

NEUMANN HOMOGENIZATION VIA INTEGRO-DIFFERENTIAL OPERATORS, PART 2: SINGULAR GRADIENT DEPENDENCE

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ABSTRACT. We continue the program initiated in a previous work, of applying integro-differential methods to Neumann Homogenization problems. We target the case of linear periodic equations with a singular drift, which includes (with some regularity assumptions) divergence equations with *non-co-normal* oscillatory Neumann conditions. Our analysis focuses on an induced integro-differential homogenization problem on the boundary of the domain. Also, we use homogenization results for regular Dirichlet problems to build barriers for the oscillatory Neumann problem with the singular gradient term. We note that our method allows to recast some existing results for fully nonlinear Neumann homogenization into this same framework. This version is the “director’s cut”.

1. INTRODUCTION

In this paper, we study the periodic homogenization of linear elliptic equations with a singular drift with oscillatory Neumann conditions. In particular, we study the $\varepsilon \rightarrow 0$ behavior of the solutions, u^ε , to the equation

$$\begin{cases} \operatorname{Tr}(A(\frac{x}{\varepsilon})D^2u^\varepsilon(x)) + \frac{1}{\varepsilon}B(\frac{x}{\varepsilon}) \cdot \nabla u^\varepsilon = 0 & \text{in } \Sigma^1 \\ u^\varepsilon = 0 & \text{on } \Sigma_1 \\ \partial_n u^\varepsilon(x) = g(\frac{x}{\varepsilon}) & \text{on } \Sigma_0. \end{cases} \quad (1.1)$$

The equation is posed in an infinite strip domain, with a normal vector given by n , which is

$$\Sigma^1 \subset \mathbb{R}^{d+1}, \quad \Sigma^1 = \{x \in \mathbb{R}^{d+1} : 0 < x \cdot n < 1\}.$$

The “top” and “bottom” boundaries are respectively

$$\begin{aligned} \Sigma_1 &\subset \mathbb{R}^{d+1}, \quad \Sigma_1 = \{x \in \mathbb{R}^{d+1} : x \cdot n = 1\}, \\ \Sigma_0 &\subset \mathbb{R}^{d+1}, \quad \Sigma_0 = \{x \in \mathbb{R}^{d+1} : x \cdot n = 0\}. \end{aligned}$$

The main goal is to prove that the “nonlinear averaging” effects of the equation in the interior and on the boundary are “compatible”, and that u^ε will converge uniformly to \bar{u} ,

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This is the “director’s cut” of the paper, and it contains many extra details for our own sanity. The work of N. Guillen was partially supported by NSF DMS-1201413. We would like to thank Zhongwei Shen for suggesting we try our method from [20] for the oblique derivative, divergence equation.

that is the unique solution of

$$\begin{cases} \operatorname{Tr}(\bar{A}D^2\bar{u}(x)) = 0 & \text{in } \Sigma^1 \\ \bar{u} = 0 & \text{on } \Sigma_1 \\ \partial_n\bar{u}(x) = \bar{g} & \text{on } \Sigma_0. \end{cases} \quad (1.2)$$

The matrix \bar{A} is a unique constant, and it is the same one obtained in the homogenization of (1.1) without oscillatory boundary data (see Section A.3 and A.7). Due to the flat geometry of Σ^1 , \bar{g} will also be a unique constant (but in more general situations would be a function of the normal vector, $n(x)$). Thus, this is simply another way of saying that there is a unique affine function, \bar{u} , such that u^ε converges to \bar{u} uniformly in $\bar{\Sigma}^1$. We note, that for the homogenization of equations of the form (1.1), assumption 2.4 involves a compatibility condition between $A(x)$ and $B(x)$ – which is standard, and this condition also forces the limit equation, (1.2), to be independent of $\nabla\bar{u}$. The compatibility condition is trivially satisfied when $B = \operatorname{div}(A)$. Our main result is:

Theorem 1.1. *Assume that $g : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$, $B : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$, $A : \mathbb{R}^{d+1} \rightarrow \mathcal{S}(d+1)$ are all \mathbb{Z}^{d+1} periodic functions, $A \in C^\gamma(\mathbb{R}^{d+1})$ is uniformly elliptic, $B \in C^\gamma(\mathbb{R}^{d+1})$, $g \in C^\gamma(\mathbb{R}^{d+1})$ for some $\gamma \in (0, 1)$, that n is an irrational direction, and the standard compatibility condition between A and B , all of which appear in detail as Assumptions 2.1-2.4. There exists a unique constant matrix, \bar{A} , and a unique constant, \bar{g} , such that $u^\varepsilon \rightarrow \bar{u}$, uniformly in $\bar{\Sigma}^1$, where \bar{u} is the unique affine function that solves (1.2).*

Remark 1.2. *When $A \in C^{1,\gamma}$ and $B = \operatorname{div}(A)$, Theorem 1.1 includes that case of solving $\operatorname{div}(A(x/\varepsilon))\nabla u^\varepsilon = 0$ with $\partial_n u^\varepsilon(x) = g(x/\varepsilon)$. The Neumann condition is oblique with respect to the operator $\operatorname{div}A(y)\nabla u$ as opposed to the more frequently studied co-normal condition, $(n, A(x/\varepsilon)\nabla u) = g(x/\varepsilon)$. This means that variational methods are not appropriate for (1.1), and so we appeal to those based on comparison principles and other non-divergence type techniques.*

Remark 1.3. *In Section 7 we mention how a modification of Theorem 1.1 and its proof give a new proof of a similar result for nonlinear equations without gradient dependence that was proved in Choi-Kim [9].*

Our treatment of Theorem 1.1 follows that of our earlier result in [20], and the main idea is to recast (1.1) as a global, interior, almost periodic integro-differential homogenization problem that takes place on Σ_0 only. The simple geometry of Σ^1 , although particular, is now known to be the most important one for resolving solutions of (1.1) in more general domains. The issues of resolving the homogenization of (1.1) for all irrational directions, and proving homogenization in general domains are separate ones. In Σ^1 , (1.1) can be thought of as a fundamental corrector problem, whereas studying (1.1) in general domains relies upon how the effective normal condition depends upon the normal direction, n , and is of a different nature than the corrector problem.

The *interior* homogenization of (1.1) with regular boundary conditions that don't depend on ε is well understood, and \bar{A} is determined by the same analysis that does not see the oscillatory boundary condition (see Section A.3). The new part of Theorem 1.1 is of course the ability to also have the oscillatory condition, $\partial_n u^\varepsilon = g(x/\varepsilon)$.

2. BACKGROUND AND ASSUMPTIONS

2.1. Background. We note that despite the many existing results about boundary homogenization (some listed below) to the best of our knowledge, there are none that can treat (1.1) when n is irrational. The field of homogenization of elliptic, parabolic, and Hamilton-Jacobi equations in the periodic, almost periodic, and random settings is by now an enormous field with contributions from many authors. The background for general homogenization is not presented here, and we simply try to give the references for oscillatory *boundary* problems. The interested reader can consult the books of Bensoussan-Lions-Papanicolaou [6] and Jikov-Kozlov-Oleinik [22] for an introduction to the subject and the survey of Engquist-Souganidis [12] for a somewhat current list of results. Since [12], there has been a large amount of activity, especially in the case of random coefficients and rates of convergence, which is not represented in [12]. Due to the irrationality of n , the problem we study is related to that of Ishii [21] for almost periodic Hamilton-Jacobi equations.

Two origins of (1.1). An equation with oscillatory Neumann data, as in (1.1), naturally arises in the study of a diffusion in an oscillatory environment with a prescribed reflection at the boundary whose strength is dictated by $g(x/\varepsilon)$. Studying this equation with a regular boundary condition (no x/ε dependence) in the periodic setting goes back at least to Freidlin [17] and Bensoussan-Lions-Papanicolaou [6]. Probabilistically, the oscillatory reflection is a natural consideration, for example in Tanaka [29]. Another place where the condition $\partial_n u^\varepsilon = g(x/\varepsilon)$ arises is in the study of boundary layers for (1.1) with *regular* boundary data. In this case, when one uses a corrector to expand the ε behavior of u^ε around the smooth, effective solution, \bar{u} , they must investigate $u^\varepsilon - \bar{u} - \varepsilon u^1$, where vaguely u^1 is a first corrector. However, this function no longer satisfies the original boundary conditions of u^ε and \bar{u} , and in general it will be a function of x/ε on Σ_0 . Hence, one is forced to consider oscillatory terms on Σ_0 , as in (1.1). For the Dirichlet problem, the appearance of the oscillatory boundary term due to the use of a corrector in the boundary layer expansion was already present in [6, Chp. 3, Sec. 5] and Avellaneda-Lin [2, Sec. 3.2], although in [6] the precise boundary behavior was not under consideration, and the oscillatory terms were not further studied because they were of lower order.

The rational case. The earliest results for linear non-divergence equations with oscillatory oblique Neumann data appear to go back to Tanaka [29], where the boundary contains a subspace of the \mathbb{Z}^{d+1} lattice, and the Neumann data is periodic with respect to this subspace. Next came the works of Arisawa [1] and subsequently Barles-Da Lio-Lions-Souganidis [4]. These works treat the oscillatory Neumann (and more general) boundary conditions in a situation where effectively the hyperplane, Σ_0 , would share a periodic sublattice with \mathbb{Z}^{d+1} . Basically, the corresponding results in those papers would match up with a choice of n as a rational vector in (1.1). The work [4] also treats fully nonlinear versions of (1.1) in this periodic, or rational, situation. Both [4] and [29] include the singular gradient term that appears in (1.1). This common periodicity between the boundary condition and the interior equation seems to be crucial in those works, as they solve coupled corrector (or “ergodic”) problems with both the interior and boundary simultaneously. Our method, as outlined in Section 3 effectively de-couples these two equations and searches for an effective limit on the boundary of Σ^1 by itself, which subsequently determines \bar{g} a posteriori. In this same shared interior/boundary periodicity set-up, with a rational choice of n , some problems of Dirichlet homogenization were studied in Barles-Mironescu [5].

The co-normal case. In the case of divergence equations, with an oscillatory *co-normal* condition, $(n, A(x/\varepsilon)\nabla u^\varepsilon) = g(x/\varepsilon)$, (1.1) was treated in the classical work of Bensoussan-Lions-Papanicolaou [6, Chp. 1 Sec. 7.3]. It in fact follows from the divergence structure of the equation, exactly as in the same way as the problem with regular boundary conditions. There, they identify the difference between the rational and irrational cases of n , in domains that are more general than Σ^1 . The problem becomes more challenging when one wishes to know more precise information about the boundary layer of u^ε near Σ_0 . In that case, the analysis is much more delicate, and some works on this boundary homogenization structure in the Dirichlet and *co-normal* Neumann setting for elliptic systems include Avellaneda-Lin [2], Gérard-Varet-Masmoudi [18] and Kenig-Lin-Shen [23], [24], where estimates for the boundary layer are determined.

The gradient independent case. Recently, the boundary homogenization for both cases of oscillatory Dirichlet and Neumann conditions with fully nonlinear uniformly elliptic equations in the interior (with x/ε dependence) has been solved in many situations, including some general domains. All of the following works treat problems with purely second order equations in the interior, hence the presence of the term $(1/\varepsilon)B(x/\varepsilon)$ in (1.1) is one aspect that separates this work from the following (also, they treat nonlinear equations, while we work with linear ones). The work of Choi-Kim-Lee [8] solved the problem in a strip domain, as Σ^1 , with a translation invariant nonlinear equation and oscillatory Neumann condition (subsequently the authors, in [20], gave a different proof along the lines presented in this work). Then Choi-Kim [9] treated the problem in more general domains with also an oscillatory equation, $F(D^2u^\varepsilon, x/\varepsilon) = 0$, in the interior. In [9] it became clear that equations in the simple domain, like (1.1), can serve as a sort of corrector problem for the boundary behavior of u^ε in more general domains, and they also noted there are two separate questions that must be addressed: the solvability of the corrector problem, such as Theorem 1.1, and the continuity properties of \bar{g} as a function of the normal $n(x)$, for x in the boundary. The related Dirichlet homogenization in general domains was solved by Feldman [14], where crucially it was noted that there may possibly be instances in which the effective boundary condition will be discontinuous, but nonetheless one can control the size of the set of discontinuities in a way that still guarantees the effective Dirichlet problem admits unique solutions. The issue of continuity / discontinuity of the effective boundary conditions has been further studied by Feldman-Kim [15]. Finally, the homogenization of random oscillatory boundary data with a translation invariant uniformly elliptic equation in the interior was recently obtained for some types of random environments by Feldman-Kim-Souganidis [16].

