natural map

$$
\check{H}^{n}\left(Y, f_{*} \mathcal{F}\right) \rightarrow \check{H}^{n}(X, \mathcal{F})
$$

for all $n \geq 0$, where $\mathcal{F}$ is a sheaf on $X$. We want something similar for sheaf cohomology, which motivates the following propositition.

Proposition 10.6. Let $f: X \rightarrow Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on $X$.

1. There are natural homomorphisms

$$
H^{n}\left(Y, f_{*} \mathcal{F}\right) \rightarrow H^{n}(X, \mathcal{F})
$$

for all $n \geq 0$.
2. If $f$ is an embedding with closed image, the natural map above is an isomorphism.

Remark 10.7. Before the proof, let us clarify the meaning of "natural" in the previous proposition. It means that given continuous map $f: X \rightarrow Y$ and a morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ on $X$, the following diagram commutes.


The horizontal maps are the maps from the proposition, and the vertical maps are the somewhat simpler induced maps on cohomology from the morphism $\mathcal{F} \rightarrow \mathcal{G}$, or in the case of the left vertical map, first there is an induced map $f_{*} \mathcal{F} \rightarrow f_{*} \mathcal{G}$, then an induced map on cohomology.

Proof. We start by working with an arbitrary sheaf $\mathcal{G}$ on $Y$. Later we will replace $\mathcal{G}$ with $f_{*} \mathcal{F}$, but for the moment, just use $\mathcal{G}$. Take an injective resolution $J^{\bullet}$ of $\mathcal{G}$.

$$
0 \rightarrow \mathcal{G} \rightarrow J^{0} \rightarrow J^{1} \rightarrow J^{2} \rightarrow \cdots
$$

where $J^{i}$ is an injective sheaf on $Y$, so we can use this to compute sheaf cohomology of $\mathcal{G}$, in the usual way - apply $\Gamma(Y,-)$, drop the first term, and take cohomology.

$$
H^{i}(Y, \mathcal{G}) \cong H^{i}\left(\Gamma\left(Y, J^{\bullet}\right)\right) \quad 0 \rightarrow \Gamma\left(Y, J^{0}\right) \rightarrow \Gamma\left(Y, J^{1}\right) \rightarrow \cdots
$$

Returning to the injective resolution $J^{\bullet}$ of $\mathcal{G}$, we apply the inverse image functor $f^{-1}$ : $\operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)$, which is exact, to get a long exact sequence of sheaves on $X$.

$$
0 \rightarrow f^{-1} \mathcal{G} \rightarrow f^{-1} J^{0} \rightarrow f^{-1} J^{1} \rightarrow \cdots
$$

Note that $f^{-1} J^{i}$ may or may not be injective (in $\operatorname{Sh}(X)$ ). However, by Theorem 9.87, there is a cochain complex $I^{\bullet}$ and a quasi-isomorphism $\phi: f^{-1} J^{\bullet} \rightarrow I^{\bullet}$ so that $\phi^{i}: f^{-1} J^{i} \rightarrow I^{i}$ is a monomorphism, and so that each $I^{i}$ is an injective sheaf (on $X$ ).


Since $\phi$ is a quasi-isomorphism and the upper sequence is exact, the lower sequence is also exact, which is to say, $I^{\bullet}$ is an injective resolution. Using exactness of both sequences, we may identify $f^{-1} \mathcal{G}$ with the zeroth cohomology of $f^{-1} J^{\bullet}$ and $I^{-1}$ with the zeroth homology of $I^{\bullet}$, so the fact that $\phi$ is a quasi-isomorphism means that $\phi^{-1}: f^{-1} \mathcal{G} \rightarrow I^{-1}$ is an isomorphism. That is,

$$
0 \rightarrow f^{-1} \mathcal{G} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \cdots
$$

is an injective resolution of $f^{-1} \mathcal{G}$ (in the category $\operatorname{Sh}(X)$.) In particular, we can use this resolution to compute $H^{n}\left(X, f^{-1} \mathcal{G}\right)$ by the usual procedure - apply $\Gamma(X,-)$, drop the first term, and take cohomology. Thus $H^{i}\left(X, f^{-1} \mathcal{G}\right)$ is the $i$ th cohomology of $\Gamma\left(X, I^{\bullet}\right)$.

$$
H^{i}\left(X, f^{-1} \mathcal{G}\right) \cong H^{i}\left(\Gamma\left(X, I^{\bullet}\right)\right) \quad 0 \rightarrow \Gamma\left(X, I^{0}\right) \rightarrow \Gamma\left(X, I^{1}\right) \rightarrow \cdots
$$

Also, applying $\Gamma(X,-)$ to the previous diagram involving $\phi$ and droppin the first terms, we get a morphism of complexes


Set this aside for now.
Now we take a moment to recall how the inverse image sheaf $f^{-1} J^{n}$ is defined. Let $\widetilde{J}^{n}$ be the auxiliary presheaf (on $X$ ) defined by

$$
\widetilde{J}^{n}(U)=\underset{f(U) \subset V}{\lim _{\vec{\prime}}} J^{n}(V)
$$

and then $f^{-1} J^{n}$ is, by definition, the sheafification of $\widetilde{J}^{n}$. In particular,

$$
\widetilde{J}^{n}(X)={\underset{f(\overrightarrow{X) \subset}}{ } J^{n}(V), ~(V)}
$$

and $Y$ is one such $V$, so there is a canonical map to the direct limit $J^{n}(Y) \rightarrow \widetilde{J}^{n}(X)$. Let $\theta: \widetilde{J}^{n} \rightarrow\left(\widetilde{J^{n}}\right)^{+}=f^{-1} J^{n}$ be the canonical map associated with sheafification, so in particular we have a map (of abelian groups) $\theta_{X}: \widetilde{J}^{n}(X) \rightarrow f^{-1} J^{n}(X)$. Composing these, we obtain a map $J^{n}(Y)=\Gamma\left(Y, J^{n}\right) \rightarrow f^{-1} J^{n}(X)=\Gamma\left(X, f^{-1} J^{n}\right)$, which we denote by $\psi^{n}$. We omit the details, but the maps $\psi^{n}$ give a morphism of cochain complexes $\psi: \Gamma\left(Y, J^{\bullet}\right) \rightarrow \Gamma\left(X, f^{-1} J^{\bullet}\right)$.


Composing $\psi$ with $\Gamma(\phi)$ form earlier, we get a morphism of complexes $\Gamma\left(Y, J^{\bullet}\right) \rightarrow \Gamma\left(X, I^{\bullet}\right)$, which then induces morphisms on cohomology groups

$$
H^{i}(Y, \mathcal{G})=H^{i}\left(\Gamma\left(Y, J^{\bullet}\right)\right) \rightarrow H^{i}\left(\Gamma\left(X, I^{\bullet}\right)\right)=H^{i}\left(X, f^{-1} \mathcal{G}\right)
$$

Recall that this was all for an arbitrary sheaf $\mathcal{G}$ on $Y$. Now take the sheaf $\mathcal{F}$ of the proposition, and let $\mathcal{G}=f_{*} \mathcal{F}$. The induced maps above give maps

$$
H^{i}\left(Y, f_{*} \mathcal{F}\right) \rightarrow H^{i}\left(X, f^{-1} f_{*} \mathcal{F}\right)
$$

Recall that we have an adjunction $\left(f^{-1}, f_{*}\right)$ which comes with a counit natural transformation $\eta: f^{-1} f_{*} \mathcal{F} \rightarrow \mathcal{F}$. Of course, a morphism of sheaves induces a morphism on sheaf cohomology,

$$
H^{i}\left(X, f^{-1} f_{*} \mathcal{F}\right) \rightarrow H^{i}(X, \mathcal{F})
$$

Composing this with the previous induced map, we obtain the desired induced map.

$$
H^{i}\left(X, f_{*} \mathcal{F}\right) \rightarrow H^{i}(X, \mathcal{F})
$$

This completes the proof of (1).
(2) Now suppose that $f$ is an embedding with closed image. We want to show that the induced map from part (1) is an isomorphism in this case. Let $\mathcal{F}$ be a sheaf on $X$, and take an injective resolution $I^{\bullet}$ of $\mathcal{F}$.

$$
I^{\bullet} \quad 0 \rightarrow \mathcal{F} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \cdots
$$

Since $f$ is a closed embedding, the direct image functor $f_{*}$ is exact (Proposition 8.4). Also, $f_{*}$ preserves injectives (Corollary 8.41). So applying $f_{*}$ to $I^{\bullet}$, we obtain an injective resolution of $f_{*} \mathcal{F}$.

$$
f_{*} I^{\bullet} \quad 0 \rightarrow f_{*} \mathcal{F} \rightarrow f_{*} I^{0} \rightarrow f_{*} I^{1} \rightarrow \cdots
$$

Now we apply $\Gamma(X,-)$ to $f_{*} I^{\bullet}$.

$$
0 \rightarrow \Gamma\left(X, f_{*} \mathcal{F}\right) \rightarrow \Gamma\left(X, f_{*} I^{0}\right) \rightarrow \Gamma\left(X, f_{*} I^{1}\right) \rightarrow \cdots
$$

The $i$ th cohomology of this complex $\Gamma\left(X, f_{*} I^{\bullet}\right)$ is $H^{i}\left(X, f_{*} \mathcal{F}\right)$. In particular, the first term is

$$
\Gamma\left(X, f_{*} \mathcal{F}\right)=\mathcal{F}\left(f^{-1}(Y)\right)=\mathcal{F}(X)=\Gamma(X, \mathcal{F})
$$

and for the other terms,

$$
\Gamma\left(X, f_{*} I^{0}\right)=I^{0}\left(f^{-1}(Y)\right)=I^{0}(X)=\Gamma\left(X, I^{0}\right)
$$

So in fact, the complex $\Gamma\left(X, f_{*} I^{\bullet}\right)$ has cohomology which computes $H^{i}(X, \mathcal{F})$. Hence $H^{i}\left(X, f_{*} \mathcal{F}\right) \cong H^{i}(X, \mathcal{F})$. By waving our hands and combining this fact with the naturality of the morphisms in part (1), those morphisms must be isomorphisms ${ }^{19}$. This completes the proof of (2).

