Proposition 0.1 (Exercise 4). Let $f$ be integrable on $[0, b]$. Define

$$g(x) = \int_x^b \frac{f(t)}{t} \, dt$$

for $0 < x \leq b$. Then

$$\int_0^b g(x) \, dx = \int_0^b f(t) \, dt$$

and hence $g$ is integrable on $[0, b]$.

Proof. Note that

$$\chi_{[x, b]}(t)\chi_{[0, b]}(x) = \chi_{[0, b]}(t)\chi_{[0, t]}(x)$$

By applying Fubini and Tonelli theorems repeatedly, we have

$$\int_0^b g(x) \, dx = \int_0^b \int_x^b \frac{f(t)}{t} \, dt \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[0, b]}(x)\chi_{[x, b]}(t)\frac{f(t)}{t} \, dt \, dx$$

$$= \int_{\mathbb{R}^2} \chi_{[0, b]}(t)\chi_{[0, t]}(x)f(t) \, dm = \int_0^b \int_0^t \frac{f(t)}{t} \, dx \, dt$$

Now $f(t)/t$ does not depend on $x$, so

$$\int_0^b g(x) \, dx = \int_0^b \frac{f(t)}{t} \left( \int_0^t dx \right) \, dt$$

And noting that

$$\int_0^t dx = t$$

we have

$$\int_0^b g(x) \, dx = \int_0^b \frac{f(t)}{t} \, t \, dt = \int_0^b f(t) \, dt$$

As $f$ is integrable on $[0, b]$, the integral is finite, so $g$ is also integrable on $[0, b]$. \qed
Proposition 0.2 (Exercise 7). Let \( f \) be a measurable function on \( \mathbb{R}^d \) and define
\[
\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : f(x) = y\}
\]
Then \( \Gamma \) is measurable and \( m(\Gamma) = 0 \).

Proof. First we show that \( \Gamma \) is measurable. Define \( F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) by \( (x, y) \mapsto f(x) \) and define \( \pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) by \( (x, y) \mapsto y \). By Corollary 3.7, \( F \) is measurable. It is obvious that \( \pi \) is measurable, as \( \pi^{-1}((a, \infty)) = \mathbb{R}^d \times (a, \infty) \). Thus \( F - \pi \) is measurable. Note that
\[
(F - \pi)^{-1}(\{0\}) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : f(x) - y = 0\} = \Gamma
\]
so \( \Gamma \) is the preimage of a closed set under a measurable function. Thus by Property 1 of measurable functions (Chapter 1, page 28), \( \Gamma \) is measurable.

Now consider the slice
\[
\Gamma^x = \{y \in \mathbb{R} : f(x) = y\} = \{f(x)\}
\]
Then \( \Gamma^x \) is a finite set, so \( m(\Gamma^x) = 0 \). By Corollary 3.3,
\[
m(\Gamma) = \int_{\mathbb{R}^d} m(\Gamma^x) dx = \int_{\mathbb{R}^d} 0 \, dx = 0
\]

\( \square \)

Proposition 0.3 (Exercise 8, repeated from Homework 5). Let \( f \) be integrable on \( \mathbb{R} \) and define \( F : \mathbb{R} \to \mathbb{R} \) by
\[
F(x) = \int_{-\infty}^{\infty} f(t)dt
\]
Then \( F \) is uniformly continuous.

Proof. (repeated verbatim from Homework 5)
We need to show that for \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
|x - y| < \delta \implies |F(x) - F(y)| < \epsilon
\]
Let \( \epsilon > 0 \). Without loss of generality, assume that \( x < y \). Note that
\[
\int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{x} f(t)dt + \int_{x}^{y} f(t)dt
\]
So then
\[
|F(x) - F(y)| = \left| \int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{y} f(t)dt \right|
= \left| \int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{x} f(t)dt - \int_{x}^{y} f(t)dt \right|
= \left| \int_{x}^{y} f(t)dt \right|
\leq \int_{x}^{y} |f(t)|dt
\]
By proposition 1.12(ii), there exists $\delta > 0$ such that

$$\int_E |f| < \epsilon$$

whenever $m(E) < \delta$. Then if $|x - y| < \delta$, $m((x, y)) < \delta$ so

$$\int_x^y |f(t)|dt < \epsilon$$

so combining our inequalities, we reach the desired inequality.

$$|F(x) - F(y)| \leq \int_x^y |f(t)|dt < \epsilon$$

whenever $|x - y| < \delta$. Thus $F$ is uniformly continuous.

