Homework 3
Real Analysis

Joshua Ruiter

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Proposition 0.1 (Exercise 13a)**.** *Let $A \subset \mathbb{R}^d$ be closed and $B \subset \mathbb{R}^d$ be open. Then A is a G_δ set and B is an F_σ set.*

Proof. Let $A_n = \{x : d(x, A) < 1/n\}$. We know that A_n is open because we can write is as a union of open balls,

$$A_n = \bigcup_{a \in A} B\left(a, \frac{1}{n}\right)$$

We pause to justify this equality. If $x \in A_n$, then $d(x, a) < 1/n$ for some $a \in A$, so $x \in B(a, 1/n)$. If $x \in \bigcup_a B(a, 1/n)$, then $d(x, a) < 1/n$ for all $a \in A$, so $d(x, A) < 1/n$.

Now we claim that $A = \cap_{n \in \mathbb{N}} A_n$. Let $a \in A$. Then $B(a, 1/n) \subset A_n$ for all n, so $a \in \cap_n B_(a, 1/n) \subset \cap_n A_n$. Thus $A \subset \cap_n A_n$.

Now suppose that $x \in \cap_n A_n$. Then $d(x, A) < 1/n$ for all n, so $d(x, A) = 0$. Then since A is closed, and $\{x\}$ is compact, by the contrapositive of Lemma 3.1, $\{x\}$ and A are not disjoint. But the only point at which they might intersect is x, hence $x \in A$. Putting this together, we have established that A can be written as a countable intersection of open sets; hence A is a G_δ set.

Now let B be open. Then let $A = \mathbb{R}^d \setminus B$ be the complement. As A is closed, A is a countable intersection of open sets ($A = \cap_n A_n$), as shown above. Then let B_n be the closed set $\mathbb{R}^d \setminus A_n$, and we have

$$B = \mathbb{R}^d \setminus A = \mathbb{R}^d \setminus \bigcap_n A_n = \bigcup_n (\mathbb{R}^d \setminus A_n) = \bigcup_n B_n$$

Thus we have written B as a countable union of closed sets, so B is an F_σ set. □

Proposition 0.2 (Exercise 13b)**.** *There exists a set, namely $\mathbb{Q}$, which is F_σ but not G_δ.*

Proof. Consider the set $\mathbb{Q}$ of rationals. We can enumerate the rationals as $\{q_i\}_{i=1}^\infty$. Then we can write $\mathbb{Q}$ as the countable union of singleton sets:

$$\mathbb{Q} = \bigcup_{i=1}^\infty \{q_i\}$$

Singleton sets are closed, so $\mathbb{Q}$ is an F_σ set.

We claim that $\mathbb{Q}$ is not a G_δ set. Suppose to the contrary that $\mathbb{Q}$ can be written as a countable intersection of open sets, $\mathbb{Q} = \cap_n \mathcal{O}_n$. Then $\mathcal{O}_n$ is an open set containing all rationals, and we know that because $\mathcal{O}_n$ is an open set in $\mathbb{R}$, it can be written as a disjoint union of open intervals,

$$\mathcal{O}_n = (a_1, b_1) \cup (a_2, b_2) \dots$$

where $a_1 \le b_1 \le a_2 \le b_2 \le \dots$. If for any i we have $b_i \neq a_{i+1}$, then there is a rational between b_i and a_{i+1}, but then $\mathcal{O}_n$ would not contain $\mathbb{Q}$. Hence each $b_i = a_{i+1}$. Clearly, we cannot have $b_i \in \mathbb{Q}$, since $\mathbb{Q} \subset \mathcal{O}_n$, so $\mathcal{O}$ is a union of disjoint open intervals covering all of $\mathbb{R}$ except for countably many irrationals. Then $\cap_n \mathcal{O}_n$ countains all but countably many irrationals. Hence $\cap_n \mathcal{O}_n$ is not equal to $\mathbb{Q}$, which contradicts our starting assumption. Hence $\mathbb{Q}$ is not a G_δ set. $\square$

Corollary 0.3 (to Exercise 13b)**.** *The set $\mathbb{R} \setminus \mathbb{Q}$ is not G_δ.*

Proof. If $\mathbb{R} \setminus \mathbb{Q}$ were G_δ, then its complement, $\mathbb{Q}$ would be F_σ, but we just showed that $\mathbb{Q}$ is not F_σ. $\square$