As one final note, we mention that in Bal-Jing [3], a Dirichlet-to-Neumann mapping was utilized— as we do here— for an equation that is translation invariant in the interior and has a Robin boundary condition with a random, stationary ergodic term.

2.2. Assumptions.

Assumption 2.1. $A : \mathbb{R}^{d+1} \rightarrow \mathcal{S}(d+1)$ (symmetric $(d+1) \times (d+1)$ matrices) and for some $\lambda < \Lambda$, for all x , $\lambda \text{Id} \leq A(x) \leq \Lambda \text{Id}$. $A \in C^\gamma(\mathbb{R}^{d+1})$ and is \mathbb{Z}^{d+1} periodic.

Assumption 2.2. $B : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$, $B \in C^\gamma(\mathbb{R}^{d+1})$, and B is \mathbb{Z}^{d+1} periodic.

Remark 2.3. Assume that $Lu(y) = A_{ij}(y)u_{y_i y_j}(y) + B(y) \cdot \nabla u(y)$ and that $L^*u = (A_{ij}(\cdot)u)_{y_i y_j} - \text{div}(uB(\cdot))$ is the formal adjoint of L . Under Assumptions 2.1 and 2.2, there exists a unique

invariant measure, $m \in L^2([0, 1]^{d+1})$, which is periodic and solves

$$L^*m = 0,$$

subject to $m > 0$ and

$$\int_{[0,1]^{d+1}} m(y)dy = 1.$$

(We mean that

$$\int m(y)L\phi(y)dy = 0$$

for all periodic $\phi \in C^2 \cap L^\infty$.) Some details of m are briefly expanded upon in the appendix.

Assumption 2.4. B must satisfy the compatibility (or “centering”) condition that

$$\int_{[0,1]^{d+1}} B(y)m(y)dy = 0.$$

It prevents trajectories of the related rescaled stochastic process from “escaping to infinity”, and gives compactness of the measures for the law of the processes.

The main point of our method is that we study an almost periodic homogenization problem on Σ_0 . We use the following definition for almost periodic functions.

Definition 2.5. $f : \Sigma_0 \rightarrow \mathbb{R}$ is almost periodic if for any $\delta > 0$, the set of δ -almost periods of f ,

$$E_\delta := \{\tau \in \Sigma_0 : \sup_{x \in \Sigma_0} |f(x + \tau) - f(x)| < \delta\},$$

satisfies the following property: there exists a compact set, $K \subset \Sigma_0$, such that

$$(z + K) \cap E_\delta \neq \emptyset \text{ for all } z \in \Sigma_0.$$

Remark 2.6. [27, Proposition 1.2] Two other equivalent and classical formulations are that $f : \Sigma_0 \rightarrow \mathbb{R}$ is almost periodic if

- (i) f can be uniformly approximated on Σ_0 by trigonometric polynomials.
- (ii) The set $\{f(\cdot + z) : z \in \Sigma_0\}$ is precompact in the space $L^\infty(\Sigma_0)$.

2.3. Notation.

- We will assume that $x \in \mathbb{R}^{d+1}$ is written in the coordinates relative to Σ_0 , that is $x = (\hat{x}, x_{d+1})$, where $\hat{x} \in \Sigma_0$ and $x_{d+1} \in \text{span}(n)$.
- $\Sigma^r = \{x \in \mathbb{R}^{d+1} : 0 < x \cdot n < r\}$
- $\Sigma_r = \{x \in \mathbb{R}^{d+1} : x \cdot n = r\}$
- $\Sigma_0 = \{x \in \mathbb{R}^{d+1} : x \cdot n = 0\}$
- typically, x , is the original macroscopic variable, as in (1.1)
- typically, y , is the microscopic variable, as in (3.2), which results from a rescaling such as $v(y) = (1/\varepsilon)u^\varepsilon(\varepsilon y)$.
- $Lu(y) = A_{ij}(y)u_{y_i y_j}(y) + B(y) \cdot \nabla u(y)$
- $B_R \subset \mathbb{R}^{d+1}$, $B'_R \subset \Sigma_0$, $B_R^+ = B_R \cap \Sigma^r$ for $r > R$.

3. MAIN IDEAS OF THE PROOF

In this section, we give the sketch of the proof, without explicit justifications. The details appear in Sections 4, 5, 6. Our strategy in this approach centers on the Dirichlet-to-Neumann (D-to-N) operator at two different scales. First is the original scale of the macroscopic variables, with W_u^ε the unique (classical) solution of

$$\begin{cases} \operatorname{Tr}(A(\frac{x}{\varepsilon})D^2W_u^\varepsilon) + \frac{1}{\varepsilon}B(\frac{x}{\varepsilon}) \cdot \nabla W_u^\varepsilon = 0 & \text{in } \Sigma^1 \\ W_u^\varepsilon = 0 & \text{on } \Sigma_1 \\ W_u^\varepsilon = u & \text{on } \Sigma_0, \end{cases} \quad (3.1)$$

and we define

$$\mathcal{I}^1 : C^{1,\gamma}(\Sigma_0) \rightarrow C^{\tilde{\gamma}}(\Sigma_0)$$

via

$$\mathcal{I}^1(u, x) = \partial_n W_u^\varepsilon(x).$$

Similarly, we will also need the D-to-N mapping for the equation in microscopic variables. This changes the domain of definition of the equation, to the larger set, $\Sigma^{1/\varepsilon}$:

$$\begin{cases} \operatorname{Tr}(A(y)D^2V_u^\varepsilon) + B(y) \cdot \nabla V_u^\varepsilon = 0 & \text{in } \Sigma^{1/\varepsilon} \\ V_u^\varepsilon = 0 & \text{on } \Sigma_{1/\varepsilon} \\ V_u^\varepsilon = u & \text{on } \Sigma_0. \end{cases} \quad (3.2)$$

We define

$$\mathcal{I}^{1/\varepsilon} : C^{1,\gamma}(\Sigma_0) \rightarrow C^{\tilde{\gamma}}(\Sigma_0)$$

via

$$\mathcal{I}^{1/\varepsilon}(u, y) = \partial_n V_u^\varepsilon(y). \quad (3.3)$$

We note that $u \in C^{1,\gamma}(\Sigma_0)$, implies that $V_u^\varepsilon, W_u^\varepsilon \in C^{1,\gamma}(\overline{\Sigma^1})$, which makes \mathcal{I}^1 and $\mathcal{I}^{1/\varepsilon}$ well defined.

Many subsequent arguments will invoke various regularity estimates involving u^ε and $g(\cdot/\varepsilon)$. However, these will not be useful unless it can first be shown that $\|u^\varepsilon\|_{L^\infty} \leq C$, independently from ε . Without assumptions on B , such as the ‘‘centering condition’’ in Assumption 2.2, such estimates are false. However, it turns out that one can use the interior homogenization for an auxiliary barrier equation to derive the needed control on $\|u^\varepsilon\|_{L^\infty}$. Thus, since $g \in C^\gamma(\Sigma_0)$, standard regularity theory (see Proposition A.2 as well as the L^∞ estimate in Section 4.2) indicates that $u^\varepsilon \in C^{1,\gamma}(\overline{\Sigma^1})$ with $[u^\varepsilon]_{C^\gamma}$ independent of ε and $[\nabla u^\varepsilon]_{C^\gamma}$ depending on ε . Thus, this says that up to subsequences, u^ε will have local uniform limits in $\overline{\Sigma^1}$. The main question is, can we characterize all possible such limits?

For the moment, *assume* that there is a *unique* $\bar{v} \in C^\gamma(\Sigma_0)$ such that $u^\varepsilon|_{\Sigma_0} \rightarrow \bar{v}$. Letting \bar{u} be any possible local uniform limit of u^ε in Σ^1 , we know by the perturbed test function method for viscosity solutions of (1.1), that \bar{u} must solve (see Proposition A.8)

$$\operatorname{Tr}(\bar{A}D^2\bar{u}) = 0 \text{ in } \Sigma^1,$$

where \bar{A} is the unique constant matrix defined (A.7). Hence, by the global C^γ continuity of \bar{u} and the *assumed* convergence of $u^\varepsilon|_{\Sigma_0} \rightarrow \bar{v}$ for a unique \bar{v} , we see that \bar{u} is the unique solution of

$$\begin{cases} \operatorname{Tr}(\bar{A}D^2\bar{u}) = 0 & \text{in } \Sigma^1, \\ \bar{u} = 0 & \text{on } \Sigma_1 \\ \bar{u} = \bar{v} & \text{on } \Sigma_0. \end{cases}$$

Thus, assuming $\bar{v} \in C^{1,\gamma}(\Sigma_0)$, then $\bar{u} \in C^{1,\gamma}(\bar{\Sigma}^1)$. We obtain, a posteriori that the unique effective Neumann condition is then

$$\bar{g} = \partial_n \bar{u}.$$

(We note that in the flat geometry of Σ^1 , it will be that \bar{g} is a constant.)

Now, why will it be true that there is a unique \bar{v} , such that $u^\varepsilon|_{\Sigma_0} \rightarrow \bar{v}$? Because $u^\varepsilon \in C^{1,\gamma}$, the equation (1.1) implies that $u^\varepsilon|_{\Sigma_0}$ is the unique classical solution of

$$\mathcal{I}^1(u^\varepsilon|_{\Sigma_0}, x) = g\left(\frac{x}{\varepsilon}\right). \quad (3.4)$$

However, it is well known that \mathcal{I}^1 , as an operator with the global comparison principle, is an integro-differential operator on functions on Σ_0 , [10]. This, combined with the fact that we have periodic ingredients restricted to the irrational hyperplane, Σ_0 , puts (3.4) into the realm of almost periodic integro-differential homogenization on Σ_0 , which we can view in a similar light to [21]. The main contribution of this work is the analysis of (3.4), and the main result is

Proposition 3.1. *There exists a unique constant, \bar{c} , such that $u^\varepsilon|_{\Sigma_0} \rightarrow \bar{c}$ uniformly in Σ_0 .*

The majority of this paper is focused on establishing Proposition 3.1. We note that an interesting aspect of our work is the need to establish estimates on $\|u^\varepsilon\|_{L^\infty}$ and $\|\partial_n u^\varepsilon\|_{L^\infty}$, which begins in the next section.

4. THE CONSTRUCTION OF BARRIERS AND A BOUND FOR $\|u^\varepsilon\|_{L^\infty}$

This section is dedicated to a deceptively simple result; the construction of barriers, contained in Proposition 4.1, below. The interesting part is that when $B \equiv 0$ in (1.1), even for fully nonlinear equations, this proposition is trivial from the observation that affine functions solve the equation in Σ^1 (more on this in Section 7). Even more intriguing is that the outcome of Proposition 4.1 is false, unless there is homogenization occurring in the interior of Σ^1 . In fact, as the reader will see, the result utilizes the convenient result that the interior homogenization of the regular Dirichlet problem enjoys a *global* rate up to the boundary of Σ^1 ; this was proved in the original work of Bensoussan-Lions-Papanicolaou for the non-divergence equation, (1.1), [6, Chp. 3, Sec 5, Thm. 5.1] and in the divergence setting by Avellaneda-Lin [2]. We are not certain if such heavy machinery is necessary, but it sufficed for the present investigation.

We then use the barriers to obtain the crucial estimate that is needed to begin the homogenization procedure, namely that

$$\|u^\varepsilon\|_{L^\infty(\Sigma^1)} \leq C, \text{ uniformly in } \varepsilon.$$

It is interesting in this case, that the ‘‘centering’’ assumption on B and the homogenization for regular boundary conditions are used to get and estimate for $\|u^\varepsilon\|_{L^\infty}$. This seems to not be an issue in existing works.