[^17]
### 10.2 Higher direct images

Definition 10.8. Let $f: X \rightarrow Y$ be a continuous map of topological spaces. Recal that the direct image functor $f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$ is left exact. The higher direct image functors are the right derived functors of $f_{*}$. We just denote them by $R^{i} f_{*}$ for $i \geq 0$.

Remark 10.9. In the next proposition, we will give a helpful interpretation of $R^{i} f_{*}$ in terms of some sheafifications, but before the proof we should recall some facts. Given an open embedding $j: U \hookrightarrow X$, the functor $j$ ! is exact, and we have an adjunction $\left(j!, j^{-1}\right)$, which implies that $j^{-1}$ preserves injectives. In particular, if $\mathcal{I}$ is an injective in $\operatorname{Sh}(X)$, then $\left.\mathcal{I}\right|_{U}=j^{-1} \mathcal{I}$ is injective in the category $\operatorname{Sh}(U)$.

Proposition 10.10. Let $f: X \rightarrow Y$ be a continuous map and $\mathcal{F}$ a sheaf on $X$. Consider the presheaf $\mathcal{G}$ on $Y$ defined by

$$
\mathcal{G}(V)=H^{i}\left(f^{-1}(V),\left.\mathcal{F}\right|_{f^{-1}(V)}\right)
$$

with restriction maps given by the natural maps of Proposition 10.6. The higher direct image of $\mathcal{F}, R^{i} f_{*}(\mathcal{F})$, is (isomorphic to) the sheafification of $\mathcal{G}$.

Proof. This will be a very nice consequence of Grothendieck's universality theorem (9.80) on cohomological $\delta$-functors.

Let $\mathcal{G}$ be the presheaf on $Y$ defined in the proposition, and let $\mathcal{H}^{i}(X, \mathcal{F})$ be the sheafification of $\mathcal{G}$, with canonical sheafificaiton morphism $\theta: \mathcal{G} \rightarrow \mathcal{H}^{i}(X, \mathcal{F})$. Then think of $\mathcal{H}^{i}(X,-)$ as a functor $\operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y)$. Since sheafification is an exact functor $\operatorname{PSh}(Y) \rightarrow \operatorname{Sh}(Y)$, the functors $\left\{\mathcal{H}^{i}(X,-)\right\}$ form a $\delta$-functor from $\operatorname{Sh}(X)$ to $\operatorname{Sh}(Y)$. We also have a $\delta$-functor $\left\{R^{i} f_{*}(-)\right\}$ with the same domain and range.

$$
\begin{aligned}
\mathcal{H}^{i}(X,-): \operatorname{Sh}(X) & \rightarrow \operatorname{Sh}(Y) \\
R^{i} f_{*}(-): \operatorname{Sh}(X) & \rightarrow \operatorname{Sh}(Y)
\end{aligned}
$$

In the case $i=0$, it is clear that

$$
R^{i} f_{*} \mathcal{F}=f_{*} \mathcal{F}=\mathcal{H}^{0}(X, \mathcal{F})
$$

If we can show that $R^{i} f_{*}(-)$ and $\mathcal{H}^{i}(X,-)$ are both effaceable functors for all $i \geq 1$, then it follows from Grothendieck's universality theorem that they are both universal $\delta$-functors, and then since they agree in degree zero, for each $i \geq 1$ we have a natural isomorphism $R^{i} f_{*}(-) \cong \mathcal{H}^{i}(X, \mathcal{F})$, which would complete the proof. So it suffices to show effaceability, which we now do.

We know that any sequence of right derived functors vanishes on injective objects and is hence effaceable, so $R_{i} f_{*}(-)$ is effaceable. On the other hand, let $I$ be an injective sheaf on $X$, and let us show that $\mathcal{H}^{i}(X, I)=0$. For any subset $V \subset Y$, we have an inclusion

$$
j: f^{-1}(V) \hookrightarrow X
$$

Since $f^{-1}(V)$ is open in $X$, the functors $\left(j!, f^{-1}\right)$ form an adjoint pair (Remark 8.51) and $j$ ! is exact (Proposition 8.47), so we also know $j^{-1}$ preserves injectives (Proposition 8.40). Hence $j^{-1} I$, otherwise denoted $I_{f^{-1}(V)}$, is an injective sheaf on $f^{-1}(V)$. Hence
 that $\mathcal{H}^{i}(X,-)$ is effaceable, which completes the proof.

### 10.3 Acyclic sheaves

Recall that we can compute derived functors using somewhat more general types of resolutions than injective resolutions, we can use acyclic resolutions (Proposition 9.85).

Definition 10.11. A sheaf $\mathcal{F}$ on a space $X$ is acyclic if $H^{i}(X, \mathcal{F})=0$ for all $i \geq 1$. In terms of the general definition of acyclic objects, this is saying that $\mathcal{F}$ is $\Gamma(X,-)$-acyclic.

Example 10.12. Injective sheaves are acyclic, since injective objects in any category are acyclic with respect to any left exact functor. Flasque sheaves are also acyclic (proof to come). Fine sheaves are also acyclic (definition of fine sheaves and proof to come).

### 10.3.1 Flasque sheaves

Recall that a sheaf $\mathcal{F}$ on $X$ is flasque if the restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ are surjective for any open subsets $V \subset U \subset X$. Before we prove that flasque sheaves are acyclic, we need a lemma from Hartshorne's book on algebraic geometry, to which we refer the reader for proof.

Lemma 10.13. Let $\left(X, \mathcal{O}_{X}\right)$ be a ringed space. Then any injective $\mathcal{O}_{X}$-module is flasque.
Proof. Hartshorne chapter III, Lemma 2.4.
In particular, we are just dealing with the special case of sheaves of abelian groups, which are the case of thinking of $X$ as a ringed space $\left(X, \mathcal{O}_{X}\right)$ where $\mathcal{O}_{X}$ is the locally constant sheaf with value group $\mathbb{Z}$. In this case, the lemma tells us that injective sheaves are flasque, in the sense of both words meaning what we have been using them to mean, regardless of whatever Hartshorne may mean by these words.

Theorem 10.14 (Flasque sheaves are acyclic). Let $\mathcal{F}$ be a flasque sheaf on a space $X$. Then $\mathcal{F}$ is acyclic.

Proof. The outline of the proof is as follows: we embed (take a monomorphism) $\mathcal{F}$ into an injective sheaf, consider the cokernel of that embedding, and examine the long exact sequence on sheaf cohomology. Then we can do some "dimension shifting" induction to get a lot of vanishing of cohomology.

Let $\mathcal{F}$ be a flasque sheaf on $X$. Since $\operatorname{Sh}(X)$ has enough injectives, let $\mathcal{F} \rightarrow \mathcal{I}$ be a monomorphism of $\mathcal{F}$ to an injective sheaf $\mathcal{I}$. Let $\mathcal{G}$ be the cokernel of this morphism, so we have a short exact sequence of sheaves

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0
$$

By the Hartshorne lemma above (10.13), $\mathcal{I}$ is flasque, so it follows that $\mathcal{G}$ is also flasque (Theorem 4.23). From the same theorem, since $\mathcal{F}$ is flasque, we get a short exact sequence on global sections.

$$
0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{I}(X) \rightarrow \mathcal{G}(X) \rightarrow 0
$$

Also, $\mathcal{I}$ is injective, so $H^{i}(X, \mathcal{I})=0$ for all $i \geq 1$. Now consider the long exact sequence on sheaf cohomology associated to $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0$.

$$
0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{I}(X) \rightarrow \mathcal{G}(X) \rightarrow H^{1}(X, \mathcal{F}) \rightarrow H^{1}(X, \mathcal{I}) \rightarrow H^{1}(X, \mathcal{G}) \rightarrow H^{2}(X, \mathcal{F}) \rightarrow \cdots
$$

As noted above, $\mathcal{I}(X) \rightarrow \mathcal{G}(X)$ is surjective, so the morphism $\mathcal{G}(X) \rightarrow H^{1}(X, \mathcal{F})$ is the zero morphism. Also, $H^{1}(X, \mathcal{I})=0$. From this it follows that $H^{1}(X, \mathcal{F})=0$, for any flasque sheaf $\mathcal{F}$. Additionally, every 3 rd term of the sequence vanishes starting with $H^{1}(X, \mathcal{I})$, so the LES gives isomorphisms

$$
H^{i}(X, \mathcal{G}) \cong H^{i+1}(X, \mathcal{F}) \quad \forall i \geq 1
$$

Since we also noted that $\mathcal{G}$ is flasque, we know that $H^{1}(X, \mathcal{G})=0$. Then by induction, $H^{i}(X, \mathcal{F})$ and $H^{i}(X, \mathcal{G})$ are both zero for all $i \geq 1$.

Remark 10.15. The trick at the end of the previous proof is a technique known as "dimension shifting," where every third term of some long exact sequence vanishes and that combines with induction to get some powerful result. This technique frequently arises in group cohomology.

We have now shown that flasque sheaves are acyclic, so we can compute sheaf cohomology using flasque resolutions instead of injective resolutions, which may be more convenient.

The question now arises, can we always find a flasque resolution? Ok, yes, we know from the Hartshorne lemma above that any injective sheaf is also flasque, so we can just take an injective resolution and consider it as a flasque resolution, but this is unhelpful. We know that injective resolutions always exist and that the proof is sort of constructive, but the construction of the proof often constructs injective sheaves which are way too big to get a handle on.

So we should amend our question: is there some sort of canonical flasque resolution which is simpler to describe? The answer is yes, and this canonical resolution is called the Godement resolution. Unfortunately, it isn't that much more practically useful, but we describe it anyway.