Proposition 0.4 (Exercise 13c)**.** *There exists a set which is a Borel set but not G_δ or F_σ.*

Proof. Let $B = (-\infty, 0) \cap \mathbb{Q}$ and let $C = (0, \infty) \setminus \mathbb{Q}$, and set $A = B \cup C$. We claim that A is neither G_δ nor F_σ. Suppose that A is F_σ, that is, A can be written as the countable union of closed sets, $A = \cup_n A_n$. For each A_n, set

$$A_n^+ = A_n \cap [0, \infty)$$
$$A_n^- = A_n \cap (-\infty, 0]$$

Then we can rewrite A as

$$A = \bigcup_n A_n^+ \cup \bigcup_n A_n^-$$

In particular, since B, C are disjoint, $B = \bigcup_n A_n^-$ and $C = \bigcup_n A_n^+$. But then C is an F_σ set, but C cannot be a F_σ set for the same reasons that $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ. Thus A is not F_σ.

Now suppose that A is G_δ. Then $\mathbb{R} \setminus A$ is F_σ, so $((-\infty, 0) \setminus \mathbb{Q}) \cup ((0, \infty) \cap \mathbb{Q}) \cup \{0\}$ is F_σ. We can then do the same construction as above, supposing $\mathbb{R} \setminus A$ to be a countable union of A_n, and defining A_n^+, A_n^-. Then again by the disjoint-ness, we would have to conclude that $(0, \infty) \cap \mathbb{Q}$ is F_σ, but once again, this set is not F_σ for the same reasons that $\mathbb{R} \setminus \mathbb{Q}$ is not F_σ. Hence A is not G_δ.

We have shown that A is neither F_σ nor G_δ. But we claim that A is a Borel set. As shown in part (b), $\mathbb{Q}$ is F_σ, so likewise B is F_σ. For analogous reasons, $\mathbb{R} \setminus C$ is F_σ, so C is G_δ. Hence A is a union of an F_σ set and a G_δ set. But both the F_σ and G_δ sets are Borel sets, so their union is contained in the Borel sets (as the Borel sets form a σ-algebra). Hence A is a Borel set. $\square$

Lemma 0.5 (for Exercise 14a)**.** *Let X be a topological space with subsets $A_1, \dots A_n$. Then*

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

that is, the closure of a finite union is equal to the union of the closures.

Proof. Suppose $x \in \overline{\bigcup_{i=1}^n A_i}$. Then $x \in \bigcup_i A_i$ or x is a limit point of this union. If x is in this union, then $x \in A_i$ for some i, so then $x \in \overline{A_i}$ and we're done. So suppose x is a limit point of this union. Then there exists a sequence x_n converging to x where each $x_n \in A_i$ for some i. Since there are only finitely many A_i, there must be some A_i containing infinitely many x_n. Choose the subsequence of x_n lying within A_i. Then this subsequence also converges to x, so $x \in \overline{A_i}$. Hence $\overline{\bigcup_{i=1}^n A_i} \subset \bigcup_{i=1}^n \overline{A_i}$.

Now suppose $x \in \bigcup_i \overline{A_i}$. Then $x \in A_i$ for some i or x is a limit point of some A_i. If $x \in A_i$, then x is in the union over all A_i and hence in the closure of that union. If x is a limit point of A_i, then x is also a limit point of the union over all A_i. Hence $\overline{\bigcup_{i=1}^n A_i} \supset \bigcup_{i=1}^n \overline{A_i}$.

We have shown two-way containment of these sets, thus they are equal. □

Proposition 0.6 (Exercise 14a)**.** *The outer Jordan content $J_*(E)$ of a set $E \subset \mathbb{R}$ is defined by*

$$J_*(E) = \inf \left\{ \sum_{j=1}^{N} |I_j| \right\}$$

where I_j are intervals such that $E \subset \bigcup_{j=1}^N I_j$. We claim that $J_*(E) = J_*(\overline{E})$ for $E \subset \mathbb{R}$.

Proof. Every covering of $\overline{E}$ by intervals $\{I_j\}$ is a covering of E, so $J_*(\overline{E})$ is an infimum over a subset of the set over which $J_*(E)$ is an infimum, hence $J_*(E) \leq J_*(\overline{E})$.