**Proposition 0.4** (Exercise 15, repeated from Homework 6). Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Fix an enumeration $\{r_n\}_{n=1}^\infty$ of $\mathbb{Q}$ and let

$$F(x) = \sum_{n=1}^\infty 2^{-n} f(x - r_n)$$

Then $F$ is integrable, and the series defining $F$ converges almost everywhere. Also, $F$ is unbounded on every interval, and any function $\tilde{F}$ that agrees with $F$ almost everywhere is unbounded on any interval.

**Proof.** (repeated verbatim from Homework 6)

By Corollary 1.10 (Stein),

$$\int F(x)dx = \int \sum_{n=1}^\infty 2^{-n} f(x - r_n) = \sum_{n=1}^\infty \int 2^{-n} f(x - r_n)dx$$

Consider the integral on the far right. Since $f(x - r_n) = 0$ outside of $(r_n, r_n + 1)$, we have

$$\int 2^{-n} f(x - r_n)dx = 2^{-n} \int f(x - r_n)dx = 2^{-n} \int_{r_n}^{r_n+1} (x - r_n)^{-1/2}dx$$

The integral on the far right is Riemann integrable, and we have

$$\int_{r_n}^{r_n+1} (x - r_n)^{-1/2}dx = 2$$
So we have
\[
\int F(x)dx = \sum_{n=1}^{\infty} 2^{-n}(2) = 2
\]
Thus \( F \) is integrable. Then also by Corollary 1.10, since the series of integrals converges, the series defining \( F \) converges almost everywhere.

Now we show that any function \( \tilde{F} \) that agrees with \( F \) almost everywhere is unbounded on any interval. Let \( A = (a, b) \) be an interval in \( \mathbb{R} \). Then there exists a rational \( r_k \) in \( A \). As \( f(x) \) is unbounded near \( x = 0 \), we have \( f(x - r_k) \) unbounded near \( x = r_k \). Since \( \tilde{F} \) differs from \( F \) only on a set of measure zero, there is some irrational \( y \) in \((r_k, r_k + 1)\) such that \( F(y) = \tilde{F}(y) \), and we may choose \( y \) to be as close to \( r_k \) as we like. Thus \( \tilde{F} \) is unbounded on every interval.

Proposition 0.5 (Exercise 18). Let \( f \) be a measurable finite-valued function on \([0, 1] \). Define \( g(x, y) = \chi_{[0,1]}(x)\chi_{[0,1]}(y)|f(x) - f(y)| \) and suppose \( g \) is integrable on \( \mathbb{R}^d \). Then \( f \) is integrable on \([0, 1] \).

Proof. By Fubini’s Theorem, \( g^y \) is integrable for a.e. \( y \), so
\[
\infty > \int_{\mathbb{R}} g^y(x)dx = \int_{\mathbb{R}} \chi_{[0,1]}(x)\chi_{[0,1]}(y)|f(x) - f(y)| = \chi_{[0,1]}(y) \int_0^1 |f(x) - f(y)|dx
\]
By the triangle inequality, \(|f(x)| \leq |f(x) - f(y)| + |f(y)|\) for all \( y \), so
\[
\int_0^1 |f(x)|dx \leq \int_0^1 |f(x) - f(y)| + |f(y)|dx \leq \int_{\mathbb{R}} g^y(x)dx + \int_0^1 |f(y)|dx = \int_{\mathbb{R}} g^y(x)dx + |f(y)| < \infty
\]
because \( f \) is finite-valued so we have \(|f(y)| < \infty \). Thus \( \int_0^1 |f(x)|dx \) is finite so \( f \) is integrable on \([0, 1] \). 

Proposition 0.6 (Exercise 19). Let \( f \) be integrable on \( \mathbb{R}^d \). For \( \alpha > 0 \), let
\[
E_\alpha = \{x \in \mathbb{R}^d : |f(x)| > \alpha \}
\]
Then
\[
\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha
\]
Proof. Note that $m(E_{\alpha}) = \int_{\mathbb{R}^d} \chi_{E_{\alpha}}(x)dx$. Then by Fubini’s Theorem,

$$\int_0^\infty m(E_{\alpha})d\alpha = \int_0^\infty \left( \int_{\mathbb{R}^d} \chi_{E_{\alpha}}dx \right) d\alpha = \int_{\mathbb{R}^d} \left( \int_0^\infty \chi_{E_{\alpha}}(x)d\alpha \right)dx$$

We have

$$\chi_{E_{\alpha}}(x) = \begin{cases} 
0 & |f(x)| \geq \alpha \\
1 & |f(x)| < \alpha 
\end{cases}$$

thus

$$\int_0^\infty \chi_{E_{\alpha}}(x)d\alpha = \int_0^{\|f(x)\|} 1d\alpha + \int_{\|f(x)\|}^\infty 0d\alpha = |f(x)|$$

so finally we get

$$\int_0^\infty m(E_{\alpha})d\alpha = \int_{\mathbb{R}^d} \int_0^\infty \chi_{E_{\alpha}}(x)d\alpha dx = \int_{\mathbb{R}^d} |f(x)|dx$$

Lemma 0.7 (for Exercise 21a). Let $f, g$ be measurable function on $\mathbb{R}^d$. Then $fg$ is measurable on $\mathbb{R}^d$.