Definition 10.16 (Preliminary to Godement resolution). Let $\mathcal{F}$ be a sheaf, and $\pi: E_{\mathcal{F}} \rightarrow X$ be the étale space of $\mathcal{F}$. Recall that

$$
E_{\mathcal{F}}=\prod_{x \in X} \mathcal{F}_{x}
$$

with projection map $\pi$ given by $\pi(e)=x$ where $e \in \mathcal{F}_{x}$. Also recall that we have an isomorphism of sheaves

$$
\mathcal{F}(-) \cong \Gamma(-, \pi)
$$

where $\Gamma(U, \pi)$ is continuous sections of $\pi$ over $U$.

$$
\Gamma(U, \pi)=\left\{s: U \rightarrow E_{\mathcal{F}} \mid \pi s=\operatorname{Id}_{U}, s \text { is continuous }\right\}
$$

For $U \subset X$ open, define $C^{0} \mathcal{F}$ to be sheaf of sections of $\pi$ which are not necessarily continuous.

$$
C^{0} \mathcal{F}(U)=\left\{s: U \rightarrow E_{\mathcal{F}} \mid \pi s=\operatorname{Id}_{U}\right\}
$$

Note that $C^{0} \mathcal{F}(U)$ can be identified with $\prod_{x \in U} \mathcal{F}_{x}$ via

$$
C^{0} \mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_{x} \quad s \mapsto(s(x))_{x \in U}
$$

For $V \subset U \subset X$ open subset, we have a natural restriction map

$$
\operatorname{res}_{V}^{U}: C^{0} \mathcal{F}(U)=\prod_{x \in U} \mathcal{F}_{x} \rightarrow C^{0} \mathcal{F}(V)=\prod_{x \in V} \mathcal{F}_{x}
$$

which is just the map which acts as the identity map on $\mathcal{F}_{x}$ for $x \in V$, and acts as the zero map for $x \in U \backslash V$. Note that $\Gamma(-, \pi) \cong \mathcal{F}$ is a subsheaf of $C^{0} \mathcal{F}$.
Lemma 10.17. Let $\mathcal{F}$ be a sheaf on $X$, and $C^{0} \mathcal{F}$ be the object defined above. Then $C^{0} \mathcal{F}$ is a flasque sheaf on $X$.

Proof. First, one would show that $C^{0} \mathcal{F}$ is a presheaf, which is fairly obvious. Verifying the sheaf axioms is also not terribly hard, and the flasque property is also obvious.

Now we return to defining the Godement resolution.
Definition 10.18 (Godement resolution). Let $\mathcal{F}$ be a sheaf on a space $X$, and let $C^{0} \mathcal{F}$ be the sheaf defined above. We previously noted that $\mathcal{F}$ is a subsheaf of $C^{0} \mathcal{F}$, meaning that we have a monomorphism $\mathcal{F} \rightarrow C^{0} \mathcal{F}$. Let $Q^{1}$ be the cokernel of this monomorphism, so we have a short exact sequence.

$$
0 \rightarrow \mathcal{F} \rightarrow C^{0} \mathcal{F} \rightarrow Q^{1} \rightarrow 0
$$

Then consider the flasque sheaf $C^{0} Q^{1}$, and repeat this to define $Q^{2}$ as the cokernel of $Q^{1} \rightarrow$ $C^{0} Q^{1}$, obtaining a short exact sequence

$$
0 \rightarrow Q^{1} \rightarrow C^{0} Q^{1} \rightarrow Q^{2} \rightarrow 0
$$

Then iterate this construction to define $Q^{n}$ for $n \geq 1$. Define $C^{i} \mathcal{F}=C^{0} Q^{i}$. Then we can splice together all of these short exact sequences to obtain a long exact sequence

$$
0 \rightarrow \mathcal{F} \rightarrow C^{0} \mathcal{F} \rightarrow C^{1} \mathcal{F} \rightarrow C^{2} \mathcal{F} \rightarrow \cdots
$$

By construction/previous lemma, this is a flasque resolution of $\mathcal{F}$. It is called the Godement resolution of $\mathcal{F}$.

Remark 10.19. Unfortunately, the Godement resolution isn't really that much more useful than the usual injective resolution which is guaranteed to exist, since the flasque sheaves in the resolution are still really big and unwieldy for practical calculation.

### 10.3.2 Fine sheaves

Definition 10.20. Let $\phi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on $X$, and for $x \in X$, let $\phi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ be the morphism on stalks. The support of $\phi$ is

$$
\operatorname{supp}(\phi)=\overline{\left\{x \in X: \phi_{x} \neq 0\right\}}
$$

where the overline denotes the topological closure.
Definition 10.21. Let $X$ be a space. An open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$ is locally finite if every $x \in X$ has a neighborhood that has nonemtpy intersection with only finitely many $U_{i}$.

Definition 10.22. Let $X$ be a space. $X$ is paracompact if $X$ is Hausdorff and every open cover of $X$ admits a locally finite refinement.

Example 10.23. Any smooth real manifold is paracompact.
Definition 10.24. Let $X$ be a space and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ a locally finite cover. Let $\mathcal{F}$ be a sheaf on $X$. A partition of unity for $\mathcal{F}$ subordinate to $\mathcal{U}$ is given by a collection of morphisms

$$
\eta_{i}: \mathcal{F} \rightarrow \mathcal{F} \quad i \in I
$$

such that $\operatorname{supp}\left(\eta_{i}\right) \subset U_{i}$ and for each $x \in X$,

$$
\sum_{i \in I} \eta_{i, x}=\operatorname{Id}_{\mathcal{F}_{x}}
$$

Note that since $\mathcal{U}$ is locally finite, each $\eta_{i, x}$ is nonzero only for finitely many $i \in I$, so the sum above has only finitely many nonzero terms for a fixed $x \in X$.

Definition 10.25. Let $\mathcal{F}$ be a sheaf on a space $X$. The sheaf $\mathcal{F}$ is fine if for every locally finite open cover of $X$, there exists a partition of unity for $\mathcal{F}$ subordinate to $\mathcal{U}$.

Theorem 10.26. Let $X$ be a topological space such that every open subset of $X$ is paracompact. For any fine sheaf $\mathcal{F}$ on $X$, and an open subset $U \subset X$, the restriction $\left.\mathcal{F}\right|_{U}$ is an acyclic sheaf on $U$.

Proof. Omitted.
Example 10.27. An example of a space $X$ satisfying the hypothesis above is if $X$ is a smooth real manifold. That is, fine sheaves are acyclic on smooth manifolds.

Proposition 10.28. Let $(X, \mathcal{O})$ be a ringed space, such that for every locally finite cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}, \mathcal{O}$ has a partition of unit subordinate to $\mathcal{U}$. Then any sheaf of $\mathcal{O}$-modules is fine.

Proof. Omitted.
Example 10.29. A situation where the previous proposition holds is when $X$ is a smooth manifold, $\mathcal{O}$ is the sheaf of smooth $\mathbb{R}$-valued functions, and $\mathcal{F}$ is the sheaf of differential forms.

### 10.4 Leray's theorem

Our next goal is to set up and prove Leray's theorem, which establishes an isomorphism between Čech cohomology and sheaf cohomology, under certain hypotheses. The first step is to describe a map from Čech cohomology to sheaf cohomology.

Definition 10.30. Let $X$ be a topological space and $\mathcal{F}$ a sheaf of abelian groups on $X$. Let $n \in \mathbb{Z}_{\geq 0}$. We will define a map

$$
\check{H}^{n}(X, \mathcal{F}) \rightarrow H^{n}(X, \mathcal{F})
$$

Later, Leray's theorem will establish conditions for this to be an isomorphism. Consider the Cech resolution of $\mathcal{F}$.

$$
0 \rightarrow \mathcal{F} \rightarrow \check{\mathcal{C}}^{0}(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \cdots
$$

where $\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})$ is the sheaf on $X$ determined by its value on an open subset $U \subset X$ as below. ${ }^{20}$

$$
\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(U)=\prod_{\left(i_{0}, \ldots, i_{n}\right) \in I^{n}} \mathcal{F}\left(U \cap U_{i_{0} \cdots i_{n}}\right)
$$

The Čech resolution is a long exact sequence of sheaves of abelian groups on $X$. Now consider an injective resolution of $\mathcal{F}$.

$$
0 \rightarrow \mathcal{F} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \cdots
$$

From general homological principles, the identity map Id: $\mathcal{F} \rightarrow \mathcal{F}$ extends to a morphism of chain complexes $\check{\mathcal{C}}^{\bullet}(\mathcal{U}, \mathcal{F}) \rightarrow I^{\bullet}$.


Now take global sections of the previous diagram. That is, apply the functor $\Gamma(X,-)$ to it. We also drop the $\mathcal{F}$ term. What we obtain is


[^18]The above is a morphism of chain complexes, and the cohomology of the bottom row is $H^{n}(X, \mathcal{F})$. On the other hand, ${ }^{21}$

$$
\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})(X)=\prod_{\left(i_{0}, \cdots i_{n}\right) \in I^{n+1}} \mathcal{F}\left(U_{i_{0} \cdots i_{n}}\right)
$$

That is to say, the top row of the previous diagram is the chain complex of Čech cochains, whose cohomology computes $\check{H}^{n}(\mathcal{U}, \mathcal{F})$. Hence our chain map induces maps on cohomology groups,

$$
\check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow H^{n}(X, \mathcal{F})
$$

This map depends on the cover $\mathcal{U}$, but the maps are compatible in the necessary manner to take the direct limit, so we obtain a map on the direct limit over such covers (partially ordered by refinement), and obtain the desired morphism

$$
\check{H}^{n}(X, \mathcal{F}) \rightarrow H^{n}(X, \mathcal{F})
$$

We will call this the Leray map. We may also refer to the various maps $\check{H}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow$ $H^{n}(X, \mathcal{F})$ as a Leray maps.