Let $\{I_j\}$ be a covering of E by intervals. Then $\overline{E}$ is covered by the intervals $\{\overline{I}_j\}$, since the closure of a finite union is equal to the union of the closures (see above lemma). But $|I_j| = |\overline{I_j}|$, so every covering of E extends to a covering of $\overline{E}$ with the same sum, so $J_*(\overline{E}) \leq J_*(E)$.

Thus we have inequalities going both ways, so the quantities $J_*(E), J_*(\overline{E})$ are equal. □

Proposition 0.7 (Exercise 14b)**.** *There exists a countable subset $E \subset [0,1]$ such that $J_*(E) = 1$ and $m_*(E) = 0$. In particular, $E = \mathbb{Q} \cap [0,1]$.*

Proof. Let $E = \mathbb{Q} \cap [0,1]$. As we know, $m_*(\mathbb{Q}) = 0$ so $m_*(E) = 0$. Suppose that $\{I_j\}$ is a covering of E by finitely many intervals,

$$E = \bigcup_{i=1}^{n} (a_i, b_i)$$

(The intervals might not be open, they might be of the form $[a_i, b_i)$ or something, but this doesn't affect the argument.) This complement of this union must not contain any intervals contained in $[0,1]$, since if there is such an interval, then that interval contains a rational, contradicting the fact that E contains all rationals in $[0,1]$. Thus the intervals covering E must contain something of the form

$$[0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \ldots \cup (a_n, 1]$$

where $a_1, \ldots a_n$ are irrational. But the sum over the lengths of these intervals is one, so for any covering of E by intervals, the sum over the lengths of those intervals is at least one. Hence $J_*(E) \geq 1$. Of course, the single interval $[0,1]$ is a covering for E, so $J_*(E) \leq 1$, which with the previous inequality yields $J_*(E) = 1$. $\square$

Proposition 0.8 (Exercise 15)**.** *Define, for $E \subset \mathbb{R}^d$,*

$$m_*^{\mathcal{R}}(E) = \inf \left\{ \sum_{j=1}^{\infty} |R_j| : E \subset \bigcup_{j=1}^{\infty} R_j \right\}$$

where R_j are closed rectangles. Then for $m_*(E) = m_*^{\mathcal{R}}(E)$ for $E \subset \mathbb{R}^d$.

Proof. Let $E \subset \mathbb{R}^d$. Since every covering of E by closed cubes is also a covering by closed rectangles, it is immediate that $m_*^{\mathcal{R}}(E) \leq m_*(E)$ for all E, since the former is an infimum over a subset of the set over which the latter is an infimum. We just need to show that the opposite inequality also holds.

If $m_*^{\mathcal{R}}(E) = \infty$, then $m_*(E) \leq m_*^{\mathcal{R}}(E)$ holds trivially, so suppose that $m_*^{\mathcal{R}}(E) < \infty$. Let $\epsilon > 0$. Then there exists a covering $\{R_j\}_{j=1}^{\infty}$ of E by closed rectangles such that

$$\sum_{j=1}^{\infty} |R_j| < m_*^{\mathcal{R}}(E) + \epsilon$$

For each rectangle R_j, $m_*(R_j) = |R_j|$, so by definition of m_*, there exists a covering $\{Q_{ij}\}_{i=1}^{\infty}$ of R_j by closed cubes such that

$$\sum_{i=1}^{\infty} |Q_{ij}| < |R_j| + \epsilon/2^j$$

Then $\{Q_{ij}\}$ is a countable covering of E by closed cubes, so

$$m_*(E) \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |Q_{ij}| \leq \sum_{j=1}^{\infty} \left(|R_j| + \epsilon/2^j \right) = m_*^{\mathcal{R}}(E) + \epsilon + \sum_{j=1}^{\infty} \epsilon/2^j = m_*^{\mathcal{R}}(E) + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, we get $m_*(E) \leq m_*^{\mathcal{R}}(E)$, and hence we have equality. $\square$

Proposition 0.9 (Exercise 16, Borel-Cantelli Lemma)**.** *Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^d$ such that*

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Define

$$E = \{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\} = \limsup_{k \to \infty}(E_k)$$

Then E is measurable, and $m(E) = 0$.

Proof. First, we claim that

$$E = \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} E_k \right)$$

Suppose that $x \in E$, that is, $x \in E_k$ for infinitely many k. Then for any $n \in \mathbb{N}$, $x \in \bigcup_{k \geq n} E_k$, hence x is in the intersection over all such sets. For the other inclusion, suppose that $x \in \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$. If x were in only finitely many E_k, then we could take the max over those k and we would have $x \notin \cup_{k \geq N}$ where N is that maximum. But by assumption, x is in every such union, so it must be that x is in infinitely many E_k. Hence $x \in E$. Thus we have two way containment, and hence equality of sets.