4.1. The barrier.

Proposition 4.1 (Barrier). *There exist (universal) constants, c_1 and c_2 , $0 < c_1 < c_2$, such that*

$$-c_2 \leq \partial_n \phi^\varepsilon \leq -c_1 < 0, \quad (4.1)$$

where ϕ^ε is the unique solution of the problem

$$\begin{cases} \operatorname{Tr}(A(y)D^2\phi^\varepsilon(y)) + B(y) \cdot \nabla\phi^\varepsilon(y) = 0 & \text{in } \Sigma^{1/\varepsilon} \\ \phi^\varepsilon = 0 & \text{on } \Sigma_{1/\varepsilon} \\ \phi^\varepsilon = 1/\varepsilon & \text{on } \Sigma_0. \end{cases} \quad (4.2)$$

An immediate consequence of the barrier behavior in Proposition 4.1 is

Proposition 4.2 (Estimates for The Neumann Problem). *If for some $c \geq 0$, η^ε solves*

$$\begin{cases} \operatorname{Tr}(A(y)D^2\eta^\varepsilon(y)) + B(y) \cdot \nabla\eta^\varepsilon(y) = 0 & \text{in } \Sigma^{1/\varepsilon} \\ |\eta^\varepsilon| \leq c & \text{on } \Sigma_{1/\varepsilon} \\ |\partial_n \eta^\varepsilon| \leq c & \text{on } \Sigma_0, \end{cases} \quad (4.3)$$

then it holds that

$$\|\eta^\varepsilon\|_{L^\infty(\overline{\Sigma^{1/\varepsilon}})} \leq C \frac{c}{\varepsilon}.$$

First we will prove the estimates for the Neumann problem, Proposition 4.2, and then we will prove the Barrier, Proposition 4.1.

Proof of Proposition 4.2. We will only prove one side of the bound for η^ε , and the reverse inequality is analogous. We begin with a note that due to the positive 1-homogeneity of the operator, L , any positive multiple of ϕ^ε will be a supersolution, i.e. $L\phi^\varepsilon \leq 0$. We claim that for an appropriate $b > 0$, the function,

$$\tilde{\phi}^\varepsilon = b\phi^\varepsilon + c \quad (4.4)$$

is a supersolution of (4.3). Indeed, choosing $b > 0$ large enough (w.l.o.g. $b > 1$), we obtain

$$\partial_n \tilde{\phi}^\varepsilon < -bc_1 < -c \leq \partial_n \eta \text{ on } \Sigma_0,$$

where c_1 is the constant appearing in Proposition 4.1. Furthermore, by the construction of ϕ^ε ,

$$\tilde{\phi}^\varepsilon|_{\Sigma_{1/\varepsilon}} = c \geq \eta|_{\Sigma_{1/\varepsilon}}.$$

Finally, by the positive 1-homogeneity of (4.2), we have not changed the equation for ϕ^ε and $\tilde{\phi}^\varepsilon$.

The upper bound on η^ε is now immediate from the comparison of sub and super solutions of Neumann problems (Remark A.1), which implies

$$\eta^\varepsilon \leq \tilde{\phi}^\varepsilon \leq 1/\varepsilon + c \text{ in } \overline{\Sigma^{1/\varepsilon}}.$$

□

Proof Proposition 4.1. The key component for our proof of Proposition 4.1 is the result in [6, Chp. 3, Sec 5, Thm 5.1] regarding rates of convergence in homogenization, which shows that ϕ^ε must stay within a fixed distance of order 1 of a hyperplane. (Note, we state the rates result of [6] as Proposition A.3 and provide modifications to allow for our assumptions on A and B , which do not require as much regularity as the presentation in [6].)

Let us define the rescaling of ϕ^ε as

$$\rho^\varepsilon(x) = \varepsilon \phi^\varepsilon\left(\frac{x}{\varepsilon}\right),$$

so we have

$$D^2 \rho^\varepsilon(x) = \varepsilon^{-1} D^2 \phi^\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \nabla \rho^\varepsilon(x) = \nabla \phi^\varepsilon\left(\frac{x}{\varepsilon}\right)$$

Hence, ρ^ε is the unique solution of

$$\begin{cases} \operatorname{Tr}(A(\frac{x}{\varepsilon})D^2 \rho^\varepsilon(y)) + \frac{1}{\varepsilon} B(\frac{x}{\varepsilon}) \cdot \nabla \rho^\varepsilon(y) = 0 & \text{in } \Sigma^1 \\ \rho^\varepsilon = 0 & \text{on } \Sigma_1 \\ \rho^\varepsilon = 1 & \text{on } \Sigma_0. \end{cases} \quad (4.5)$$

Furthermore, by standard homogenization results, e.g. Bensoussan-Lions-Papanicolaou [6, Chp. 3, Sec 4 and 5], also summarized here in Section A.3 we know that there is a unique constant matrix \bar{A} so that $\rho^\varepsilon \rightarrow \bar{\rho}$, and $\bar{\rho}$ is the unique solution of

$$\begin{cases} \operatorname{Tr}(\bar{A}D^2 \bar{\rho}(y)) = 0 & \text{in } \Sigma^1 \\ \bar{\rho} = 0 & \text{on } \Sigma_1 \\ \bar{\rho} = 1 & \text{on } \Sigma_0. \end{cases} \quad (4.6)$$

Since this equation is purely second order, we see that $\bar{\rho}(x) = 1 - x_{d+1}$. Now we use the fact that there is a global rate of convergence of ρ^ε , for example in [6, Chp. 3, Thm 5.1] (see also Proposition A.3). Namely, there is a universal constant, C , so that

$$\|\rho^\varepsilon - \bar{\rho}\|_{L^\infty(\overline{\Sigma^1})} \leq C\varepsilon.$$

We note also, for the divergence case ($B = \operatorname{div}(A)$), we could utilize the global rate from [2, Theorem 5]. In the microscopic variables, this says

$$\|\phi^\varepsilon - \frac{1}{\varepsilon} \bar{\rho}(\varepsilon \cdot)\|_{L^\infty(\overline{\Sigma^1/\varepsilon})} \leq C. \quad (4.7)$$

Finally, we note that since we are ultimately concerned with only $\partial_n \phi^\varepsilon$, we assume that we have subtracted $1/\varepsilon$ from both ϕ^ε and $\varepsilon^{-1} \bar{\rho}(\varepsilon \cdot)$ so that

$$\phi^\varepsilon = 0 \text{ on } \Sigma_0, \quad \phi^\varepsilon = -1/\varepsilon \text{ on } \Sigma_{1/\varepsilon}, \quad \text{and } \bar{\rho}(y) = -y_{d+1}.$$

These observations are now enough to build a barrier for ϕ^ε . The most important observation of ϕ^ε and $\bar{\rho}$ is that (4.7) shows that there will be a *fixed* distance, t^* , that is independent of ε so that $\phi^\varepsilon(y) \leq -1$ for all $y_{d+1} = t^*$. The fact that t^* is independent of ε allows to construct a good barrier. Indeed, we simply choose $t^* = C + 1$, so from (4.7)

$$\phi^\varepsilon(y) \leq -1 \text{ for all } y_{d+1} = t^*.$$

We will construct an upper barrier, M , for ϕ^ε . M will be a super solution in Σ^{t^*} , with $M = 0$ on Σ_0 , and $M = -1$ on Σ_{t^*} . We note that M is constructed to be a function only of y_{d+1} and $M' \leq 0$ globally. We need

$$L(M) = \text{Tr}(A(y)D^2M) + B(y) \cdot \nabla M \leq 0 \text{ in } \Sigma^{t^*}.$$

By the uniform ellipticity of A ,

$$L(M) = \text{Tr}(A(y)D^2M) + B(y) \cdot \nabla M \leq \Lambda M'' - \|B\|M',$$

and so we seek a solution of the form

$$M'' - \frac{C}{\Lambda}M' = 0, \quad M(0) = 0, \quad M(t^*) = -1.$$

Let us denote $C_2 = (C/\Lambda)$. The good choice of M will be

$$M(t) = \frac{a_0}{C_2}(e^{C_2 t} - 1),$$

where a_0 is chosen to obtain $M(t^*) = -1$. That means that

$$a_0 = \frac{-C_2}{(e^{C_2 t^*} - 1)} < 0,$$

and without loss of generality, $C_2 \geq 1$, and $t^* \geq 1$. Thus M is a super solution of $L(M) \leq 0$ in Σ^{t^*} , ϕ^ε is a solution of $L(\phi^\varepsilon) = 0$ in Σ^{t^*} , $\phi^\varepsilon = M$ on Σ_0 , and $M \geq \phi^\varepsilon$ on Σ_{t^*} . Thus

$$\phi^\varepsilon \leq M \text{ in } \overline{\Sigma^{t^*}}, \text{ and } \partial_n \phi^\varepsilon(y) \leq M'(0) = a_0 < 0 \text{ for } y \in \Sigma_0.$$

This concludes the claimed upper bound of the proof, with $c_1 = a_0$.

Next, we construct a lower barrier for ϕ^ε , which we will call m . For the lower bound construction, we can simply choose $s^* = 10$. We will build m so that

$$m' \leq 0 \text{ globally,}$$

and

$$L(m) = \text{Tr}(A(y)D^2M) + B(y) \cdot \nabla M \geq 0, \quad m = 0 \text{ on } \Sigma_0, \quad m = -1/\varepsilon \text{ on } \Sigma^{s^*}$$

the sign of m' plus the ellipticity says

$$L(m) \geq \lambda m'' + \|B\|m'$$

so we choose it concretely to solve

$$m'' + (C/\lambda)m' = 0 \text{ with } m(0) = 0, \text{ and } m(10) = -10 - C.$$

Hence,

$$m(t) = \frac{a_1}{C_3}(1 - e^{-C_3 t}),$$

where $C_3 = (C/\lambda)$.

We make the choice

$$a_1 = \frac{C_2(-10 - C)}{(1 - e^{-C_2 * 10})} < 0.$$

Furthermore, the rates of convergence of $\phi^\varepsilon \rightarrow \bar{\rho}$ show that

$$\phi^\varepsilon(y) \geq -10 - C \text{ for all } y \text{ such that } y_{d+1} = 10,$$

and hence

$$m(10) = -10 - C \leq \phi^\varepsilon(y) \text{ for } y_{d+1} = 10.$$

So, again, we conclude that m is a subsolution in Σ^{s^*} , and so $\phi^\varepsilon \geq m$. We note

$$m'(0) = a_1 < 0,$$

and thus the lower bound holds with $c_2 = a_1$. \square

Remark 4.3. *The proof of the previous proposition is basically the one feature that separates the proofs for the case where L is independent from the gradient and the case of (1.1). When the equation is independent of the gradient, Proposition 4.1 is trivial because affine functions are solutions of the equation.*

We conclude this section with one last result that captures the limiting behavior of $\phi^\varepsilon - 1/\varepsilon$ as $\varepsilon \rightarrow 0$. It will be useful for proving the uniqueness of the limiting boundary condition in Theorem 1.1.

Lemma 4.4. *Assume that ψ^ε solves*

$$\begin{cases} \operatorname{Tr}(A(y)D^2\psi^\varepsilon(y)) + B(y) \cdot \nabla\psi^\varepsilon(y) = 0 & \text{in } \Sigma^{1/\varepsilon} \\ \psi^\varepsilon = -1/\varepsilon & \text{on } \Sigma_{1/\varepsilon} \\ \psi^\varepsilon = 0 & \text{on } \Sigma_0. \end{cases}$$

Then there exists ψ^∞ such that $\psi^\varepsilon \rightarrow \psi^\infty$ uniformly in $\overline{\Sigma^1}$ as $\varepsilon \rightarrow 0$.

Proof. This is a consequence of uniqueness in various classes of functions for

$$\begin{cases} \operatorname{Tr}(A(y)D^2u(y)) + B(y) \cdot \nabla u(y) = 0 & \text{in } \Sigma^\infty \\ u = u_0 & \text{on } \Sigma_0, \end{cases} \quad (4.8)$$

where

$$\Sigma^\infty = \{x \in \mathbb{R}^{d+1} : 0 < x \cdot n\}.$$

In particular, there is a unique *bounded* solution of (4.8). Thus if there are u and v that solve (4.8) and for a fixed constant, satisfy

$$|u(y) - y_{d+1}| \leq C, \text{ and } |v(y) - y_{d+1}| \leq C \text{ for all } y \in \Sigma^\infty,$$

then $u - v$ is bounded, solves (4.8) with $u - v = 0$ on Σ_0 , and hence is the unique solution, which is $u - v \equiv 0$.