Remark 10.31. Tracing through the construction, it is not too hard to see that the map $\breve{H}^{n}(X, \mathcal{F}) \rightarrow H^{n}(X, \mathcal{F})$ is natural in the sense that a morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ of sheaves induces a commutative square

where the horizontal maps are the maps constructed in the previous definition.
Definition 10.32. Let $\mathcal{F}$ be a sheaf on a space $X$. $\mathcal{F}$ is Čech ayclic if $\check{H}^{n}(\mathcal{U}, \mathcal{F})=0$ for all $n \geq 1$ and any open cover $\mathcal{U}$ of $X$. Taking the direct limit, this obviously implies that $\check{H}^{n}(X, \mathcal{F})=0$ for all $n \geq 1$, but note that it might happen that $\check{H}^{n}(X, \mathcal{F})=0$ for all $n \geq 1$ without $\mathcal{F}$ being Čech acyclic.

Proposition 10.33. Let $\mathcal{F}$ be a flasque sheaf on a space $X$. Then $\mathcal{F}$ is Čech acyclic.
Proof. Let $\mathcal{U}$ be an open cover of $X$. Since $\mathcal{F}$ is flasque, by Lemma 6.51, the Čech resolution of $\mathcal{F}$ is a flasque resolution.

$$
0 \rightarrow \mathcal{F} \rightarrow \check{\mathcal{C}}^{\bullet}(\mathcal{U}, \mathcal{F})
$$

Hence this is an acyclic resolution, and may be used to compute sheaf cohomology, by Proposition 9.85. On the other hand, this complex computes the Čech cohomology for the

[^19]cover $\mathcal{U}$ as well. Hence Čech cohomology using $\mathcal{U}$ is isomorphic to sheaf cohomology, which vanishes as $\mathcal{F}$ is ayclic (because it is flasque).
$$
\check{H}^{n}(\mathcal{U}, \mathcal{F}) \cong H^{n}(X, \mathcal{F})=0
$$

Definition 10.34. Let $X$ be a topological space, and $\mathcal{F}$ a sheaf on $X$, and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ an open cover of $X$. The cover $\mathcal{U}$ is acyclic for $\mathcal{F}$ if for any $(n+1)$-tuple $\left(i_{0}, \ldots, i_{n}\right) \in I^{n+1}$ (where $n \geq 0$ ), we have

$$
H^{k}\left(U_{i_{0} \cdots i_{n}},\left.\mathcal{F}\right|_{U_{i_{0} \cdots i_{n}}}\right)=0
$$

for all $k \geq 1$.
Theorem 10.35 (Leray). Let $\mathcal{F}$ be a sheaf of abelian groups on a space $X$, and let $\mathcal{U}$ be an open cover of $X$ which is acyclic for $\mathcal{F}$. Then the Leray maps are isomorphisms for $p \geq 0$.

$$
\check{H}^{p}(\mathcal{U}, \mathcal{F}) \stackrel{\cong}{\leftrightarrows} H^{p}(X, \mathcal{F})
$$

We will give an attempt at a proof, but first a remark on the utility of the theorem. Leray's theorem is useful in algebraic geometry, since the hypotheses of the theorem in the following situation: $X$ is a Noetherian separated scheme with Zariski topology, and $\mathcal{F}$ is a quasicoherent sheaf on $X$, and $\mathcal{U}$ is any open cover.

Proof. We proceed by induction on $p$. The case $p=0$ is straightforward, since both cohomology groups are just global sections of $\mathcal{F}$.

$$
\check{H}^{0}(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X) \cong H^{0}(X, \mathcal{F})
$$

To understand why this isomorphism is actually induced by the Leray map, just trace through the construction of the Leray map.

Now for the induction. Let $\mathcal{F}, \mathcal{U}=\left\{U_{i}\right\}_{i \in I}, X$ be as in the statement of the theorem, and assume the theorem holds for some fixed value $p_{0}$. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, and let $\mathcal{Q}$ be the cokernel of the embedding $\mathcal{F} \hookrightarrow \mathcal{I}$. That is, we have a short exact sequence of sheaves (of abelian groups) on $X$.

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0
$$

Let $U \subset X$ be some intersection of elements of the open cover $\mathcal{U}$.

$$
U=U_{i_{0} \cdots i_{n}}=U_{i_{0}} \cap \cdots \cap U_{i_{n}}
$$

Let $j: U \hookrightarrow X$ be the inclusion. The functor

$$
j^{-1}=\left.(-)\right|_{U}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(U) \quad \mathcal{F} \mapsto j^{-1} \mathcal{F}=\left.\mathcal{F}\right|_{U}
$$

is exact, and takes injectives to injectives, so we have a short exact sequence

$$
\begin{equation*}
\left.\left.\left.0 \rightarrow \mathcal{F}\right|_{U} \rightarrow \mathcal{I}\right|_{U} \rightarrow \mathcal{Q}\right|_{U} \rightarrow 0 \tag{10.1}
\end{equation*}
$$

of sheaves on $U$, where $\left.\mathcal{I}\right|_{U}$ is injective as a sheaf on $U$. By the hypothesis, $\mathcal{F}$ is acyclic for $\mathcal{U}$, so $H^{p}\left(U,\left.\mathcal{F}\right|_{U}\right)=0$ for all $p \geq 1$. Also, since $\mathcal{I}$ is injective, $H^{p}\left(U,\left.\mathcal{I}\right|_{U}\right)=0$ for all $p \geq 1$, as injective sheaves are acyclic. Now consider the long exact sequence on sheaf cohomology associated to the sequence 10.1. Two out of every three terms are zero.

$$
0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{I}(U) \rightarrow \mathcal{Q}(U) \rightarrow 0 \rightarrow 0 \rightarrow H^{1}\left(U,\left.\mathcal{Q}\right|_{U}\right) \rightarrow 0 \rightarrow 0 \rightarrow H^{2}\left(U,\left.\mathcal{Q}\right|_{U}\right) \rightarrow 0 \rightarrow \cdots
$$

By exactness of this, we get that $H^{p}\left(U,\left.\mathcal{Q}\right|_{U}\right)=0$ for all $p \geq 1$. That is, $\mathcal{Q}$ is also acyclic for $\mathcal{U}$. We will use this later.

Recall the sheafified Čech complex, defined as

$$
\check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F})=\prod_{\left(i_{0}, \ldots, i_{n}\right) \in I^{n+1}} \mathcal{F}\left(U \cap U_{i_{0} \cdots i_{n}}\right)
$$

Fix $n \geq 1$. Taking the product over all $(n+1)$-tuples $\left(i_{0}, \ldots, i_{n}\right)$, the short exact sequences $0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{I}(U) \rightarrow \mathcal{Q}(U) \rightarrow 0$ give a short exact sequence of sheaves

$$
0 \rightarrow \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{I}) \rightarrow \check{\mathcal{C}}^{n}(\mathcal{U}, \mathcal{Q}) \rightarrow 0
$$

Furthermore, these maps are compatible with the boundary maps of the sheafified Čech complex $\check{C} \bullet(\mathcal{U}, \mathcal{F})$ (and same for $\mathcal{I}, \mathcal{Q})$, so we obtain a short exact sequence of chain complexes of sheaves on $X$.

$$
0 \rightarrow \check{\mathcal{C}}^{\bullet}(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}} \bullet(\mathcal{U}, \mathcal{I}) \rightarrow \check{\mathcal{C}} \bullet(\mathcal{U}, \mathcal{Q}) \rightarrow 0
$$

Then by the usual procedure, from a short exact sequence of chain complexes, we obtain a long exact sequence on cohomology. In particular, the cohomology of each of the complexes involved above is Čech cohomology, so the resulting LES is

$$
0 \rightarrow \check{H}^{0}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{0}(\mathcal{U}, \mathcal{I}) \rightarrow \check{H}^{0}(\mathcal{U}, \mathcal{Q}) \rightarrow \check{H}^{1}(\mathcal{U}, \mathcal{F}) \rightarrow \cdots
$$

Since $\mathcal{I}$ is injective, it is Cech acyclic, for $p \geq 1$, the $\mathcal{I}$ terms vanish in the long exact sequence above. Hence we obtain isomorphisms from the boundary maps above,

$$
0 \rightarrow \check{H}^{i}(\mathcal{U}, \mathcal{Q}) \stackrel{\cong}{\leftrightarrows} \check{H}^{i+1}(\mathcal{U}, \mathcal{F}) \rightarrow 0
$$

for $i \geq 1$. Using naturality of the Leray maps, we have a commutative diagram below, where the vertical maps are Leray maps. Note that the degree zero Leray maps are isomorphisms by the base case $p=0$.


Then by a diagram chasing argument, the arrow on the far right must also be an isomorphism. ${ }^{22}$ Again using naturality of the Leray maps, we get the following commutative square, where the vertical maps are the Leray maps.


As $\mathcal{Q}$ is acyclic for $\mathcal{U}$, the left vertical Leray map above is an isomorphism by inductive hypothesis, so the right map must also be an isomorphism. Since we have such commutative squares for all $i \geq 1$, this completes the induction.

### 10.5 Unification of de Rham cohomology, singular cohomology, and sheaf cohomology

The final goal of this course is to give an account of De Rham cohomology, singular cohomology, and an isomorphism between them for smooth manifolds. This isomorphism will be obtained by identifying each with a particular sheaf cohomology, the sheaf cohomology of the sheaf of locally constant real valued functions.

There are simpler treatments of the correspondence between de Rham and singular cohomology, but we will approach it from the perspective of using the powerful tools of sheaf theory. Many proofs will be omitted, especially details which are more geometric in nature. Even some of the definitions will be a little bit less rigorous than usual. The main result is the following.

Theorem 10.36 (De Rham). Let $M$ be a smooth real manifold. Let $\mathcal{C}$ be the sheaf of locally constant real valued functions on $M$. Then for all $n \in \mathbb{Z}_{\geq 0}$,

$$
H_{\mathrm{dR}}^{n}(M) \cong H_{\mathrm{sing}}^{n}(M, \mathbb{R}) \cong H^{n}(M, \mathcal{C})
$$

Philosophically speaking, this result is a powerful justification for the great abstraction and generality of sheaf cohomology. It tells us that sheaf cohomology is so much more powerful and general than de Rham and singular cohomology that the sheaf cohomology of one of the simplest sheaves imaginable contains the entirety of these two theories, at least for smooth manifolds.