Each E_k is measurable by hypothesis, so $\bigcup_{k \geq n} E_k$ is measurable, and hence the countable intersection of such sets is also measurable. Thus E is measurable. Now we claim that $m(E) = 0$.

Let $\epsilon > 0$. Since we have that the series $\sum_{k=1}^{\infty} m(E_k)$ converges and is finite, we can always go out far enough that the tail of the series is less than ϵ. Formally, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} m(E_k) < \epsilon$$

Using our established expression for E, we have the inclusion

$$E = \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} E_k \right) \subset \bigcup_{k \geq N} E_k$$

And then by monotonicity of measure,

$$m(E) \leq m \left(\bigcup_{k \geq N} E_k \right) \leq \sum_{k=N}^{\infty} m(E_k) < \epsilon$$

Thus $m(E) < \epsilon$ for every $\epsilon > 0$, hence $m(E) = 0$. □

Proposition 0.10 (Exercise 17). *Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. x. Then there exists a sequence c_n of positive real numbers such that*

$$\frac{f_n(x)}{c_n} \to 0 \text{ a.e. } x$$

Proof. For any sequence c_n of positive reals, let E_n be the sequence of sets given by

$$E_n = \left\{ x : \left| \frac{f_n(x)}{c_n} \right| > \frac{1}{n} \right\}$$

We claim that for every n there exists $c_n \in \mathbb{R}$ such that

$$m_*(E_n) < 2^{-n}$$

Suppose no such c_n exists. Then for every $c_n \in \mathbb{R}^+$, we have

$$m_* \left(\left\{ x : |f_n(x)| > \frac{c_n}{n} \right\} \right) \geq 2^{-n}$$

In particular, for $c_n = n, 2n, 3n, \ldots$ we have

$$\{x : |f_n(x)| > 1\} \supset \{x : |f_n(x)| > 2\} \supset \{x : |f_n(x)| > 3\} \supset \ldots$$
$$2^{-n} \leq m(\{x : |f_n(x)| > 1\}) \leq m(\{x : |f_n(x)| > 2\}) \leq m(\{x : |f_n(x)| > 3\}) \leq \ldots$$

Then if we take the intersection over all such sets, the measure is still at least 2^{-n}, that is,

$$2^{-n} \leq m \left(\bigcap_{k=1}^{\infty} \{x : |f_n(x)| > k\} \right)$$

But this intersection is $\{x : |f(x)| = \infty\}$, which by hypothesis has measure zero. Hence there must be a $c_n \in \mathbb{R}$ such that

$$m(E_n) < 2^{-n}$$

(Specifically, we just showed that we can take c_n to be a positive integer multiple of n.) Now we consider the set

$$E = \{x : x \in E_k \text{ for infinitely many } k\}$$

Suppose $x \notin E$. Then x is in at most finitely many E_k, that is,

$$\left| \frac{f_k(x)}{c_k} \right| \leq \frac{1}{k}$$

for infinitely many k. Hence for $x \in [0, 1] \setminus E$, we have

$$\lim_{n \to \infty} \frac{f_n(x)}{c_n} = 0$$

By the Borel-Cantelli lemma, since we showed that $\sum_{n=1}^{\infty} m(E_n) \leq \sum_n 2^{-n} < 1 < \infty$, we have $m(E) = 0$. Hence

$$\lim_{n \to \infty} \frac{f_n(x)}{c_n} = 0 \text{ a.e. } x$$

□

Proposition 0.11 (Exercise 18)**.** *Every measurable function is the limit a.e. of a sequence of continuous functions.*

Proof. First we prove this in the case that f is finite-valued. Let $f : A \to \mathbb{R}$ be a measurable function (where $A \subset \mathbb{R}^d$ is measurable). By Theorem 4.3, there exists a sequence of step functions ψ_k such that $\lim_k \psi_k(x) = f(x)$ for almost every x, that is, there exists $B \subset A$ such that $\psi_k(x) \to f(x)$ on B and $m(A \setminus B) = 0$.