Proof. By Theorem 4.2, there exists sequences $f_n, g_n$ of simple functions such that $f_n(x) \rightarrow f(x)$ and $g_n(x) \rightarrow g(x)$ for all $x \in \mathbb{R}^d$. Then by the properties of limits,

$$(f_n g_n)(x) = f_n(x)g_n(x) \rightarrow f(x)g(x) = (fg)(x)$$

for all $x$. As $f_n, g_n$ are simple functions, their product is a simple function. Thus $fg$ is a pointwise limit of the simple functions $f_n g_n$, so by Property 4 (pg 29), $fg$ is measurable. 

Proposition 0.8 (Exercise 21a). Let $f, g$ be measurable functions on $\mathbb{R}^d$. Then $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

Proof. By Proposition 3.9, $f(x-y)$ is measurable on $\mathbb{R}^{2d}$. By Corollary 3.7, $g(y) = \tilde{g}(x, y)$ is measurable on $\mathbb{R}^{2d}$. Then by the above lemma, the product $f(x-y)g(y)$ is measurable.

Proposition 0.9 (Exercise 21b). Let $f, g$ be integrable on $\mathbb{R}^d$. Then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

Proof. By Fubini’s Theorem,

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)| = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)|dx \, dy = \int_{\mathbb{R}^d} |g(y)| \left( \int_{\mathbb{R}^d} |f(x-y)|dx \right)dy$$

By translation invariance of the integral,

$$\int_{\mathbb{R}^d} |f(x-y)|dx = \int_{\mathbb{R}^d} |f(x)|dx$$
so the integral under consideration is equal to
\[
\int_{\mathbb{R}^d} |g(y)| \left( \int_{\mathbb{R}^d} |f(x)|dx \right) dy = \int_{\mathbb{R}^d} |f(x)|dx \int_{\mathbb{R}^d} |g(y)|dy
\]
As \(f, g\) are integrable, each of the integrals on the RHS are finite, so the integral of 
\(f(x - y)g(y)\) is also finite.

**Proposition 0.10** (Exercise 21c). Let \(f, g\) be integrable functions on \(\mathbb{R}^d\). Then the convolution of \(f\) and \(g\), given by
\[
(f \ast g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy
\]
is well defined for a.e. \(x\). That is, \(f(x - y)g(y)\) is integrable on \(\mathbb{R}^d\) for a.e. \(x\).

**Proof.** Let \(h(x, y) = f(x - y)g(y)\). By part (b), \(h\) is integrable on \(\mathbb{R}^d\). Then by part (i) of Fubini’s Theorem (Theorem 3.1), the slice
\[
h_x(y) = f(x - y)g(y)
\]
is integrable with respect to \(y\) for a.e. \(x\).

**Proposition 0.11** (Exercise 21d). Let \(f, g\) be integrable on \(\mathbb{R}^d\). Then \(f \ast g\) is integrable on \(\mathbb{R}^d\) and
\[
\|f \ast g\| \leq \|f\|\|g\|
\]
(all norms are \(L^1(\mathbb{R}^d)\).) Equality holds if \(f, g\) are non-negative.

**Proof.** Let \(h(x, y) = f(x - y)g(y)\). As \(h\) is integralbe on \(\mathbb{R}^{2d}\) by part (b), by part (ii) of Fubini’s Theorem (Theorem 3.1),
\[
\int_{\mathbb{R}^d} h^y(x)dy = \int_{\mathbb{R}^d} f(x - y)g(y)dy = (f \ast g)(x)
\]
is integrable for a.e. \(x\). Then we compute
\[
\|f \ast g\| = \int_{\mathbb{R}^d} |(f \ast g)(x)|dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x - y)g(y)dy \right| dx
\]
\[
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x - y)g(y)|dy \ dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x - y)||g(y)| \ dx \ dy
\]
\[
= \int_{\mathbb{R}^d} |g(y)| \left( \int_{\mathbb{R}^d} |f(x - y)| \ dx \right) dy = \int_{\mathbb{R}^d} |g(y)| \left( \int_{\mathbb{R}^d} |f(x)| \ dx \right) dy
\]
\[
= \int_{\mathbb{R}^d} |g(y)|dy \int_{\mathbb{R}^d} |f(x)|dx = \|f\|\|g\|
\]
Thus \(\|f \ast g\| \leq \|f\|\|g\|\). If \(f, g\) are non-negative, then
\[
\left| \int_{\mathbb{R}^d} f(x - y)g(y)dy \right| = \int_{\mathbb{R}^d} |f(x - y)g(y)|dy
\]
so the one step that introduces an inequality becomes an equality. Thus if \(f, g\) are non-negative, we have equality.
Proposition 0.12 (Exercise 21e). Let \( f, g \) be integrable on \( \mathbb{R}^d \). Then for \( \xi \in \mathbb{R}^d \),