In another sense, this result is very intimidating. It tells us that sheaf cohomology is very very difficult to compute in any sort of generality. The sheaf of locally constant real valued functions is much simpler and easier to understand than a random sheaf, and a smooth manifold already has a lot of structure. And still, to compute sheaf cohomology, we need to

[^20]compute de Rham or singular cohomology, things which are in principle not too hard, but still not trivial to compute.

What then, of computing sheaf cohomology for some less well understood sheaf, on some less understood space than a smooth manifold? Given the above discussion, it seems nearly impossible. And in some sense, it is. Nevertheless, we do not give up, since it is clear that sheaf cohomology is a powerful tool, and any attempts to compute it should give a of useful information, whether or not they succeed.

### 10.5.1 De Rham cohomology

Definition 10.37. Let $M$ be a smooth real manifold of dimension $n$. The structure sheaf of $M$, denoted $\mathcal{O}$, is the sheaf of smooth $\mathbb{R}$-valued functions. That is, for $U \subset M, \mathcal{O}(U)$ is the ring of smooth functions $U \rightarrow \mathbb{R}$. (The ring structure comes from adding and multiplying functions pointwise.) Thus $\mathcal{O}$ is a sheaf of commutative rings. The restriction maps are literal function restrictions.

Definition 10.38. Let $M$ be a smooth real manifold of dimension $n$, and let $p \in M$. The tangent space of $M$ at $p$ is a real vector space of dimension $n$, denoted $T_{p} M$. One can think of it as derivations at $p$, or various other things, but we omit the details here.

A choice of local chart near $p$, that is, a diffeomorphism $h: U \rightarrow \mathbb{R}^{n}$ where $U$ is an open neighborhood of $p$, gives a basis for $T_{p} M$. We will write this basis as

$$
\frac{\partial}{\partial x_{1}}(p), \cdots, \frac{\partial}{\partial x_{n}}(p)
$$

Definition 10.39. The tangent bundle of $M$, denoted $T M$, is a smooth manifold of dimension $2 n$, with a map

$$
\pi: T M \rightarrow M
$$

such that for each $p \in M$, the fiber $\pi^{-1}(p)$ is a real vector space of dimension $n$, identified with the tangent space $T_{p} M$.

We have not been nearly precise enough to really understand $T M$ and its topology, but we're just going for the general ideas here.

Definition 10.40. Let $M$ be a smooth manifold with tangent bundle $\pi: T M \rightarrow M$. Smooth sections of $\pi$ are called vector fields on $M$. That is, a vector field on $M$ is a smooth map

$$
s: M \rightarrow T M
$$

such that $\pi s=\operatorname{Id}_{M}$. Given a local chart $h: U \rightarrow \mathbb{R}^{n}$ where $U$ is a neighborhood of $p$, we have our basis of $\frac{\partial}{\partial x_{i}}(p)$ for $T_{p} M$, and using these we can write a vector field/section of $\pi$ locally as

$$
s_{p}=f_{1}(p) \frac{\partial}{\partial x_{1}}(p)+\cdots+f_{n}(p) \frac{\partial}{\partial x_{n}}(p)
$$

where the $f_{i}$ are functions $U \rightarrow \mathbb{R}$. In particular, the condition that $s$ is smooth is equivalent to the functions $f_{i}$ all being smooth.

Note that for $U \subset M$ open, $U$ is a submanifold, and the tangent bundle for $U$ is just the restriction of $\pi$.

$$
\left.\pi\right|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U
$$

Definition 10.41. Let $M$ be a smooth manifold of dimension $n$, with tangent bundle $\pi$ : $T M \rightarrow M$. The collection of all vector fields (sections of $\pi$ ) on $M$ is denoted $\operatorname{Vect}(M)$.

For an open subset $U \subset M$ (which we might also refer to as an open submanifold), we may also talk about the vector fields on $U$, which we call $\operatorname{Vect}(U)$. Since $\operatorname{Vect}(U)$ is a collection of sections, we get a sheaf on $M$, which we denote $\mathcal{V}$.

$$
\mathcal{V}(U)=\operatorname{Vect}(U)
$$

The restriction maps for $\mathcal{V}$ are just literal function restrictions.
Remark 10.42. $\operatorname{Vect}(U)$ is an abelian group under pointwise addition, so $\mathcal{V}$ is a sheaf of abelian groups on $M$.
Definition 10.43. Let $M$ be a smooth real manifold, with structure sheaf $\mathcal{O}$. For $U \subset M$, we have a local coordinate function $h: U \rightarrow \mathbb{R}^{n}$. We can give $\mathcal{V}(U)$ the structure of an $\mathcal{O}(U)$-module, as follows. Recall that $s \in \mathcal{V}(U)$ can be written as

$$
s_{p}=f_{1}(p) \frac{\partial}{\partial x_{1}}(p)+\cdots+f_{n}(p) \frac{\partial}{\partial x_{n}}(p)
$$

where $f_{i}$ are smooth functions $U \rightarrow \mathbb{R}$. Then given any $g \in \mathcal{O}(U)$, a smooth function $U \rightarrow \mathbb{R}$, we can multiply $g$ pointwise with each $f_{i}$, and define

$$
g s_{p}=g(p) f_{1}(p) \frac{\partial}{\partial x_{1}}(p)+\cdots+g(p) f_{n}(p) \frac{\partial}{\partial x_{n}}(p)
$$

This makes $\mathcal{V}(U)$ into a $\mathcal{O}(U)$-module. We omit the structural details, but this makes $\mathcal{V}$ into a sheaf of $\mathcal{O}$-modules on $M$.

Now we dualize a lot of what just happened.
Definition 10.44. Let $M$ be a smooth manifold of dimension $n$ with $p \in M$, and tangent space $T_{p} M$. The cotangent space at $p$ is the dual space of $T_{p} M$, which we denote $\left(T_{p} M\right)^{*}$.

$$
\left(T_{p} M\right)^{*}=\operatorname{Hom}_{\mathbb{R}}\left(T_{p} M, \mathbb{R}\right)
$$

where the homomorphisms are $\mathbb{R}$-linear maps. If one chooses a local chart $h: U \rightarrow \mathbb{R}^{n}$ where $U \subset M$ is an open neighborhood of $p$, then there is a basis of $\left(T_{p} M\right)^{*}$, which is just the dual basis of the basis that $h$ determines for $T_{p} M$. We denote this basis

$$
d x_{1}, \ldots, d x_{n}
$$

That this is dual to the basis $\frac{\partial}{\partial x_{1}}(p), \ldots, \frac{\partial}{\partial x_{n}}(p)$ just means that

$$
d x_{i}\left(\frac{\partial}{\partial x_{j}}(p)\right)=\delta_{i j}
$$

We omit the full details of the definition, but there is a dualized version of the tangent bundle, called the cotangent bundle. We denote the cotangent bundle by $T^{*} M$. It has a map

$$
\pi^{*}: T^{*} M \rightarrow M
$$

$T^{*} M$ is also a smooth manifold of dimension $2 n$, where $n=\operatorname{dim} M$. As one would expect, the map $\pi^{*}$ is such that the fiber above $p \in M$ is identified with the cotangent space $\left(T_{p} M\right)^{*}$. The cotangent bundle as a whole is the disjoint union of these,

$$
T^{*} M=\bigsqcup_{p \in M}\left(T_{p} M\right)^{*}
$$

But of course this is not the full story, since then one needs to put the appropriate topology and smooth structure on $T^{*} M$ so that $\pi^{*}$ is a smooth map.

Definition 10.45. Let $M$ be a smooth manifold with cotangent bundle $\pi^{*}: T^{*} M \rightarrow M$. Smooth sections of $\pi^{*}$ are called differential 1-forms. So a differential 1-form is a smooth map

$$
\omega: M \rightarrow T^{*} M \quad p \mapsto \omega_{p} \in\left(T_{p} M\right)^{*}
$$

The set of such differential 1-forms is denoted $A^{(1)}(M)$. For $U \subset M$ an open submanifold (just an open subset), we then have $A^{(1)}(U)$, the differential 1-forms on $U$. Combining this data, we get a sheaf $\mathcal{A}^{(1)}$ on $M$, given by

$$
\mathcal{A}^{(1)}(U)=A^{(1)}(U)
$$

where restriction maps are just literal function restriction. The sheaf $\mathcal{A}^{(1)}$ is called the sheaf of differential 1-forms.

Definition 10.46. If we fix a local chart $h: U \rightarrow \mathbb{R}^{n}$ where $U \subset M$ is an open neighborhood of $p$, we can write $\omega_{p} \in\left(T_{p} M\right)^{*}$ in terms of the basis $d x_{1}, \ldots, d x_{n}$.

$$
\omega_{p}=\phi_{1}(p) d x_{1}+\ldots+\phi_{n}(p) d x_{n}
$$

where $\phi_{1}, \ldots, \phi_{n}$ are smooth functions $U \rightarrow \mathbb{R}$, that is, $\phi_{1}, \ldots, \phi_{n} \in \mathcal{O}(U)$. Then we can pointwise multiply $g \in \mathcal{O}(U)$ by $\omega_{p}$, hence making $A^{(1)}(U)$ into a $\mathcal{O}(U)$-module. We omit some details, but this is compatible in such a way to make $\mathcal{A}^{(1)}$ into a sheaf of $\mathcal{O}$-modules on $M$.

Now things start get really sketchy - we're going to start using somethings which we really aren't prepared to define or discuss in detail.