Let $E_n = B(0,n) \cap A$ for $n \in \mathbb{N}$. Then ψ_k is measurable and finite-valued on E_n so by Lusin's theorem there exists a closed set F_n such that $F_n \subset E_n$ and $m(E_n \setminus F_n) \leq 2^{-n}$, and $\psi_k|_{F_n}$ is continuous. By the Tietze extension theorem, there exists a continuous function $f_n : A \to \mathbb{R}$ such that $f_n(a) = f(a)$ for $a \in F_n$. So we have $f_n(x) \to f(x)$ for $x \in F_n$.

Suppose that f_n does not converge to f almost everywhere, in particular, suppose that $f_n(x)$ does not converge to $f(x)$ for some $x \in A$. Then $x \in A \setminus F_n$ for infinitely many n. We know that

$$A \setminus F_n \subset \big((E_n \setminus F_n) \cup (\mathbb{R}^d \setminus B(0,n))\big)$$

and since $x \in \mathbb{R}^d \setminus B(0,n)$ for only finitely many n, we have $x \in E_n \setminus F_n$ for infinitely many n. Since $\sum_{n=1}^{\infty} m(E_n \setminus F_n) \leq 1 < \infty$, by the Borel-Cantelli lemma (Exercise 16 above), we have

$$m\left(\{x : x \in E_k \setminus F_k \text{ for infinitely many } k\}\right) = 0$$

But

$$\begin{aligned}\{x : x \in E_k \setminus F_k \text{ for infinitely many } k\} &= \bigcup_n \{x \in A \setminus F_n\} \\ &= \{x \in A : f_n(x) \text{ does not converge to } f(x)\}\end{aligned}$$

hence the set of non-convergence of f_n to f has measure zero, so $f_n \to f$ a.e.

Now suppose that $f : A \to [-\infty, \infty]$ is a measurable function. Define a function $g_n : A \to [-\infty, \infty]$ by

$$g_n(x) = \begin{cases} f(x) & f(x) < \infty \\ n & f(x) = \infty \\ -n & f(x) = -\infty \end{cases}$$

The sequence of functions g_n converges pointwise to f on A. Furthermore, each g_n is measurable and finite-valued, so by the above there exist continuous functions h_{nk} such that $\lim_k h_{nk}(x) = g_n(x)$ a.e. We claim that

$$\lim_{n\to\infty} h_{nn}(x) = f(x) \text{ a.e.}$$

for all $x \in A$. Let $\epsilon > 0$. If $f(x)$ is finite, then there exists N such that for all $n \in \mathbb{N}$,

$$k \geq N \implies |g_n(x) - h_{nk}(x)| = |f(x) - h_{nk}(x)| < \epsilon \text{ (this holds for } x \text{ a.e.)}$$

Since this holds for all $n \in \mathbb{N}$, it holds in particular for $n \geq N$. Thus

$$k \geq N \implies |f(x) - h_{kk}(x)| < \epsilon \text{ (a.e.)}$$

and hence $h_{nn}(x) \to f(x)$ a.e. for $f(x)$ finite. Now suppose that $f(x) = \infty$ and let $M > 0$. Then there exist N_1, N_2 such that

$$\begin{aligned} n \geq N_1 &\implies g_n(x) > M \\ k \geq N_2 &\implies |g_n(x) - h_{nk}(x)| < 1 \text{ (a.e.)} \end{aligned}$$

Then let $N = \max(N_1, N_2)$. So then

$$n \geq N \implies h_{nn}(x) > M - 1 \text{ (a.e.)}$$

and hence $h_{nn}(x) \to \infty = f(x)$ a.e.. A symmetric argument shows that $h_n n(x) \to -\infty$ a.e. if $f(x) = -\infty$. Thus the sequence of continuous functions h_{nn} converges pointwise to f for almost every $x \in A$, so every measurable function is the limit a.e. of continuous functions. □

Proposition 0.12 (Exercise 19a). *Let $A, B \subset \mathbb{R}^d$ such that either A or B is open. Then $A + B$ is open.*

Proof. Let $x \in A + B$. Then there exist $a \in A, b \in B$ such that $a + b = x$. Since A is open, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. Then $b + B(a, \epsilon) \subset A + B$. Note that $b + B(a, \epsilon) = B(b + a, \epsilon)$, since translating the center of a ball is the same as translating the ball by the same amount. Hence $B(a + b, \epsilon) = B(x, \epsilon) \subset A + B$, so $A + B$ is open. By interchanging the labels for A and B, we can see that if B is open the $A + B$ must be open. □