To identify ψ^∞ , we note that ψ^ε does indeed have local uniform limits. This is due to the fact that its oscillation in any ball of a fixed radius is uniformly bounded— as seen in the proof of Proposition 4.1— and hence enjoys a global estimate on $[\psi^\varepsilon]_{C^\gamma}$, by uniformly using the local estimate to cover $\Sigma^{1/\varepsilon}$. But then if ψ^∞ is *any* local uniform limit of ψ^ε , it follows as in the proof of Proposition 4.1 that

$$|\psi^\varepsilon(y) - y_{d+1}| \leq C,$$

as well as ψ^∞ satisfies (4.8) by the stability of (viscosity) solutions. Hence any possible ψ^∞ will indeed satisfy

$$|\psi^\infty - y_{d+1}| \leq C,$$

and is thus unique.

Now, to conclude, we need to check that the convergence is in fact uniform in the strip $\overline{\Sigma^1}$. We note that in $\Sigma^{1/\varepsilon}$, $\psi^\varepsilon - \psi^\infty$ is a bounded solution of

$$\operatorname{Tr}(A(y)D^2u(y)) + B(y) \cdot \nabla u(y) = 0 \text{ in } \Sigma^{1/\varepsilon}.$$

Thus by the maximum principle, $\max_{\overline{\Sigma^{1/\varepsilon}}}(\psi^\varepsilon - \psi^\infty)$, occurs on the boundary of $\Sigma^{1/\varepsilon}$. Since $\psi^{1/\varepsilon} - \psi^\infty = 0$ on Σ_0 and $|\psi^{1/\varepsilon} - \psi^\infty| \leq C$ on $\Sigma^{1/\varepsilon}$, we conclude that

$$\varepsilon C(\phi^\varepsilon - 1/\varepsilon) \text{ and } -\varepsilon C(\phi^\varepsilon - 1/\varepsilon)$$

are respectively sub and super solutions of the equation and boundary data, where ϕ^ε is defined in (4.2). Hence

$$\varepsilon C(\phi^\varepsilon - 1/\varepsilon) \leq \psi^\varepsilon - \psi^\infty \leq -\varepsilon C(\phi^\varepsilon - 1/\varepsilon) \text{ in } \overline{\Sigma^{1/\varepsilon}}.$$

Since $|\phi^\varepsilon - 1/\varepsilon| \leq C$ in $\overline{\Sigma^1}$ (again, due to the arguments in the proof of Proposition 4.1), we conclude that

$$\sup_{y \in \Sigma^1} |\psi^{1/\varepsilon} - \psi^\infty| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

□

4.2. Uniform bound for $\|u^\varepsilon\|_{L^\infty}$. We will use, in a crucial way, the C^γ and $C^{1,\gamma}$ estimates for u^ε , and various other related functions (see Corollary 4.6 and Proposition 4.2 and A.2). For these to be useful, we first need to know that u^ε are uniformly bounded. It is interesting that it seems in most– if not all– of the existing literature, this bound comes from basic observations using directly the structure of the equation being homogenized. For (1.1), however, this is not the case, and such an estimate will not be true without the centering condition on B , Assumption 2.4. Thus, the estimate we need for the Neumann problem in fact utilizes the homogenization of (1.1) with *regular Dirichlet* data.

Lemma 4.5. *There exists a universal $C > 0$ such that $\|u^\varepsilon\|_{L^\infty(\Sigma^1)} \leq C\|g\|_{L^\infty(\Sigma_0)}$, where u^ε is the unique solution of (1.1).*

Proof. This statement is immediate from Proposition 4.2. If we define η^ε as

$$\eta^\varepsilon(y) := \frac{1}{\varepsilon}u^\varepsilon(\varepsilon y),$$

then we see that η^ε satisfies (4.3). Indeed,

$$|\partial_n \eta^\varepsilon(y)| = |\partial_n u^\varepsilon(\varepsilon y)| = |g(y)| \leq \|g\|_{L^\infty(\Sigma_0)} \text{ for } y \in \Sigma_0,$$

and

$$|\eta^\varepsilon(y)| = |u^\varepsilon(y)| = 0 \text{ for } y \in \Sigma_{1/\varepsilon}.$$

Also, the rescaling of u^ε in this way transforms the equation in (1.1) in Σ^1 into (4.3) in $\Sigma^{1/\varepsilon}$. Hence Proposition 4.2 implies that

$$\|u^\varepsilon\|_{L^\infty(\Sigma^1)} \leq \varepsilon \|\eta^\varepsilon\|_{L^\infty(\Sigma^{1/\varepsilon})} \leq C\|g\|_{L^\infty(\Sigma_0)}.$$

□

By Proposition A.2, we have the following corollary.

Corollary 4.6. u^ε satisfies the following estimates

$$[u^\varepsilon]_{C^\gamma(\Sigma^1)} \leq C \|g\|_{L^\infty(\Sigma_0)}$$

and

$$[u^\varepsilon]_{C^{1,\gamma}(\Sigma^1)} \leq \frac{C}{\varepsilon^\gamma} (1 + [g]_{C^\gamma(\Sigma_0)}).$$

5. STRUCTURAL FEATURES OF $\mathcal{I}^{1/\varepsilon}$

In this section, we collect several features of the operator, $\mathcal{I}^{1/\varepsilon}$ (defined above, in (3.2), (3.3)), which will be used to prove the existence and uniqueness of the limiting constant, \bar{g} , in Section 6. The most important result for this section is the following

Lemma 5.1. *There exists a function $f^\varepsilon \in C(\Sigma_0) \cap L^\infty(\Sigma_0)$ such that for all $\varphi \in C^{1,\gamma}(\Sigma_0)$ and constants, c ,*

$$\mathcal{I}^{1/\varepsilon}(\phi + c, y) = \mathcal{I}^{1/\varepsilon}(\phi, y) - cf^\varepsilon(y).$$

Furthermore, $f^\varepsilon \geq 0$, and for $0 < c_1 < c_2$ (the universal constants from Proposition 4.1)

$$\varepsilon c_1 \leq f^\varepsilon(y) \leq \varepsilon c_2.$$

Remark 5.2. *This is the counterpart to [20, Lemma 3.6]. The difference is that there the operator in the bulk, F , and subsequently $\mathcal{I}^{1/\varepsilon}$ were invariant by addition of hyperplanes. In that case, this lemma is trivial, as the equation is invariant by the addition of hyperplanes to solutions. Here hyperplanes are not solutions, nor can they serve as sub/super solution barriers, and so we appeal to the barrier provided by Proposition 4.1.*

Proof of Lemma 5.1. Let us call V_ϕ^ε the solution of (3.2) with boundary data, ϕ , and $\tilde{V}_\phi^\varepsilon$ the solution of (3.2) with data $\phi + c$. If ϕ^ε is as in Proposition 4.1, then since (3.2) is linear, we see that

$$\tilde{\Phi} = \Phi + \varepsilon c \phi^\varepsilon.$$

Hence,

$$\mathcal{I}^{1/\varepsilon}(\phi + c, y) = \partial_n \tilde{\Phi}(y) = \partial_n \Phi(y) + \varepsilon c \partial_n \phi^\varepsilon.$$

The lemma follows immediately from Proposition 4.1 with the choice

$$f^\varepsilon = -\varepsilon \partial_n \phi^\varepsilon. \tag{5.1}$$

□

Lemma 5.3 (Rescaling). *For all $v \in C^{1,\gamma}(\Sigma_0)$,*

$$\mathcal{I}^1(\varepsilon v(\frac{\cdot}{\varepsilon}), x) = \mathcal{I}^{1/\varepsilon}(v, \frac{x}{\varepsilon})$$

Proof. We let U_v^ε solve (3.1) with data $\varepsilon v(\cdot/\varepsilon)$, let V_v^ε solve (3.2) with data v , and define

$$\tilde{U}^\varepsilon(y) = \varepsilon V_v^\varepsilon(\frac{x}{\varepsilon}).$$

Hence

$$\nabla \tilde{U}^\varepsilon(x) = \nabla V_v^\varepsilon(\frac{x}{\varepsilon}), \text{ and } D^2 \tilde{U}^\varepsilon(x) = \frac{1}{\varepsilon} D^2 V_v^\varepsilon(\frac{x}{\varepsilon}),$$

and thus \tilde{U}^ε solves (3.1), with data $\varepsilon v(\cdot/\varepsilon)$. By the uniqueness of solutions to (3.1), we conclude

$$\tilde{U}^\varepsilon = U_v^\varepsilon,$$

and hence

$$\mathcal{I}^1(\varepsilon v(\frac{\cdot}{\varepsilon}), x) = \partial_n U_v^\varepsilon(x) = \partial_n \tilde{U}^\varepsilon(x) = \partial_n V_v^\varepsilon(\frac{x}{\varepsilon}) = \mathcal{I}^{1/\varepsilon}(v, x).$$

□

The following auxiliary functions will be useful for localizing points of maxima and minima. Let

$$\phi_1(x) := \frac{|x|^2}{1 + |x|^2}, \text{ for } x \in \Sigma_0$$

and for $R > 0$ we will consider the functions

$$\phi_R(x) := \phi(x/R). \quad (5.2)$$

As we shall see below, the Dirichlet to Neumann maps for the standard extremal operators \mathcal{M}^\pm will be of use. The extremal operators are not essential for this paper, as we don't solve a nonlinear equation. However, we choose to use them here for possible application in the future. They are defined as follows, given $\phi : \Sigma_0 \rightarrow \mathbb{R}$, define,

$$M^{r,\pm}(\phi, y) := \partial_n U_\phi^{r,\pm}, \quad (5.3)$$

where $U_\phi^{r,\pm} = U^{r,\pm} : \Sigma^r \rightarrow \mathbb{R}$ are the unique viscosity solutions of

$$\begin{cases} \mathcal{M}^+(D^2 U^{r,+}) + C |\nabla U^{r,+}| = 0 & \text{in } \Sigma^r, \\ U^{r,+} = 0 & \text{on } \Sigma_r, \\ U^{r,+} = \phi & \text{on } \Sigma_0, \end{cases} \quad (5.4)$$

and

$$\begin{cases} \mathcal{M}^-(D^2 U^{r,-}) - C |\nabla U^{r,-}| = 0 & \text{in } \Sigma^r, \\ U^{r,-} = 0 & \text{on } \Sigma_r, \\ U^{r,-} = \phi & \text{on } \Sigma_0. \end{cases} \quad (5.5)$$

The operators \mathcal{M}^\pm are the standard Pucci extremal operators of the second order fully nonlinear theory [7, Chapter 2], and they correspond to the linear ellipticity assumption that $\lambda \text{Id} \leq A(x) \leq \Lambda \text{Id}$. We mention that by, e.g. [28], if $\phi \in C^{1,\gamma}(\Sigma_0)$, then $U^{r,\pm} \in C^{1,\gamma}(\bar{\Sigma}^r)$. Hence the definition (5.3) holds classically in the pointwise sense.

Proposition 5.4 (Bump function). *Assume that $r \geq 1$ and $R_0 > 0$ are both fixed. Then*

$$\lim_{R \rightarrow \infty} \sup_{x \in B'_{R_0}} |M^{r,\pm}(\phi_R, x)| = 0$$

(we recall that $B'_{R_0} \subset \Sigma_0$).