Definition 10.47. Let $M$ be a smooth manifold of dimension $n$. Generalizing the cotangent bundle $\pi^{*}: T^{*} M \rightarrow M$, for $1 \leq k \leq n$ there are bundles

$$
\pi^{(k)}: E^{(k)} \rightarrow M
$$

where $\pi^{(1)}$ is the cotangent bundle. For $p \in M$, the fiber of $\pi^{(k)}$ looks like

$$
\left(\pi^{(k)}\right)^{-1}(p) \cong \Lambda^{k}\left(T_{p} M\right)^{*}
$$

where $\Lambda^{k}$ is the $k$-fold exterior algebra of $\left(T_{p} M\right)^{*}$. In analogy with $A^{(1)}(M)$, we define $A^{(k)}(M)$ to be smooth sections of $\pi^{(k)}$. Hence $\omega \in A^{(k)}(M)$ is multilinear alternating function

$$
\omega: \prod_{i=1}^{k} T_{p} M \rightarrow \mathbb{R}
$$

As before, given a local chart $h: U \rightarrow \mathbb{R}^{n}$, we can write $\omega \in A^{(k)}(U)$ in terms of coordinates, although it gets very notationally confusing to do so.

$$
\omega=\sum_{1 \leq i_{1} \leq \cdots \leq i_{k} \leq n} \phi_{i_{1}} \cdots \phi_{i_{k}} d x_{i_{1}} \wedge \cdots \wedge d_{i_{k}}
$$

And as before, we sheafify this construction to obtain a sheaf $\mathcal{A}^{(k)}$ on $M$, given by

$$
\mathcal{A}^{(k)}(U)=A^{(k)}(U)
$$

where restriction maps are literal function restrictions. Following the same song and dance, this is once again a sheaf of $\mathcal{O}$-modules. This defines sheaves $\mathcal{A}^{(k)}$ for $k=1, \ldots, n$. By convention, we set $\mathcal{A}^{(0)}=\mathcal{O}$.

This mostly concludes our whirlwind tour of setup and sloppy definitions for building up de Rham cohomology in the language of sheaves. We'll need a few more definitions, but now we can start stating some results.

Theorem 10.48. Let $M$ be a smooth real manifold.

1. $M$ is paracompact.
2. For every locally finite cover $\mathcal{U}$ of $M$, there exists a partition of unity for $\mathcal{O}$ subordinate to $\mathcal{U}$.

Proof. Standard result from intro differential geometry, see various textbooks.
Remark 10.49. From the previous theorem and Proposition 10.28 , every sheaf of $\mathcal{O}$-modules on $M$ is fine, hence acyclic. In particular, the sheaves $\mathcal{A}^{(k)}$ defined above are acyclic.

$$
H^{n}\left(M, \mathcal{A}^{(k)}\right)=0 \quad \forall k \geq 0, n \geq 1
$$

Definition 10.50. There is a map called the exterior derivative

$$
d: A^{(k)} M \rightarrow A^{(k+1)} M
$$

which in local coordinates looks like

$$
d\left(\phi d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}\right)=d \phi \wedge d x_{i_{1}} \wedge \cdots \wedge d x_{i_{k}}
$$

where

$$
d \phi=\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_{i}} d x_{i}
$$

It is mostly straightforward from these definitions that $d^{2}=0$, along with properties of the exterior algebra. The details are not so important for us, just mainly the fact that $d$ gives a morphism of sheaves

$$
d: \mathcal{A}^{(k)} \rightarrow \mathcal{A}^{(k+1)}
$$

Note that $d$ is NOT a morphism of $\mathcal{O}$-modules, merely a morphism of sheaves of abelian groups.

Definition 10.51. Fix $U \subset M$ an open subset, and let $d=d^{(k)}: A^{(k)}(U) \rightarrow A^{(k+1)}(U)$ be the exterior derivative. A differential $k$-form $\omega \in A^{(k)}(U)$ is closed if it lies in the kernel of $d$. A differential $k$-form $\eta \in A^{(k)}$ is exact if it lies in the image of $d: A^{(k-1)}(U) \rightarrow A^{(k)}(U)$.

Since $d^{2}=0$, the differnetial $k$-forms form a chain complex,

$$
0 \rightarrow A^{(0)}(U) \rightarrow A^{(1)}(U) \rightarrow \cdots \rightarrow A^{(n)}(U) \rightarrow 0
$$

The $k$ th de Rham cohomology of $M$ is the $k$ th cohomology of this complex, that is, the quotient of closed $k$-forms on $U$ by exact $k$-forms on $U$.

$$
H_{\mathrm{dR}}^{k}(U)=\frac{\text { closed } k \text {-forms on } U}{\text { exact } k \text {-forms on } U}=\frac{\operatorname{ker} d^{(k)}}{\operatorname{imd} d^{(k-1)}}
$$

Remark 10.52. For $\phi \in \mathcal{O}(U)$, if $d \phi=0$, then $\phi$ is locally constant.
Definition 10.53. As always, let $M$ be a smooth real manifold. Let $\mathcal{C}$ be the locally constant sheaf on $M$ with value ring $\mathbb{R}$. It is then clear that we have a short exact sequence of sheaves (of abelian groups) on $M$

$$
0 \rightarrow \mathcal{C} \rightarrow \mathcal{A}^{(0)} \rightarrow \mathcal{A}^{(1)}
$$

This is exact as a sequence of presheaves since the sequence on sections over an open subset $U \subset M$ is exact, hence it is exact as a sequence of sheaves.

Next we want to give a big result about how to continue the above exact sequence using the other sheaves $\mathcal{A}^{(k)}$, but first we should state Poincaré's lemma, since it is involved in the proof.

Definition 10.54. A subset $X \subset \mathbb{R}^{n}$ is star shaped with respect to $x \in X$ if for any $y \in X$, the line segment from $x$ to $y$ lies in $X . X$ is star shaped if it star shaped with respect to any point $x \in X$.

Lemma 10.55 (Poincaré). Let $X \subset \mathbb{R}^{n}$ be star shaped and open. Then $X$ has no de Rham cohomology, that is,

$$
H_{\mathrm{dR}}^{k}(X)=0 \quad \forall k \geq 1
$$

Proof. This is essentially a computation in multivariable calculus.
Remark 10.56. If $M$ is a smooth manifold, then every point $p \in M$ has a neighborhood which is isomorphic to a star shaped region in $\mathbb{R}^{n}$, so locally speaking, every closed form on $M$ is exact. Of course, there may be globally closed forms which are not exact, on regions such as the circle or sphere.

Theorem 10.57. Let $M$ be a smooth manifold. The following sequence of abelian groups on $M$ is exact.

$$
0 \rightarrow \mathcal{C} \rightarrow \mathcal{A}^{(1)} \rightarrow \mathcal{A}^{(2)} \rightarrow \mathcal{A}^{(3)} \rightarrow \cdots \rightarrow \mathcal{A}^{(n)} \rightarrow 0
$$

Proof. The fact that $d^{2}=0$ is easy enough, this basically follows from the description of $d$ in local coordinates and the definition of the exterior algebra. The proof involves using local coordinates, Clairaut's theorem on mixed partials, and the fact that the wedge product is alternating.

The proof of exactness needs to be checked on stalks, and involves Poincaré's lemma about the vanishing of de Rham cohomology for star-shaped regions.

Remark 10.58. For our purposes, we will think of the exact sequence above as a resolution of $\mathcal{C}$, the locally constant sheaf on $M$, by acyclic (fine) sheaves. In particular, it is a resolution which can be used to compute sheaf cohomology groups $H^{k}(M, \mathcal{C})$ by the usual process take global sections, drop the first term, take cohomology.

On the other hand, if we take sections over $M$ of the above exact sequence (and drop the first term), then we get an exact sequence of abelian groups

$$
0 \rightarrow \mathcal{A}^{(0)}(M) \rightarrow \mathcal{A}^{(1)}(M) \rightarrow \cdots \rightarrow \mathcal{A}^{(n)}(M) \rightarrow 0
$$

and this sequence is exactly the chain complex used to compute the de Rham cohomology groups of $M$. That is, we obtain a isomorphisms

$$
H_{\mathrm{dR}}^{k}(M) \cong H^{k}(M, \mathcal{C})
$$

Since this is the main result of this section, we codify it as a theorem.
Theorem 10.59. Let $M$ be a smooth manifold, and $\mathcal{C}$ be the sheaf of locally constant $\mathbb{R}$ valued functions. Then sheaf cohomology of $\mathcal{C}$ computes de Rham cohomology of M.

$$
H_{\mathrm{dR}}^{k}(M) \cong H^{k}(M, \mathcal{C})
$$

### 10.5.2 Singular cohomology

In the previous section, the main result was that we may compute de Rham cohomology of a smooth manifold using sheaf cohomology of the locally constant sheaf with $\mathbb{R}$-valued functions. In this section, we pursue a similar goal of identifying singular cohomology with the same sheaf cohomology. From this, we will obtain the well-known result that, under appropriate circumstances, de Rham cohomology and singular cohomology are isomorphic.

As we reviewed de Rham cohomology in the previous section, we start with a review of singular cohomology in this section.
Definition 10.60. Let $n \in \mathbb{Z}_{\geq 1}$. The $n$-simplex $\Delta^{n}$ in $\mathbb{R}^{n}$ is the convex hull of $n$ points. For concreteness, one may think of it as the convex hull of $(1,0, \ldots),(0,1,0, \ldots), \ldots,(0, \ldots, 0,1)$.

Definition 10.61. Let $M$ be a smooth real manifold and let $n \in \mathbb{Z}_{\geq 1}$. A singular $n$ simplex in $M$ is a continuous map $f: \Delta^{n} \rightarrow M$. The word "singular" is used to emphasize that $f$ need not be smooth, merely continuous, so the image may look much stranger than if smoothness were required.

Definition 10.62. We define $C_{n}(M)$, or sometimes written $C_{n}(M, \mathbb{R})$ to be the $\mathbb{R}$-vector space spanned by all singular $n$-simplices in $M$. That is, $C_{n}(M)$ consists of formal $\mathbb{R}$-linear combinations of continuous maps $\Delta^{n} \rightarrow M$. It is a very large, infinite dimensional vector space.