Lemma 0.13 (for Exercise 19b). *If $A, B \subset \mathbb{R}^d$ are compact, then $A + B$ is compact.*

Proof. Let x_n be a sequence in $A + B$. Then for each n, there exist $a_n \in A, b_n \in B$ such that $x_n = a_n + b_n$. Since A, B are compact, they are sequentially compact, thus a_n, b_n have convergent subsequences a_{n_k}, b_{n_k}. Then x_{n_k} is a convergent subsequence of x_n. Hence $A + B$ is sequentially compact, so it is compact. □

Lemma 0.14 (for Exercise 19b). *Every closed set in $\mathbb{R}^d$ can be written as a countable union of compact sets.*

Proof. Let $A \subset \mathbb{R}^d$ be closed. Then $A \cap \overline{B}(0, n)$ is compact, and $A = \bigcup_{n=1}^{\infty} (A \cap \overline{B}(0, n)$. □

Proposition 0.15 (Exercise 19b). *Let $A, B \subset \mathbb{R}^d$ be closed. Then $A + B$ is measurable.*

Proof. We will show that $A + B$ is F_σ. Each A, B is closed, so they can be written as a countable union of compact sets, $A = \bigcup_i A_i, B = \bigcup_j B_j$. Then

$$A + B = \bigcup_{i,j} (A_i + B_j)$$

By a previous lemma, since A_i, B_j are compact, $A_i + B_j$ is compact. Hence we have written $A + B$ as a countable union of closed sets, so $A + B$ is F_σ. Thus $A + B$ is measurable. □

Proposition 0.16 (Exercise 19c). *There exist closed sets A, B such that $A + B$ is not closed.*

Proof. Let

$$A = \{2 + 1/2, 3 + 1/3, 4 + 1/4, \ldots\} = \{n + 1/n : n = 2, 3, \ldots\}$$
$$B = \{-1, -2, -3, -4, \ldots\} = \{-n : n \in \mathbb{N}\}$$

Then A and B are both closed. But $A + B$ is not closed, since it does not contain all of its limit points. In particular, the sequence

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$

is in $A + B$. However, its limit (zero) is not in $A + B$, since there are no positive integers in A. □

Proposition 0.17 (Exercise 20a)**.** *There exist closed sets $A, B \subset \mathbb{R}$ such that $m(A) = m(B) = 0$ and $m(A + B) > 0$.*

Proof. Let $A = \mathcal{C}$ be the middle thirds Cantor set, and let $B = (1/2)\mathcal{C}$ (the dilation of the Cantor set by 1/2). Let $x \in [0, 1]$. Then x has a ternary representation,

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where each $a_k \in \{0, 1, 2\}$. We form a partition of K_0, K_1, K_2 of $\mathbb{N}$ where $K_i = \{k \in \mathbb{N} : a_k = i\}$. Since the infinite sum for x converges and the terms are positive, we can rearrange terms to rewrite it as

$$\sum_{k=1}^{\infty} a_k 3^{-k} = \sum_{k \in K_0} (0)3^{-k} + \sum_{k \in K_1} (1)3^{-k} + \sum_{k \in K_2} (2)3^{-k} = \sum_{k \in K_1} 3^{-k} + \sum_{k \in K_2} (2)3^{-k}$$

Then notice that $\sum_{k \in K_1} (2)3^{-k} \in \mathcal{C}$ by Exercise 4a, and $\sum_{k \in K_2} 3^{-k} \in (1/2)\mathcal{C}$ (since multiplying it by 2 puts it in $\mathcal{C}$ for the same reason). Hence x is a sum of something in A and B, so $x \in A + B$. Thus $[0, 1] \subset A + B$. Then by monotonicity, $m(A + B) \geq 1$. However, as we know, $m(A) = m(B) = 0$. □

Proposition 0.18 (Exercise 20b)**.** *There exist closed sets $A, B \subset \mathbb{R}^2$ such that $m(A) = m(B) = 0$ and $m(A + B) > 0$.*

Proof. Let $A = [0, 1] \times \{0\}$ and $B = \{0\} \times [0, 1]$. Then $A + B = [0, 1] \times [0, 1]$ since any $(x, y) \in [0, 1] \times [0, 1]$ is equal to $(x, 0) + (0, y)$. Hence $m(A + B) = 1$, but $m(A) = m(B) = 0$. □