Proof of Proposition 5.4. We just focus on the case of $M^{r,+}\phi$. The proof of this proposition is a result of the stability of viscosity solutions of (5.4) with respect to local uniform limits

of the Dirichlet data, combined with the $C^{1,\gamma}$ boundary regularity in [28]. To this end, we let U_{ϕ_R} denote the solution of (5.4) with data ϕ_R . We note that for each t , fixed,

$$\lim_{R \rightarrow \infty} \|\phi_R\|_{L^\infty(B_t)} = 0,$$

and since $\hat{U}(x) = 0$ is the unique solution to (5.4) with $\hat{U}|_{\Sigma_0} = 0$, the stability of (5.4) implies that $U_{\phi_R} \rightarrow 0$ locally uniformly in Σ^r . Furthermore, since $\phi_R \in C^{1,\gamma}(\Sigma_0)$, [28] shows that $U_{\phi_R} \in C^{1,\gamma}(\overline{\Sigma^r})$. Thus, [28, Theorem 1.1] implies

$$\|\partial_n U_{\phi_R}\|_{L^\infty(B_{R_0})} \leq C(\|U_{\phi_R}\|_{L^\infty(B_{2R_0}^+)} + \|\phi_R\|_{L^\infty(B_{2R_0})} + \|\nabla \phi_R\|_{C^\gamma(B_{2R_0})}). \quad (5.6)$$

We note here that $B'_{R_0} \subset \Sigma_0$ and $B_{R_0}^+ \subset \overline{\Sigma^r}$ are a ball and a half ball in respectively the boundary and the closure of Σ^r . The first term on the right of (5.6) converges to 0 by the stability of (5.4). The second and third terms converge to 0 by the rescaling that defines ϕ_R . Thus we conclude

$$\lim_{R \rightarrow \infty} \sup_{x \in B_{R_0}} |M^{r,+}(\phi_R, x)| = 0.$$

□

Lemma 5.5 (Comparison principle for smooth functions). *Let $u, v : \Sigma_0 \rightarrow \mathbb{R}$ be bounded functions such that $\mathcal{I}^{1/\varepsilon}(u, \cdot)$ and $\mathcal{I}^{1/\varepsilon}(v, \cdot)$ are classically defined and*

$$\mathcal{I}^{1/\varepsilon}(u, y) \geq \mathcal{I}^{1/\varepsilon}(v, y) \quad \forall y \in \Sigma_0.$$

Then,

$$u(y) \leq v(y) \quad \forall y \in \Sigma_0.$$

Proof of Lemma 5.5. Using Proposition 5.4, the proof of Lemma 5.5 now follows identically to the one in [20, Lemma 3.9].

□

Lemma 5.6. *If $w \in C^{1,\gamma}(\Sigma_0)$ solves*

$$\mathcal{I}^{1/\varepsilon}(w, y) = g(y) \quad \text{in } \Sigma_0,$$

then

$$-\frac{1}{c_1 \varepsilon} \|g\|_{L^\infty} \leq w \leq \frac{1}{c_1 \varepsilon} \|g\|_{L^\infty},$$

where c_1 is the constant from Lemma 5.1.

Proof of Lemma 5.6. We note that since $\mathcal{I}^{1/\varepsilon}(0, \cdot) = 0$, it follows by Lemma 5.1 that for any constant, c ,

$$\mathcal{I}^{1/\varepsilon}(c, y) = -cf^\varepsilon(y).$$

Using c_1 from Lemma 5.1, we have that

$$\mathcal{I}^{1/\varepsilon}(w, y) = g(y) \leq \|g\|_{L^\infty} = \frac{\varepsilon c_1 \|g\|}{\varepsilon c_1} \leq f^\varepsilon(y) \frac{\|g\|}{c_1 \varepsilon} = \mathcal{I}^{1/\varepsilon} \left(\frac{-1}{c_1 \varepsilon} \|g\|_{L^\infty}, y \right).$$

Thus by the comparison for $\mathcal{I}^{1/\varepsilon}$ (Lemma 5.5), we conclude that

$$\frac{-\|g\|}{c_1 \varepsilon} \leq w.$$

The reverse inequality follows analogously.

□

The following monotonicity property with respect to the domain, $\Sigma^{1/\varepsilon}$, will be useful in Section 6.

Lemma 5.7. *Suppose that $(1/\varepsilon_2) \geq (1/\varepsilon_1)$ and that $u \geq 0$, then*

$$\mathcal{I}^{1/\varepsilon_2}(u, y) \geq \mathcal{I}^{1/\varepsilon_1}(u, y) \quad \forall y \in \Sigma_0.$$

Proof. Let $V_u^{\varepsilon_1}$ and $V_u^{\varepsilon_2}$ solve (3.2) in respectively Σ^{1/ε_1} and Σ^{1/ε_2} . Note that $\Sigma^{1/\varepsilon_1} \subset \Sigma^{1/\varepsilon_2}$, so $V_u^{\varepsilon_2}$ is defined in Σ^{1/ε_1} . Since $V_u^{\varepsilon_2} = u \geq 0$ on Σ_0 , the comparison principle implies that $V_u^{\varepsilon_2} \geq 0$ in Σ^{1/ε_2} , and in particular $V_u^{1/\varepsilon_2} \geq 0$ on Σ_{1/ε_1} . Moreover, $V_u^{\varepsilon_1}$ and $V_u^{\varepsilon_2}$ agree on Σ_0 and solve the same equation in Σ^{1/ε_1} . Thus $V_u^{\varepsilon_2}$ is a supersolution for the equation solved by $V_u^{\varepsilon_1}$, so that $V_u^{\varepsilon_1} \leq V_u^{\varepsilon_2}$ everywhere in Σ^{1/ε_1} .

Since the two functions agree on Σ_0 , their normal derivatives must be ordered, namely

$$\partial_n V_u^{\varepsilon_2}(y) \geq \partial_n V_u^{\varepsilon_1}(y) \quad \forall y \in \Sigma_0,$$

and the lemma follows. \square

Lemma 5.8. *Let $(1/\varepsilon) \geq 1$ be fixed and assume f^ε is as in Lemma 5.1. Suppose that there exist bounded $w_1, w_2 \in C^{1,\gamma}(\Sigma_0)$, respectively sub and super solutions w_1 and w_2 to*

$$\mathcal{I}^{1/\varepsilon}(w_1, y) \geq -b_1 + \frac{a_1 f^\varepsilon(y)}{\varepsilon} + g(y) \quad \text{and} \quad \mathcal{I}^{1/\varepsilon}(w_2, y) \leq \frac{a_2 f^\varepsilon(y)}{\varepsilon} + g(y) \quad \text{in } \Sigma_0,$$

where $b_1 \geq 0$. Then $a_1 - a_2 - (1/c_1)b_1 \leq \varepsilon \sup_{\Sigma_0} |w_1 - w_2|$, where c_1 is the constant from Lemma 5.1.

Proof. Without loss of generality, assume that $a_1 \geq a_2$, else there is nothing to prove (since $b_1 \geq 0$). Let $\tilde{w}_2 := w_2 - (1/\varepsilon)(a_1 - a_2) + (1/c_1\varepsilon)b_1$, then by Lemma 5.1 we have

$$\begin{aligned} \mathcal{I}^{1/\varepsilon}(\tilde{w}_2, y) &= \mathcal{I}^{1/\varepsilon}(w_2, y) + \frac{(a_1 - a_2)}{\varepsilon} f^\varepsilon(y) - \frac{b_1 f^\varepsilon(y)}{c_1 \varepsilon}, \\ &\leq g(y) + \frac{a_2 f^\varepsilon(y)}{\varepsilon} + \frac{(a_1 - a_2) f^\varepsilon}{\varepsilon} - b_1 \\ &\leq g(y) + \frac{a_1 f^\varepsilon(y)}{\varepsilon} - b_1, \\ &\leq \mathcal{I}^{1/\varepsilon}(w_1, y). \end{aligned}$$

Then Lemma 5.5 yields that $w_1 \leq \tilde{w}_2$, i.e. $w_1 \leq \tilde{w}_2 = w_2 - (1/\varepsilon)(a_1 - a_2) + (1/c_1\varepsilon)b_1$. Rearranging,

$$(1/\varepsilon)(a_1 - a_2) - (1/c_1\varepsilon)b_1 \leq w_2 - w_1 \leq \sup_{\Sigma_0} |w_1 - w_2|,$$

and the lemma follows by multiplying by ε . \square

6. THE PROOFS OF PROPOSITION 3.1 AND THEOREM 1.1

We are now in a position to prove Proposition 3.1 as well as show how it implies Theorem 1.1. To this end, we define the new function

$$v^\varepsilon = u^\varepsilon|_{\Sigma_0},$$

where u^ε is the solution to (1.1). We know by Corollary 4.6 that $u^\varepsilon \in C^{1,\gamma}(\overline{\Sigma^1})$, and so

$$\partial_n u^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ classically on } \Sigma_0.$$

Furthermore, the uniqueness of solutions to (3.2) shows that when $V_{v^\varepsilon}^\varepsilon$ solves (3.2) with data v^ε , then in fact

$$V_{v^\varepsilon}^\varepsilon \equiv u^\varepsilon.$$

Thus, as pointed out in Section 3, the original homogenization problem is equivalent to

$$\mathcal{I}^1(v^\varepsilon, x) = g\left(\frac{x}{\varepsilon}\right) \text{ in } \Sigma_0.$$

It is useful to unscale this equation; thanks to Lemma 5.3 the function

$$w^\varepsilon(y) = (1/\varepsilon)v^\varepsilon(\varepsilon y), \tag{6.1}$$

solves

$$\mathcal{I}^{1/\varepsilon}(w^\varepsilon, y) = g(y) \text{ in } \Sigma_0. \tag{6.2}$$

Since Σ_0 is irrational, $g|_{\Sigma_0}$ will not be periodic, but it will be almost periodic (Definition 2.5). Furthermore, $\mathcal{I}^{1/\varepsilon}$ is effectively an ‘‘almost periodic operator’’ on Σ_0 , which is not precisely defined, but it manifests itself in the almost periodicity of w^ε , presented below as Lemma 6.1.

6.1. Almost periodicity of w^ε . Here we use almost periodicity properties of Σ_0 with respect to \mathbb{Z}^{d+1} to establish almost periodicity properties of w^ε . We remind the reader that the relevant definitions appear in Section 2.2.

Lemma 6.1. *There exists a universal $C > 0$ such that for all $\delta > 0$, there exists $R_\delta > 0$, such that if E_δ and F_δ are respectively the δ -almost periods of g and $C\left(\delta + \frac{\delta}{\varepsilon}\right)$ -periods of w^ε , then*

$$B'_{R_\delta}(z) \cap (E_\delta \cap F_\delta) \neq \emptyset \text{ for all } z \in \Sigma_0.$$

We emphasize that R_δ depends only on the irrationality of n , and in particular is independent of ε . Note that in this context, $B_{R_\delta} \subset \Sigma_0$.

Proof. Let $z \in \Sigma_0$ be fixed and $\rho > 0$ be arbitrary. We will eventually choose ρ to depend on δ to make the calculation easier. First, we note that by the irrationality of n , there exists an R_ρ such that given any $z \in \Sigma_0$, there is $\tau(z) \in \Sigma_0 \cap B_{R_\rho}(z)$ such that $\text{dist}(\tau(z), \mathbb{Z}^{d+1}) \leq \rho$ (see, e.g. [8, Lem. 2.7]). We take $\hat{\tau} \in \Sigma^{1/\varepsilon} \cap \mathbb{Z}^{d+1}$ to be any element that is within $\text{dist}(\tau(z), \mathbb{Z}^{d+1})$ to Σ_0 . We also note that by possibly re-adjusting the choice of $\tau(z)$, we can assume that $\tau(z) \perp (\hat{\tau} - \tau(z))$. Both $\tau(z)$ and $\hat{\tau}(z)$ depend on z , but we suppress this dependence for the rest of the proof.