We denote the dual space of $C_{n}(M)$ by $C^{n}(M)$. That is,

$$
C^{n}(M)=\left(C_{n}(M)\right)^{*}=\operatorname{Hom}_{\mathbb{R}}\left(C_{n}(M), \mathbb{R}\right)
$$

where the homomorphisms are just $\mathbb{R}$-linear maps.
Definition 10.63. Let $\Delta^{n}, C_{n}(M)$ be as above. The $i$ th face map is

$$
d_{i}^{n}: \Delta^{n-1} \rightarrow \Delta^{n}
$$

which includes $\Delta^{n-1}$ as the $i$ th face of $\Delta^{n}$. There is some hidden business of ordering the faces of $\Delta^{n}$ here, but don't worry about it. The $n$th boundary map

$$
\partial_{n}: C_{n}(M) \rightarrow C_{n-1}(M)
$$

is given by

$$
\partial_{n}(f)=\sum_{i=0}^{n}(-1)^{i}\left(f \circ d_{i}^{n}\right)
$$

Definition 10.64. The dualized version of the boundary map above is the coboundary map

$$
\partial^{n}: C^{n}(M) \rightarrow C^{n+1}(M)
$$

described by

$$
\left(\partial^{n} \phi\right)(f)=(-1)^{n+1} \phi\left(\partial_{n+1}(f)\right)
$$

where $f \in C_{n+1}(M)$ and $\phi \in C^{n}(M)$.

Definition 10.65. Let $C^{n}(M), \partial^{n}$ be as above. It is a somewhat tedious calculation to verify that $\partial^{n+1} \circ \partial^{n}=0$. Hence we get a chain complex

$$
C^{0}(M) \xrightarrow{\partial^{0}} C^{1}(M) \xrightarrow{\partial^{1}} C^{2}(M) \rightarrow \cdots
$$

The cohomology of this complex is called the singular cohomology of $M$ with coefficients in $\mathbb{R}$. It is denoted $H_{\text {sing }}^{n}(M, \mathbb{R})$.

Our next goal is to sheafify the above complex in a similar way to what we did with the de Rham complex, so that we may get a relation (isomorphism) with sheaf cohomology.

Definition 10.66. Let $M$ be a smooth manifold. For an open subset $U \subset M$, consider $C_{n}(U)$, the $\mathbb{R}$-vector space with basis given by singular $n$-simplices $f: \Delta^{n} \rightarrow U$. For $V \subset U$, the embedding $V \hookrightarrow U$ gives an embedding

$$
C_{n}(V) \hookrightarrow C_{n}(U)
$$

In fact, (the image of) $C_{n}(V)$ is a direct summand of $C_{n}(U)$, but this will not be important until later. For the moment, what is more important is that restriction of functions gives a map

$$
\left.C^{n}(U) \rightarrow C^{n}(V) \quad \phi \mapsto \phi\right|_{C_{n}(V)}
$$

This defines a presheav $C^{n}(-)$ on $M$. This is not in general a sheaf, so we sheafify it. Let $\mathcal{C}^{n}$ be the sheafification of $C^{n}$. The boundary maps $\partial^{n}: C^{n}(U) \rightarrow C^{n+1}(U)$ then induce sheaf maps

$$
\partial^{n}: \mathcal{C}^{n} \rightarrow \mathcal{C}^{n+1}
$$

leading to a chain complex of sheaves

$$
\mathcal{C}^{0} \xrightarrow{\partial^{0}} \mathcal{C}^{1} \xrightarrow{\partial^{1}} \mathcal{C}^{2} \rightarrow \cdots
$$

We will prove in a moment that this is exact.
Remark 10.67. For $U \subset M$, the presheaf $C^{0}$ has sections over $U$ which are just functions $U \rightarrow \mathbb{R}$. The condition $\phi \in \operatorname{ker} \partial^{0}$ is then equivalent to the property that $\phi(x)=\phi(y)$ if and only if $x, y$ can be connected by a curve in $U$. since the boundary map $\partial^{0}$ just evaluates $\phi$ as $\phi(x)-\phi(y)$. Hence $\phi \in \operatorname{ker} \partial^{0}$ if and only if $\phi$ is locally constant. Passing to sheaves does not impact this discussion, so we have a short exact sequence of sheaves of abelian groups on $M$,

$$
0 \rightarrow \mathcal{C} \rightarrow \mathcal{C}^{0} \xrightarrow{\partial^{0}} \mathcal{C}^{1}
$$

where $\mathcal{C}$, as before, is the sheaf of locally constant $\mathbb{R}$-valued functions $U \rightarrow \mathbb{R}$.
The previous remark gives some reason to hope that we may extend it to a long exact sequence using the sheaves $\mathcal{C}^{n}$, and we may. First, we need to state a lemma from algebraic topology, which gives slightly more concrete description of the sections of the sheaf $\mathcal{C}^{n}$ obtained by sheafifying $C^{n}$.

Lemma 10.68. Let $M$ be a smooth manifold and $U \subset M$ be open. Define

$$
C^{n}(U)_{0}=\left\{\phi \in C^{n}(U) \mid \exists \text { an open cover }\left\{U_{i}\right\}_{i \in I} \text { of } U \text { such that }\left.\phi\right|_{C_{n}\left(U_{i}\right)}=0 \forall i\right\}
$$

Then there is a natural isomorphism, compatible with the differentials and restriction maps,

$$
\mathcal{C}^{n}(U) \cong \frac{C^{n}(U)}{C^{n}(U)_{0}}
$$

Proof. This is purely topological, so we omit it. A proof can be found in Foundations of Differentiable Manifolds and Lie Groups, by Warmer as Proposition 5.27.

Proposition 10.69. Let $M$ be a smooth manifold, and $\mathcal{C}^{n}$ be the sheaves above. Then

$$
0 \rightarrow \mathcal{C} \rightarrow \mathcal{C}^{0} \xrightarrow{\partial^{0}} \mathcal{C}^{1} \xrightarrow{\partial^{1}} \mathcal{C}^{2} \rightarrow \cdots
$$

is a flasque (hence acyclic) resolution of $\mathcal{C}$.
The important aspect of the previous proposition is that the resolution above computes sheaf cohomology groups $H^{k}(M, \mathcal{C})$.

Proof. The exactness aspect of the claim is not too bad to prove. It must be check on stalks, so the sheafification aspect can be basically ignored. This part of the proof utilizes homotopy invariance of singular cohomology - each point $p \in M$ has a contractible open neighborhood $U$ with trivial singular homology, so the sequence

$$
C^{n-1}(U) \xrightarrow{\partial^{n-1}} C^{n}(U) \xrightarrow{\partial^{n}} C^{n+1}(U)
$$

is exact. Hence we get exactness on the stalks at $x$. Since $x$ was arbitrary, this gives exactness at every stalk, so the sequence is exact.

The flasqueness aspect of the claim is not too bad, assuming the previous lemma characterizing sections of $\mathcal{C}^{n}$ in Lemma 10.68. If $V \subset U$, then $C_{n}(V) \subset C_{n}(U)$ is a direct summand, so passing to the dual space,

$$
C^{n}(U) \rightarrow C^{n}(V)
$$

is surjective. Hence the presheaf $C^{n}$ is flasque. Then by the lemma, since $\mathcal{C}^{n}(U) \cong$ $C^{n}(U) / C^{n}(U)_{0}$ in a way which is compatible with restriction maps, so the restriction maps for $\mathcal{C}^{n}$ are also surjective, which is to say, $\mathcal{C}^{n}$ is flasque.

Taking global sections of the previous chain complex, we obtain a computation of sheaf cohomology groups for $M$ using the locally constant sheaf $\mathcal{C}$.

Corollary 10.70. The cohomology groups of the complex

$$
0 \rightarrow \mathcal{C}^{0}(M) \xrightarrow{\partial^{0}} \mathcal{C}^{1}(M) \xrightarrow{\partial^{1}} \mathcal{C}^{2}(M) \rightarrow \cdots
$$

are the sheaf cohomology groups $H^{k}(M, \mathcal{C})$.

Remark 10.71. On the other hand, we can also use the flasque resolution to compute singular cohomology, but it takes a few mores steps. Recall that singular cohomology groups $H_{\text {sing }}^{k}(M, \mathbb{R})$ are the cohomology groups of the un-sheafified complex

$$
0 \rightarrow C^{0}(M) \xrightarrow{\partial^{0}} C^{1}(M) \xrightarrow{\partial^{1}} C^{2}(M) \rightarrow \cdots
$$

Using Lemma 10.68, we have a short exact sequence of chain complexes

$$
0 \rightarrow C^{\bullet}(M)_{0} \rightarrow C^{\bullet}(M) \rightarrow \mathcal{C}^{\bullet}(M) \rightarrow 0
$$

The left term is the one defined in the lemma, the middle term computes singular cohomology, and the right term is the sheafified version, which the lemma specifies as the quotient $C^{\bullet}(M) / C^{\bullet}(M)_{0}$. Note that the compatiblity with the boundary maps is necessary to make this a sequence of chain complexes.

To complete the identification of singular cohomology and sheaf cohomology (of $\mathcal{C}$ ), we will pass to the long exact sequence associated with the above sequence of chain complexes.

$$
0 \rightarrow H^{0}\left(C^{\bullet}(M)_{0}\right) \rightarrow H^{0}\left(C^{\bullet}(M)\right) \rightarrow H^{0}\left(\mathcal{C}^{\bullet}(M)\right) \rightarrow H^{1}\left(C^{\bullet}(M)_{0}\right) \rightarrow \cdots
$$

We are going to show that $C^{\bullet}(M)_{0}$ is acyclic, which means that in that LES, every third term vanishes and we obtain isomorphisms between the cohomology of $C^{\bullet}(M)$ (which is singular cohomology) and $\mathcal{C}^{\bullet}(M)$ (which is sheaf cohomology)

First, we need to cite another result from algebraic topology.
Definition 10.72. Let $M$ be a smooth manifold, and $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ be an open cover of $M$. Define the small chains $C_{n}^{\mathcal{U}}(M)$ to be the subspace of $C_{n}(M)$ spanned by singular $n$-simplices whose image lies inside some $U_{i}$.

$$
C_{n}^{\mathcal{U}}(M)=\operatorname{span}_{\mathbb{R}}\left\{f \in C_{n}(M): \exists i \in I, \operatorname{im} f \subset U_{i}\right\}
$$

Then define the small cochains $C_{\mathcal{U}}^{n}(M)$ to be the dual space of of $C_{n}^{\mathcal{U}}$.

$$
C_{\mathcal{U}}^{n}(M)=\operatorname{Hom}_{\mathbb{R}}\left(C_{n}^{\mathcal{U}}, \mathbb{R}\right)
$$

By dualizing the inclusion $C_{n}^{\mathcal{U}}(M) \hookrightarrow C^{n}(M)$, we obtain a surjection $C^{n}(M) \rightarrow C_{\mathcal{U}}^{n}(M)$. Define $C_{\mathcal{U}}^{n}(M)_{0}$ to be the kernel of this surjection, so that we have a short exact sequence (of abelian groups)

$$
0 \rightarrow C_{\mathcal{U}}^{n}(M)_{0} \rightarrow C^{n}(M) \rightarrow C_{\mathcal{U}}^{n}(M) \rightarrow 0
$$

Lemma 10.73. Let $M$ be a smooth manifold, and for an open cover $\mathcal{U}$, let $C_{\mathcal{U}}^{n}(M)_{0}$ be as above. These form a directed system with indexing set given by open covers $\mathcal{U}$, and

$$
C^{n}(M)_{0}=\underset{\vec{u}}{\lim } C_{\mathcal{U}}^{n}(M)_{0}
$$

Proof. Omitted.