\[
\hat{f} \ast \hat{g}(\xi) = \hat{f}(\xi) \hat{g}(\xi)
\]

Proof. The proof is an uninspiring sequence of equalities, involving several applications of Fubini’s Theorem and the translation invariance of the Lebesgue integral.

\[
(\hat{f} \ast \hat{g})(\xi) = \int_{\mathbb{R}^d} (f \ast g)(x)e^{-2\pi i x \cdot \xi} dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x-y)g(y)dy \right) e^{-2\pi i x \cdot \xi} dx
\]

\[
= \int_{\mathbb{R}^d} f(x-y)g(y)e^{-2\pi i (x+y) \cdot \xi} dy = \int_{\mathbb{R}^d} g(y) \left( \int_{\mathbb{R}^d} f(x-y)e^{-2\pi i y \cdot \xi} dx \right) dy
\]

\[
= \int_{\mathbb{R}^d} g(y)e^{-2\pi i y \cdot \xi} \hat{f}(\xi) dy = \hat{f}(\xi) \int_{\mathbb{R}^d} g(y)e^{-2\pi i y \cdot \xi} dy = \hat{f}(\xi) \hat{g}(\xi)
\]

Proposition 0.13 (Exercise 22). Let \( f \in L^1(\mathbb{R}^d) \) and define

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx
\]

Then

\[
\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0
\]

Proof. First let, \( \xi' = \frac{\xi}{2|\xi|^2} \). First, note that

\[
\xi' \cdot \xi = \frac{\xi \cdot \xi}{2|\xi|^2} = \frac{1}{2}
\]

so therefore

\[
e^{-2\pi i \xi' \cdot \xi} = e^{-\pi i} = -1
\]

Then we compute

\[
\frac{1}{2} \int_{\mathbb{R}^d} [f(x) - f(x - \xi')]e^{-2\pi i x \cdot \xi} dx = \frac{1}{2} \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx - \frac{1}{2} \int_{\mathbb{R}^d} f(x - \xi')e^{-2\pi i x \cdot \xi} dx
\]

\[
= \frac{1}{2} \hat{f}(\xi) - \frac{1}{2} \int_{\mathbb{R}^d} f(x)e^{-2\pi i (x + \xi') \cdot \xi} dx
\]

\[
= \frac{1}{2} \hat{f}(\xi) - \frac{1}{2} \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} e^{-2\pi i \xi' \cdot \xi} dx
\]

\[
= \frac{1}{2} \hat{f}(\xi) + \frac{1}{2} \hat{f}(\xi) = \hat{f}(\xi)
\]
Now we have

\[
2 \hat{f}(\xi) = \int_{\mathbb{R}^d} [f(x) - f(x - \xi')]e^{-2\pi i x \cdot \xi} dx \leq \int_{\mathbb{R}^d} \left| f(x) - f(x - \xi') \right| e^{-2\pi i x \cdot \xi} dx
\]

\[
= \int_{\mathbb{R}^d} \left| f(x) - f(x - \xi') \right| e^{-2\pi i x \cdot \xi} dx = \int_{\mathbb{R}^d} \left| f(x) - f(x - \xi') \right| dx
\]

\[
= \| f - f_{\xi'} \|_{L^1(\mathbb{R}^d)}
\]

As \( \xi' = \frac{\xi}{2|\xi|^2} \to 0 \), we have \(|\xi| \to \infty\), so by Proposition 2.5,

\[
\lim_{|\xi| \to \infty} \hat{f}(\xi) = \frac{1}{2} \lim_{|\xi| \to \infty} 2 \hat{f}(\xi) = \frac{1}{2} \lim_{|\xi| \to \infty} \| f - f_{\xi'} \| = \frac{1}{2}(0) = 0
\]

□