We begin by unscaling the original equation, (1.1). To that end, define

$$V^\varepsilon(y) = \frac{1}{\varepsilon}u^\varepsilon(\varepsilon y).$$

Thus, V^ε solves the equation in the microscopic variables, i.e.

$$\begin{cases} \operatorname{Tr}(A(y)D^2V^\varepsilon) + B(y)\nabla V^\varepsilon = 0 & \text{in } \Sigma^{1/\varepsilon} \\ V^\varepsilon = 0 & \text{on } \Sigma_{1/\varepsilon} \\ \partial_n V^\varepsilon = g(y) & \text{on } \Sigma_0. \end{cases}$$

This means that in fact, $V^\varepsilon|_{\Sigma_0} = w^\varepsilon$, where w^ε is defined in (6.1) and solves (6.2). We will now shift V^ε by $\hat{\tau}$, defining

$$\hat{V}^\varepsilon(y) = V^\varepsilon(y + \hat{\tau}).$$

We define

$$\hat{z} = \hat{\tau} - \tau, \text{ note, by definition } |\hat{z}| \leq \rho,$$

and because of the orthogonality of $\tau \perp (\hat{\tau} - \tau)$, we see that \hat{V}^ε solves in the shifted domain, $(\Sigma^{1/\varepsilon} - \hat{z})$,

$$\begin{cases} \operatorname{Tr}(A(y + \hat{\tau})D^2\hat{V}^\varepsilon) + B(y + \hat{\tau})\nabla\hat{V}^\varepsilon = 0 & \text{in } (\Sigma^{1/\varepsilon} - \hat{z}) \\ \hat{V}^\varepsilon = 0 & \text{on } (\Sigma_{1/\varepsilon} - \hat{z}) \\ \partial_n \hat{V}^\varepsilon = g(y + \hat{\tau}) & \text{on } (\Sigma_0 - \hat{z}). \end{cases}$$

Since by choice, $\hat{\tau} \in \mathbb{Z}^{d+1}$, and by the periodicity of A and g we conclude

$$\begin{cases} \operatorname{Tr}(A(y)D^2\hat{V}^\varepsilon) + B(y)\nabla\hat{V}^\varepsilon = 0 & \text{in } (\Sigma^{1/\varepsilon} - \hat{z}) \\ \hat{V}^\varepsilon = 0 & \text{on } (\Sigma_{1/\varepsilon} - \hat{z}) \\ \partial_n \hat{V}^\varepsilon = g(y) & \text{on } (\Sigma_0 - \hat{z}). \end{cases}$$

Now we can compare V^ε to \hat{V}^ε . Let us define

$$\hat{W}^\varepsilon = V^\varepsilon - \hat{V}^\varepsilon \text{ in } \Sigma^{1/\varepsilon} \cap (\Sigma^{1/\varepsilon} - \hat{z}) = \Sigma^{1/\varepsilon - |\hat{z}|}.$$

We will need to observe that the oscillation of V^ε and \hat{V}^ε in the sets that are a distance less than one to $\Sigma_{1/\varepsilon}$ are bounded independently of ε . This is true because we can use the barriers involving ϕ^ε from Proposition 4.1 plus the observation that ϕ^ε stays a uniform distance from the fixed hyperplane, $1/\varepsilon - y_{d+1}$. Thus, from the C^γ estimates for the Dirichlet problem (near $\Sigma_{1/\varepsilon}$) we know that V^ε and \hat{V}^ε are C^γ in a neighborhood of $\Sigma_{1/\varepsilon}$, uniformly in ε . First, we check the boundary data on the $1/\varepsilon - |\hat{z}|$ boundary:

$$\|V^\varepsilon - \hat{V}^\varepsilon\|_{L^\infty(\Sigma_{1/\varepsilon - |\hat{z}|})} = \|V^\varepsilon - 0\|_{L^\infty(\Sigma_{1/\varepsilon - |\hat{z}|})} \leq C|\hat{z}|^\gamma.$$

Next we check the normal derivative on the Σ_0 hyperplane:

$$\|\partial_n V^\varepsilon - \partial_n \hat{V}^\varepsilon\|_{L^\infty(\Sigma_0)} = \|g - \partial_n \hat{V}^\varepsilon\|_{L^\infty(\Sigma_0)}$$

since $V^\varepsilon = g$ on Σ_0 . We would like to transfer this inequality to an evaluation on $\Sigma_0 - \hat{z}$ to utilize the boundary values of $\partial_n \hat{V}^\varepsilon$. By Corollary 4.6, we have

$$[\partial_n \hat{V}^\varepsilon]_{C^\gamma} = [\partial_n u^\varepsilon(\varepsilon \cdot)]_{C^\gamma} \leq \frac{C}{\varepsilon^\gamma} \varepsilon^\gamma = C,$$

and so for $y \in \Sigma_0$, $y - \hat{z} \in \Sigma_0 - \hat{z}$

$$\begin{aligned} & \left| \partial_n \hat{V}^\varepsilon(y) - g(y) \right| \leq \left| \partial_n \hat{V}^\varepsilon(y) - \partial_n \hat{V}^\varepsilon(y - \hat{z}) \right| + \left| \partial_n \hat{V}^\varepsilon(y - \hat{z}) - g(y) \right| \\ & = \left| \partial_n \hat{V}^\varepsilon(y) - \partial_n \hat{V}^\varepsilon(y - \hat{z}) \right| + |g(y - \hat{z}) - g(y)| \\ & \leq C |\hat{z}|^\gamma. \end{aligned}$$

We used the fact that $\partial_n \hat{V}^\varepsilon = g$ on $\Sigma_0 - \hat{z}$, and so by the Hölder regularity of g and $\partial_n \hat{V}^\varepsilon$, the difference on Σ_0 is of order $|\hat{z}|^\gamma$. (Note, $\partial_n \hat{V}^\varepsilon = g$ on $\Sigma_{-|\hat{z}|}$, not on Σ_0 .) Importantly, this constant, C , in the two estimates above, is independent of ε .

Therefore, \hat{W}^ε solves

$$\begin{cases} \operatorname{Tr}(A(y)D^2\hat{W}^\varepsilon) + B(y) \cdot \nabla \hat{W}^\varepsilon = 0 & \text{in } \Sigma^{1/\varepsilon - |\hat{z}|} \\ \left| \hat{W}^\varepsilon \right| \leq C |\hat{z}|^\gamma & \text{on } \Sigma_{1/\varepsilon - |\hat{z}|} \\ \left| \partial_n \hat{W}^\varepsilon \right| \leq C |\hat{z}|^\gamma & \text{on } \Sigma_0. \end{cases}$$

Thus, by Proposition 4.2, we conclude, in particular, that

$$\|\hat{W}^\varepsilon\|_{L^\infty(\Sigma_0)} \leq C \frac{|\hat{z}|^\gamma}{\varepsilon}.$$

Finally, we check,

$$\begin{aligned} \|w^\varepsilon(\cdot + \tau) - w^\varepsilon\|_{L^\infty(\Sigma_0)} &= \|V^\varepsilon(\cdot + \tau) - V^\varepsilon\|_{L^\infty(\Sigma_0)} \\ &\leq \|V^\varepsilon(\cdot + \hat{\tau}) - V^\varepsilon(\cdot + \tau)\|_{L^\infty(\Sigma_0)} + \|V^\varepsilon(\cdot + \hat{\tau}) - V^\varepsilon\|_{L^\infty(\Sigma_0)} \\ &\leq C |\tau - \hat{\tau}|^\gamma + \|\hat{W}^\varepsilon\|_{L^\infty(\Sigma_0)} \\ &\leq C |\hat{z}|^\gamma + C \frac{|\hat{z}|^\gamma}{\varepsilon}. \end{aligned}$$

We also confirm

$$\begin{aligned} \|g(\cdot + \tau) - g\|_{L^\infty(\Sigma_0)} &= \|g(\cdot + \hat{\tau}) - g(\cdot - \tau)\|_{L^\infty(\Sigma_0)} + \|g(\cdot + \hat{\tau}) - g\|_{L^\infty(\Sigma_0)} \\ &\leq C |\hat{\tau} - \tau|^\gamma + 0 \\ &= C |\hat{z}|^\gamma, \end{aligned}$$

since by choice, $\hat{\tau} \in \mathbb{Z}^{d+1}$, and the periodicity of g . Now, to conclude the proposition, we choose $\rho = \delta^{1/\gamma}$, and since $|\hat{z}| \leq \rho$, we have shown that indeed $\tau \in B'(R_\delta)(z)$ is an almost period for w^ε and g . \square

Remark 6.2. *We note that the proof of Proposition 4.1 utilized a similar technique to that in [9, Section 3], where there it was also very important to translate the whole equation by $\hat{\tau}$ and appeal to the periodicity to keep the coefficients unchanged.*

6.2. Limits for $\varepsilon w^\varepsilon - \varepsilon w^\varepsilon(0)$. The key lemma that establishes Proposition 3.1, in combination with the almost periodicity of Lemma 6.1, is a nonlocal version of the almost periodic arguments which appeared for Hamilton-Jacobi equations in Ishii's work, [21]. There are however, many differences between the Hamilton-Jacobi setting and our nonlocal setting here, and we give a slightly different argument.

Lemma 6.3 (Nonlocal Elliptic Version of Ishii [21]). w^ε defined in (6.1), (6.2) satisfies the decay

$$\|\varepsilon w^\varepsilon - \varepsilon w^\varepsilon(0)\|_{L^\infty(\Sigma_0)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.3)$$

Proof of Lemma 6.3. Let $\{\varepsilon_k\}_k$ be a sequence such that $\varepsilon_k \rightarrow 0^+$, and let $\{y_k\}_k$ be a sequence in Σ_0 such that for each k ,

$$|\varepsilon_k w^{\varepsilon_k}(y_k) - \varepsilon_k w^{\varepsilon_k}(0)| \geq \frac{1}{2} \|\varepsilon_k w^{\varepsilon_k} - \varepsilon_k w^{\varepsilon_k}(0)\|_{L^\infty(\Sigma_0)}.$$

Let $\delta > 0$ be given. Let F_δ be the set of $C(\delta + \delta/\varepsilon)$ -periods of w^ε . Thanks to Lemma 6.1, there is some $R_\delta > 0$ such that

$$(z + B_{R_\delta}) \cap (F_\delta) \neq \emptyset \quad \forall z \in \Sigma_0.$$

We note that it is essential that R_δ be strictly sublinear in ε for later purposes. In this case, R_δ is in fact independent of ε , which is even better.

Taking $\varepsilon = \varepsilon_k$, $z = y_k$ above, it follows that for each k there is some τ_k which is a $C(\varepsilon_k \delta + \delta)$ -almost period for $\varepsilon_k w^{\varepsilon_k}$ and such that

$$y_k - \tau_k \in B_{R_\delta}. \quad (6.4)$$

In particular,

$$|\varepsilon_k w^{\varepsilon_k}(y_k) - \varepsilon_k w^{\varepsilon_k}(0)| \leq |\varepsilon_k w^{\varepsilon_k}(y_k) - \varepsilon_k w^{\varepsilon_k}(y_k - \tau_k)| + |\varepsilon_k w^{\varepsilon_k}(y_k - \tau_k) - \varepsilon_k w^{\varepsilon_k}(0)|.$$

Since τ_k is a $C(\varepsilon_k \delta + \delta)$ -almost period for $\varepsilon_k w^{\varepsilon_k}$ the first quantity on the right is at most $C(\varepsilon_k \delta + \delta)$, and also from (6.4),

$$|\varepsilon_k w^{\varepsilon_k}(y_k) - \varepsilon_k w^{\varepsilon_k}(0)| \leq C(\varepsilon_k \delta + \delta) + \operatorname{osc}_{B_{R_\delta}} \{\varepsilon_k w^{\varepsilon_k}\}, \quad \forall k > 0.$$

Next, we note that

$$\operatorname{osc}_{B_{R_\delta}} \{\varepsilon_k w^{\varepsilon_k}\} = \operatorname{osc}_{B_{\varepsilon_k R_\delta}} \{v^{\varepsilon_k}\}.$$

Corollary 4.6 guarantees that the functions v^ε are $C^{\bar{\gamma}}$ -continuous in B_1 , uniformly in ε . Therefore (for each fixed $\delta > 0$),

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{osc}_{B_{\varepsilon R_\delta}} \{v^\varepsilon\} = 0.$$

(Here, it would be enough that $\lim \varepsilon R_\delta = 0$ if it happened that R_δ depended on ε .) Given that $\varepsilon_k \rightarrow 0$, for every large enough k (this possibly depending on δ) we have

$$\frac{1}{2} \|\varepsilon_k w^{\varepsilon_k} - \varepsilon_k w^{\varepsilon_k}(0)\|_{L^\infty(\Sigma_0)} \leq |\varepsilon_k w^{\varepsilon_k}(y_k) - \varepsilon_k w^{\varepsilon_k}(0)| \leq 2C\delta.$$

That is (as the sequence $\varepsilon_k \rightarrow 0^+$ was arbitrary)

$$\limsup_{\varepsilon \rightarrow 0^+} \|\varepsilon w^\varepsilon - \varepsilon w^\varepsilon(0)\|_{L^\infty(\Sigma_0)} \leq 4C\delta,$$

letting $\delta \rightarrow 0^+$, the lemma follows. \square

Lemma 6.4. *Given any $\varepsilon_j \rightarrow 0$, there exists a subsequence, ε'_j such that $v^{\varepsilon'_j} \rightarrow c$ uniformly on Σ_0 , for some constant c .*

Proof of Lemma 6.4. By Proposition Corollary 4.6 we know that $v^\varepsilon \in C^\gamma(\Sigma_0)$ for some $0 < \gamma < 1$. Thus since v^ε are uniformly bounded, we can take some subsequence such that $v^{\varepsilon_j}(0) \rightarrow c$. Furthermore, Lemma 6.3 shows that

$$\|v^{\varepsilon_j'} - v^{\varepsilon_j'}(0)\|_{L^\infty(\Sigma_0)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Hence $v^{\varepsilon_j} \rightarrow c$ uniformly on Σ_0 . □

6.3. Uniqueness of the limiting constant.

Lemma 6.5. *The constant, c , of Lemma 6.4 is independent of the sequence, ε_j , and hence unique.*

Proof of Lemma 6.5. We note that this is basically a consequence of the fact that (6.2) is a uniformly elliptic integro-differential equation. However, the proof is not as straightforward as in the existing literature for either second order or nonlocal elliptic equations due to the influence of the Dirichlet condition $u^\varepsilon = 0$ on Σ_1 . The necessary modifications are not serious road blocks, but we do include them for completeness.