Theorem 10.74. Let $M$ be a topological space and $\mathcal{U}$ be an open cover of $M$. The natural embedding of complexes

$$
\iota_{\bullet}: C_{\bullet}^{\mathcal{U}}(M) \rightarrow C_{\bullet}(M)
$$

is a chain equivalence. That is, there exists a morphism of complexes

$$
f: C_{\bullet}(M) \rightarrow C_{\bullet}^{U}(M)
$$

such that $f_{\bullet} \circ \iota_{\bullet}$ and $\iota_{\bullet} \circ f_{\bullet}$ are each chain homotopic to the identity.
Proof. This is very technical, involving such things as barycentric subdivision. See Algebraic Topology by Tom Dieck, Theorem 9.4.5.

Now we are set up to show that $C^{\bullet}(M)$ is acyclic, which will shortly thereafter complete the proof that $H_{\text {sing }}^{n}(M, \mathbb{R}) \cong H^{n}(M, \mathcal{C})$.

Lemma 10.75. The chain complex $C^{\bullet}(M)_{0}$ is acyclic.
Proof. Consider the short exact sequence of chain complexes

$$
0 \rightarrow C_{\mathcal{U}}^{\bullet}(M)_{0} \rightarrow C^{\bullet}(M) \rightarrow C_{\mathcal{U}}^{n}(M) \rightarrow 0
$$

Then consider the long exact sequence on cohomology.

$$
0 \rightarrow H^{0}\left(C_{\mathcal{U}}^{\bullet}(M)_{0}\right) \rightarrow H^{0}\left(C^{\bullet}(M)\right) \rightarrow H^{0}\left(C_{\mathcal{U}}^{n}(M)\right) \rightarrow H^{1}\left(C_{\mathcal{U}}^{\bullet}(M)_{0}\right) \rightarrow \cdots
$$

Passing to the dual spaces in Theorem 10.74, we get that

$$
C^{\bullet}(M) \rightarrow C_{\mathcal{U}}^{\bullet}(M)
$$

is also a chain equivalence, hence induces isomorphisms of cohomology, which implies that in the long exact sequence above the terms $H^{n}\left(C_{\dot{\mathcal{U}}}^{\bullet}(M)_{0}\right)$ are all zero for $n \geq 1$, which is to say, the complex $C_{\mathcal{U}}^{\bullet}(M)_{0}$ is acyclic. Passing to the direct limits using Lemma 10.73, the complex $\left.C^{\bullet}(M)_{0}\right)$ is acyclic, as claimed. Note that in this last step, we are utilizing an exactness property of the direct limit functor, which has not been entirely spelled out here.

Theorem 10.76. Let $M$ be a smooth manifold, and $\mathcal{C}$ be the locally constant sheaf of $\mathbb{R}$ valued functions on $M$. Singular cohomology of $M$ with coefficients in $\mathbb{R}$ may be identified with sheaf cohomology of $M$ using the sheaf $\mathcal{C}$. That is, for $k \in \mathbb{Z}_{\geq 0}$,

$$
H_{\text {sing }}^{k}(M, \mathbb{R}) \cong H^{k}(M, \mathcal{C})
$$

Proof. Following Remark 10.71, we have a long exact sequence

$$
0 \rightarrow H^{0}\left(C^{\bullet}(M)_{0}\right) \rightarrow H^{0}\left(C^{\bullet}(M)\right) \rightarrow H^{0}\left(\mathcal{C}^{\bullet}(M)\right) \rightarrow H^{1}\left(C^{\bullet}(M)_{0}\right) \rightarrow \cdots
$$

By Lemma 10.75, the $C^{\bullet}(M)_{0}$ terms vanish for $k \geq 1$, so we get isomorphisms

$$
H^{k}\left(C^{\bullet}(M)\right) \cong H^{k}\left(\mathcal{C}^{\bullet}(M)\right)
$$

Corollary 10.70 tell us that the right term is isomorphic to $H^{k}(M, \mathcal{C})$, and the left term is $H_{\text {sing }}^{k}(M, \mathbb{R})$ by definition.

As one final statement collecting everything together, we give one last theorem.
Theorem 10.77. Let $M$ be a smooth manifold and $\mathcal{C}$ the sheaf of locally constant $\mathbb{R}$-valued functions on $M$. Let $k \in \mathbb{Z}_{\geq 0}$. Then

$$
H^{k}(M, \mathcal{C}) \cong H_{\mathrm{sing}}^{k}(M, \mathbb{R}) \cong H_{\mathrm{dR}}^{k}(M)
$$


[^0]:    ${ }^{1}$ For example, in $K[x]$, take $I=\left(x^{2}\right)$. Then $X=\{0\}$, and $I(X)=(x)$.

[^1]:    ${ }^{2}$ Rational functions $U \rightarrow K$ can be added and multiplied without introducing "poles" (points where the denominator vanishes). There is some checking to do that the choice of representative doesn't matter.

[^2]:    ${ }^{3}$ I'm not entirely sure if this limit is in the category of rings, or of modules over $R$. Probably the latter.
    ${ }^{4}$ This particular example is why we don't insist that our filtered sets be partially ordered, because in this ordering $S$ is not partially ordered. In particular, the property that fails is "antisymmetry," that is, $x \leq y$ and $y \leq x$ implies $x=y$.

[^3]:    ${ }^{5}$ Topology: A First Course by Munkres, Theorem 26.9

[^4]:    ${ }^{6}$ A category is discrete if the only morphisms are identity arrows.

[^5]:    ${ }^{7}$ In the above sequence, zero refers to the zero presheaf, which is the constant sheaf with values in the trivial group.

[^6]:    ${ }^{8} E \times{ }_{X} E$ is the fiber product of $E$ and $E$ over $X$, utilizing the maps $\pi$. Concretely,

    $$
    E \times_{X} E=\left\{\left(e_{1}, e_{2}\right) \in E \times E: \pi\left(e_{1}\right)=\pi\left(e_{2}\right)\right\}
    $$

[^7]:    ${ }^{9}$ It is tempting to say the condition (2) implies that $\left\{t_{i j}\right\} \in \check{B}^{1}(\mathcal{U}, \mathcal{O})$, but that would only be true if we knew that $f_{i} \in \mathcal{O}\left(U_{i}\right)$, which is not necessarily the case, since $f_{i}$ may have a pole at $a_{i}$.

[^8]:    ${ }^{10}$ Just kidding, the sign is an important part of the fact that $d^{n+1} \circ d^{n}=0$.

[^9]:    ${ }^{11}$ Explicitly, this means that the identity morphism of presheaves induces the identity map on $\check{H}^{n}$, and that this process of inducing maps respects composition.

[^10]:    ${ }^{12}$ Recall that $I$ is the indexing set for our open cover $\mathcal{U}=\left\{U_{i}\right\}_{i \in I}$ for $X$.

[^11]:    ${ }^{13} \mathcal{T}$ ranges over open covers of $U_{i}$.

[^12]:    ${ }^{14} \mathrm{~A}$ subspace $A \subset X$ is locally closed if $A=U \cap V$ where $U \subset X$ is open and $V \subset X$ is closed.

[^13]:    ${ }^{15}$ The only difference in algebra between homology and cohomology is which direction the arrows go in the chain complexes, so it doesn't actually affect the "shape" of the diagram in a meaningful way.

[^14]:    ${ }^{16}$ As always, this is just expressed by some diagram being commutative.

[^15]:    ${ }^{17}$ This might need to be $\operatorname{Ext}_{R}^{n}(-, A)$ instead, I get the entries confused sometimes.

[^16]:    ${ }^{18} \mathrm{We}$ are somewhat abusing notation of quotients by thinking of things as modules over a ring, but there is a way to make this precise, it just takes a lot of time and effort.

[^17]:    ${ }^{19}$ Perhaps I am wrong about this. In any case, it should follow that those morphisms are isomorphisms by tracing through the construction in part (1) and combining with the argument in (2).

[^18]:    ${ }^{20}$ For the next equation, recall the notation

    $$
    U_{i_{0} \cdots i_{n}}=U_{i_{0}} \cap \cdots \cap U_{i_{n}}
    $$

[^19]:    ${ }^{21}$ Unfortunately we have used $I^{n+1}$ for two different things. It is the $(n+1)$ th term of the injective resolution of $\mathcal{F}$, but we also used $I$ as the indexing set for the open cover $\mathcal{U}$ of $X$. In the formula below, $I^{n+1}$ refers to the $(n+1)$-fold Cartesian product of this indexing set, not the injective sheaf.

[^20]:    ${ }^{22}$ It must be surjective by simple commutativity of the square. To construct an inverse map, lift to $H^{0}(X, \mathcal{Q})$, then use the isomorphism to get to $\check{H}^{0}(\mathcal{U}, \mathcal{Q})$, then apply the horizontal map. This does not depend on choice of lift by some exactness/diagram chasing, so it gives a well-defined inverse.