Let a_1 and a_2 be constants such that there are sequences $v^{\varepsilon_j} \rightarrow a_1$ and $v^{\varepsilon_k} \rightarrow a_2$ uniformly on Σ_0 . We will establish that

$$a_2 \leq a_1,$$

and since the sequences were arbitrary, this proves the lemma. If we rewrite v^{ε_j} and v^{ε_k} in the microscale variables, this says that (recall w^ε from (6.1), (6.2))

$$\varepsilon_j w^{\varepsilon_j} \rightarrow a_1 \quad \text{and} \quad \varepsilon_k w^{\varepsilon_k} \rightarrow a_2 \quad \text{uniformly on } \Sigma_0.$$

We will also define the functions

$$\hat{w}^{\varepsilon_j} = w^{\varepsilon_j} - \frac{1}{\varepsilon_j} a_1 \quad \text{and} \quad \hat{w}^{\varepsilon_k} = w^{\varepsilon_k} - \frac{1}{\varepsilon_k} a_2.$$

In anticipation of applying Lemma 5.7, we need to make sure that \hat{w}^{ε_j} and \hat{w}^{ε_k} are non-negative. We do so by shifting them up by respectively δ_j , δ_k where

$$\delta_j = \|\hat{w}^{\varepsilon_j}\| \quad \text{and} \quad \delta_k = \|\hat{w}^{\varepsilon_k}\|,$$

which gives

$$\hat{w}^{\varepsilon_j} + \delta_j \geq 0 \quad \text{and} \quad \hat{w}^{\varepsilon_k} + \delta_k \geq 0.$$

We will assume without loss of generality that j and k are such that $\varepsilon_j < \varepsilon_k$, which suffices because j and k can otherwise be chosen independently of one another. Using the

equations for w^{ε_j} and w^{ε_k} , we see that from Lemmas 5.1 and 5.7

$$\begin{aligned}
& \frac{(-\varepsilon_k \delta_k + a_2) f^{\varepsilon_j}(y)}{\varepsilon_j} + g(y) = \\
& = (-\varepsilon_k \delta_k + a_2) \left(\frac{f^{\varepsilon_j}(y)}{\varepsilon_j} - \frac{f^{\varepsilon_k}(y)}{\varepsilon_k} \right) + \frac{(-\varepsilon_k \delta_k + a_2) f^{\varepsilon_k}(y)}{\varepsilon_k} + g(y) \\
& = \mathcal{I}^{1/\varepsilon_k}(w^{\varepsilon_k} - \frac{a_2}{\varepsilon_k} + \delta_k, y) + (-\varepsilon_k \delta_k + a_2) \left(\frac{f^{\varepsilon_j}(y)}{\varepsilon_j} - \frac{f^{\varepsilon_k}(y)}{\varepsilon_k} \right) \\
& = \mathcal{I}^{1/\varepsilon_k}(\hat{w}^{\varepsilon_k} + \delta_k, y) + (-\varepsilon_k \delta_k + a_2) \left(\frac{f^{\varepsilon_j}(y)}{\varepsilon_j} - \frac{f^{\varepsilon_k}(y)}{\varepsilon_k} \right) \\
& \leq \mathcal{I}^{1/\varepsilon_j}(\hat{w}^{\varepsilon_k} + \delta_k, y) + (-\varepsilon_k \delta_k + a_2) \left(\frac{f^{\varepsilon_j}(y)}{\varepsilon_j} - \frac{f^{\varepsilon_k}(y)}{\varepsilon_k} \right) \\
& \leq \mathcal{I}^{1/\varepsilon_j}(\hat{w}^{\varepsilon_k} + \delta_k, y) + \rho_{k,j},
\end{aligned}$$

where

$$\rho_{j,k} = \max \left\{ 0, \sup_y (-\varepsilon_k \delta_k + a_2) \left(\frac{f^{\varepsilon_j}(y)}{\varepsilon_j} - \frac{f^{\varepsilon_k}(y)}{\varepsilon_k} \right) \right\}.$$

But on the other hand,

$$\frac{(-\varepsilon_j \delta_j + a_1) f^{\varepsilon_j}(y)}{\varepsilon_j} + g(y) = \mathcal{I}^{1/\varepsilon_j}(\hat{w}^{\varepsilon_j} + \delta_j, y). \quad (6.5)$$

Thus

$$\mathcal{I}^{1/\varepsilon_j}(\hat{w}^{\varepsilon_j} + \delta_j, y) \leq \frac{(-\varepsilon_j \delta_j + a_1) f^{\varepsilon_j}(y)}{\varepsilon_j} + g(y)$$

and

$$\mathcal{I}^{1/\varepsilon_j}(\hat{w}^{\varepsilon_k} + \delta_k, y) \geq \frac{(-\varepsilon_k \delta_k + a_2) f^{\varepsilon_j}(y)}{\varepsilon_j} + g(y) - \rho_{j,k},$$

where $\rho_{j,k} \geq 0$ is defined above.

Thus Lemma 5.8 tells us that

$$(-\varepsilon_k \delta_k + a_2) - (-\varepsilon_j \delta_j + a_1) - (1/c_1) \rho_{j,k} \leq \varepsilon_j \sup_{\Sigma_0} ((\hat{w}^{\varepsilon_k} + \delta_k) - (\hat{w}^{\varepsilon_j} + \delta_j)).$$

Hence

$$\begin{aligned}
a_2 - a_1 & \leq \varepsilon_j \sup_{\Sigma_0} \hat{w}^{\varepsilon_k} - \varepsilon_j \inf_{\Sigma_0} \hat{w}^{\varepsilon_j} + \varepsilon_j \delta_k - \varepsilon_j \delta_j + \varepsilon_k \delta_k - \varepsilon_j \delta_j + (1/c_1) \rho_{j,k} \\
& \leq \varepsilon_j \|\hat{w}^{\varepsilon_k}\|_{L^\infty(\Sigma_0)} + \varepsilon_j \|\hat{w}^{\varepsilon_j}\|_{L^\infty(\Sigma_0)} + \varepsilon_j \delta_k + \varepsilon_k \delta_k + (1/c_1) \rho_{j,k} \\
& \leq 3\varepsilon_k \|\hat{w}^{\varepsilon_k}\|_{L^\infty(\Sigma_0)} + \varepsilon_j \|\hat{w}^{\varepsilon_j}\|_{L^\infty(\Sigma_0)} + (1/c_1) \rho_{j,k},
\end{aligned} \quad (6.6)$$

where we have used both $\varepsilon_j \delta_j \geq 0$ and $\varepsilon_j < \varepsilon_k$.

Now, we recall the definition of f^ε , from the proof of Lemma 5.1, in (5.1) as

$$f^\varepsilon = -\varepsilon \partial_n \phi^\varepsilon.$$

Thanks to Lemma 4.4, we know that $\phi^\varepsilon - 1/\varepsilon \rightarrow \psi^\infty$ uniformly on $\overline{\Sigma^1}$, and since $\phi^{1/\varepsilon} - 1/\varepsilon$ also has uniform $C^{2,\gamma}(\overline{\Sigma^1})$ estimates, it holds that

$$\frac{f^\varepsilon}{\varepsilon} \rightarrow \partial_n \psi^\infty \text{ uniformly on } \Sigma_0.$$

Hence

$$\rho_{j,k} \rightarrow 0 \text{ as } j \rightarrow \infty \text{ and } k \rightarrow \infty.$$

Now, we preserve $\varepsilon_j < \varepsilon_k$ and first take $j \rightarrow \infty$ followed by $k \rightarrow \infty$. By construction of \hat{w}^{ε_j} and \hat{w}^{ε_k} , we have

$$\varepsilon_k \|\hat{w}^{\varepsilon_k}\|_{L^\infty(\Sigma_0)} \rightarrow 0 \text{ and } \varepsilon_j \|\hat{w}^{\varepsilon_j}\|_{L^\infty(\Sigma_0)} \rightarrow 0.$$

Hence $a_2 \leq a_1$, and this finishes the lemma. \square

6.4. The Proofs of Proposition 3.1 and Theorem 1.1. Proposition 3.1 follows immediately from Lemmas 6.4 and 6.5. We again mention that by Corollary 4.6, $\|u^\varepsilon\|_{C^\gamma} \leq C$ independently of ε . Thus, we can extract locally uniformly convergent subsequences of u^ε in Σ^1 . Let \bar{u} be any possible subsequential limit of u^ε . The perturbed test function method, as in Proposition A.8, shows that \bar{u} is a solution of

$$\text{Tr}(\bar{A}D^2\bar{u}) = 0 \text{ in } \Sigma^1,$$

(see Section A.3). Proposition 3.1 gives a unique constant, \bar{c} such that $\bar{u}|_{\Sigma_0} = \bar{c}$. The uniform Hölder continuity of u^ε gives $\bar{u}|_{\Sigma_1} = 0$. Thus by the uniqueness of solutions to (1.2) we see that there is exactly one choice for \bar{u} . Hence Theorem 1.1 is established with $\bar{g} = -\bar{c}$.

7. MODIFICATIONS TO OBTAIN THE CELL PROBLEM OF CHOI-KIM [9]

The work of Choi and Kim [9] proves the homogenization of fully nonlinear equations with oscillatory Neumann data in some more general domains. They studied (1.1) with a nonlinear operator in the interior, as

$$\begin{cases} F(\frac{x}{\varepsilon}, D^2u^\varepsilon) = 0 & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } K \\ \partial_n u^\varepsilon(x) = g(\frac{x}{\varepsilon}) & \text{on } \partial\Omega. \end{cases} \quad (7.1)$$

There are basically two main results that they establish. The first is to obtain for each possible normal, n , the constant, $\bar{g}(n)$, from Theorem 1.1 in, $\Sigma^1(n)$, where now it is written explicitly that $\bar{g}(n)$ depends on the normal direction. This is basically a cell problem for the general domain. The second is to study the continuity properties of $\bar{g}(n)$ with respect to n .

With minor adaptations of our proofs above, we can also obtain this first result about the cell problem in [9]. That is, we also have the following theorem (cf. [9, Theorem 3.1])

Theorem 7.1. *Assume that F is \mathbb{Z}^{d+1} periodic in x , uniformly elliptic, satisfies basic assumptions for existence / uniqueness of u^ε , and $F(x, 0) \equiv 0$. Assume that instead of*

(1.1), u^ε now solves

$$\begin{cases} F(\frac{x}{\varepsilon}, D^2 u^\varepsilon) = 0 & \text{in } \Sigma^1 \\ u^\varepsilon = 0 & \text{on } \Sigma_1 \\ \partial_n u^\varepsilon(x) = g(\frac{x}{\varepsilon}) & \text{on } \Sigma_0. \end{cases} \quad (7.2)$$

Then the outcome Theorem 1.1 still holds true, where instead \bar{u} solves $\bar{F}(D^2 \bar{u}) = 0$ in Σ^1 , and \bar{F} is the same effective equation from the standard periodic homogenization theory (see [13]).

There are only two statements / techniques in our proof of Theorem 1.1 that need to be slightly modified to obtain Theorem 7.1: the statement of Lemma 5.1 and the proof of Lemma 6.1. Due to the fact that $F(x, 0) \equiv 0$, any affine function is a solution of $F(y, D^2 u) = 0$. Thus, Proposition 4.1 is in fact trivial in this case. The function ϕ^ε , from Proposition 4.1, is simply

$$\phi^\varepsilon(y) = 1/\varepsilon - y_{d+1}.$$

Thus, for this case, Lemma 5.1 is true with (see (5.1))

$$f^\varepsilon(y) = \varepsilon.$$

Finally, in the proof of Lemma 6.1, we see that the function, \hat{W}^ε , solves in the viscosity sense:

$$\begin{cases} \mathcal{M}^-(D^2 \hat{W}^\varepsilon) \leq 0 & \text{in } \Sigma^{1/\varepsilon - |\hat{z}|} \\ \mathcal{M}^+(D^2 \hat{W}^\varepsilon) \geq 0 & \text{in } \Sigma^{1/\varepsilon - |\hat{z}|} \\ |\hat{W}^\varepsilon| \leq C |\hat{z}|^\gamma & \text{on } \Sigma_{1/\varepsilon - |\hat{z}|} \\ |\partial_n \hat{W}^\varepsilon| \leq C |\hat{z}|^\gamma & \text{on } \Sigma_0. \end{cases}$$

Here \mathcal{M}^\pm are the Pucci extremal operators of fully nonlinear elliptic equations, see [7, Chp. 2]. We recall also that \hat{z} was chosen so that $|\hat{z}|^\gamma \leq \delta$. Thus the functions

$$\eta_{super}^\varepsilon(y) = C\delta(1/\varepsilon - y_{d+1}) + C\delta$$

and

$$\eta_{sub}^\varepsilon(y) = -C\delta(1/\varepsilon - y_{d+1}) - C\delta$$

are respectively upper and lower barriers for \hat{W}^ε . Thus it follows that

$$|\hat{W}^\varepsilon| \leq C \frac{\delta}{\varepsilon}.$$

The rest of the proofs now follow in the same fashion as for the linear case proved in Section 6. We note that nowhere else in Sections 5 and 6 was linearity used. In fact, the rest of the details are very similar to those in [20].

8. TWO NATURAL QUESTIONS

In the proof of Lemma 6.1, it was very convenient that A and B were periodic so that the translation by $\hat{\tau}$ kept the equations for V^ε and \hat{V}^ε the same. However, it seems that this should only be a convenience, and that in fact one could establish Lemma 6.1 when A , B , and g are only almost periodic functions. Is this true?

When the non-divergence equation in (1.1) is replaced by the operator, $\operatorname{div}(A(x/\varepsilon)\nabla u^\varepsilon) = 0$, the equation still has a comparison principle, even with bounded, measurable A (see [19, Chp. 8] or [25]). Thus the idea of using the D-to-N mapping is still plausible, but the main drawback could be the $C^{1,\gamma}$ regularity (i.e. Proposition A.2 and Corollary 4.6). Assume that A is only C^γ ; can the integro-differential method be utilized to cover the case of the oscillatory oblique Neumann condition?

APPENDIX A. VARIOUS USEFUL FACTS AND EXTRA DETAILS OF AUXILIARY RESULTS

Here we collect some useful facts, expand upon the invariant measure, m , show a result about rates, and give the details of the perturbed test function method as it pertains to the *interior* homogenization of (1.1).

A.1. Useful Facts. The following can be adapted from [11], combined with some details in [20, Appendix A]

Proposition A.1 (Comparison for the Neumann problem). *Assume u (“subsolution”) and v (“supersolution”) solve the following:*

$$M^+(D^2u) + C|\nabla u| \geq F(D^2u, \nabla u, x) \geq 0 \text{ in } \Sigma^1, \quad \partial_n u = f \text{ on } \Sigma_0, \text{ and } u = u_0 \text{ on } \Sigma_1$$

$$M^-(D^2v) - C|\nabla v| \leq F(D^2v, \nabla v, x) \leq 0 \text{ in } \Sigma^1, \quad \partial_n v = g \text{ on } \Sigma_0, \text{ and } v = v_0 \text{ on } \Sigma_1.$$

If $f - g \geq 0$ on Σ_0 and $u_0 \leq v_0$ on Σ_1 , then $u \leq v$ in $\overline{\Sigma^1}$.

The next result is simply a paraphrasing of those that appear in [26, Sec. 2,3,4], and standard modifications of the arguments in [26] yield the result as stated here. We note that for our purposes, there is no harm in using a Hölder exponent for g that is lower than the one obtained in the Krylov-Safonov theorem. Thus, in the result below, we may assume it is the same γ appearing for the Hölder norms for g , v , ∇v .

Proposition A.2 (Lieberman-Trudinger [26]). *Assume that $\lambda \operatorname{Id} \leq A \leq \Lambda \operatorname{Id}$, A , B , and g are all bounded and C^γ continuous, and that v solves*

$$\begin{cases} \operatorname{Tr}(A(y)D^2v) + B(y) \cdot \nabla v = 0 & \text{in } \Sigma^1 \\ \partial_n v = g & \text{on } \Sigma_0. \end{cases}$$

There exists a universal constant, $C_1(\lambda, \Lambda, d)$ such that

$$[v]_{C^\gamma(\overline{\Sigma^{1/2}})} \leq C_1 \left(\operatorname{osc}_{\Sigma^1}(v) + \|g\|_{L^\infty(\Sigma_0)} \right)$$

and

$$[v]_{C^{1,\gamma}(\overline{\Sigma^{1/2}})} \leq C_1 \left(\operatorname{osc}_{\Sigma^1}(v) + \|g\|_{C^\gamma(\Sigma_0)} \right).$$

The next theorem is essential for our method. It is not clear that it is essential for the homogenization to occur, but we use it in a critical way (but maybe there is another approach without it). As it appears in [6], A and B should both be $C^{1,\gamma}$, however, this is not necessary, and we give some details below.

Proposition A.3 ([6, Chp. 3, Sec. 5, Thm 5.1]). *There exists a universal constant, C , such that if $f \in C^{4,\gamma}(\Sigma_0)$, \bar{A} is the unique homogenized coefficients of (A.7) and Proposition A.8, w^ε and \bar{w} are the unique solutions of respectively*

$$\begin{cases} \operatorname{Tr}(A(\frac{x}{\varepsilon})D^2w^\varepsilon) + \frac{1}{\varepsilon}B(\frac{x}{\varepsilon}) \cdot \nabla w^\varepsilon = 0 & \text{in } \Sigma^1 \\ w^\varepsilon = 0 & \text{on } \Sigma_1 \\ w^\varepsilon = f & \text{on } \Sigma_0 \end{cases}$$

and

$$\begin{cases} \operatorname{Tr}(\bar{A}D^2\bar{w}) = 0 & \text{in } \Sigma^1 \\ \bar{w} = 0 & \text{on } \Sigma_1 \\ \bar{w} = f & \text{on } \Sigma_0, \end{cases}$$

then

$$\|w^\varepsilon - \bar{w}\|_{L^\infty(\Sigma^1)} \leq C\varepsilon.$$

Remark A.4. *This theorem also holds for the case of divergence equations, due to [2].*

A.2. The invariant measure. A crucial tool for homogenization is the existence of an invariant measure for L , which is mentioned in Remark 2.3. The existence and uniqueness follows almost identically to that of [6, Chp. 3, Thm. 3.4], and we include here the main ideas required to modify their proof to suit our assumptions that A and B are C^γ . The difference between the two arguments is minor; in [6], they converted the equation to a divergence form operator, whereas here we simply use the relevant estimates for the non-divergence setting.

We first note that a reworking of the results 9.11-9.14 in [19] show that in the setting of periodic functions, when f is periodic and u is a *periodic* strong solution of

$$Lu = f \text{ in } \mathbb{R}^{d+1},$$

then

$$\|u\|_{W^{2,2}([0,1]^{d+1})} \leq C(\|u\|_{L^2([0,1]^{d+1})} + \|f\|_{L^2([0,1]^{d+1})})$$

and also for $\sigma > 0$ large enough,

$$\|u\|_{W^{2,2}([0,1]^{d+1})} \leq \|Lu - \sigma u\|_{L^2([0,1]^{d+1})}.$$

Thus, in particular, any periodic $W^{2,2}([0,1]^{d+1})$ solution of

$$-\sigma u + Lu = f$$

is unique. For existence, we can take any *periodic* and Hölder continuous approximation of f , say f^δ , such that $f^\delta \rightarrow f$ in $L^2([0,1]^{d+1})$. Thus, there exists a unique, bounded, continuous, and periodic solution u^δ to

$$-\sigma u^\delta + Lu^\delta = f^\delta,$$

and moreover, $u^\delta \in C^{2,\gamma}$, so the equation holds classically. Thus, the above $W^{2,2}$ estimates are applicable, and they show that $\{u^\delta\}$ is Cauchy in $W^{2,2}([0, 1]^{d+1})$. Hence the limit of u^δ gives existence for the equation.

Up to this point, we have basically outlined the details that show the operator,

$$L_\sigma u = -\sigma u + Lu,$$

has the property that L_σ^{-1} is well defined,

$$L_\sigma^{-1} : L^2([0, 1]^{d+1}) \rightarrow L^2([0, 1]^{d+1}),$$

with the estimate

$$\|L_\sigma^{-1} f\|_{W^{2,2}([0,1]^{d+1})} \leq C \|f\|_{L^2([0,1]^{d+1})},$$

and hence is compact. The final remaining step in the Fredholm Alternative to obtain the existence of the invariant measure, m , is to determine the dimension of the space of solutions to

$$(I + \sigma(L_\sigma^{-1}))z = 0, \text{ for } z \in L^2([0, 1]^{d+1}).$$

As noted in [6, Chp. 3, Thm. 3.4], this is equivalent to determining the set of solutions to

$$Lz = 0 \text{ with } z \in W_{per}^{2,2}([0, 1]^{d+1}).$$

Just as in [6, Chp. 3, Thm. 3.4], one can use a boot strapping argument to raise the exponent in the L^p estimate for z to show that in fact, any such z is in $W_{per}^{2,(d+1)}([0, 1]^{d+1})$. At this point, we see that z extends to a global, periodic, bounded, and $W_{loc}^{2,(d+1)}$ solution of $Lz = 0$. Applying the Krylov-Safonov Theorem in, e.g. B_1 , i.e.

$$[z_R]_{C^\gamma(B_1)} \leq C \|z_R\|_{L^\infty(B_2)},$$

to $z_R(x) = z(Rx)$, we conclude that z must be a constant. Thus, there is a one parameter family of solutions to the adjoint equation that differ by a multiplicative constant, and m is selected as the unique one that is positive and has $\int_{[0,1]^{d+1}} m = 1$. We note that the positivity of m follows as in [6, Chp. 3, Thm. 3.4].

A.3. The perturbed test function method for the interior homogenization of (1.1). Just for clarity, we include the main arguments of the perturbed test function method for u^ε . What follows is a summary of the methods of [6, Chp. 3, sec 4.2, 5.1], combined with the perturbed test function method in [13].

We begin with the formal ansatz that assumes

$$u^\varepsilon(x) = \bar{u}(x, \frac{x}{\varepsilon}) + \varepsilon v(x, \frac{x}{\varepsilon}) + \varepsilon^2 w(x, \frac{x}{\varepsilon}) + o(\varepsilon^2).$$

Plugging this into (1.1), we see that

$$\begin{aligned}
& \text{Tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2u^\varepsilon\right) + \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla u^\varepsilon \\
&= A_{ij}\left(\frac{x}{\varepsilon}\right)\bar{u}_{x_ix_j}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}A_{ij}\left(\frac{x}{\varepsilon}\right)\left(\bar{u}_{x_iy_j}\left(x, \frac{x}{\varepsilon}\right) + \bar{u}_{y_ix_j}\left(x, \frac{x}{\varepsilon}\right)\right) + \frac{1}{\varepsilon^2}A_{ij}\left(\frac{x}{\varepsilon}\right)\bar{u}_{y_iy_j}\left(x, \frac{x}{\varepsilon}\right) \\
&+ \varepsilon A_{ij}\left(\frac{x}{\varepsilon}\right)v_{x_ix_j}\left(x, \frac{x}{\varepsilon}\right) + A_{ij}\left(\frac{x}{\varepsilon}\right)\left(v_{x_iy_j}\left(x, \frac{x}{\varepsilon}\right) + v_{y_ix_j}\left(x, \frac{x}{\varepsilon}\right)\right) + \frac{1}{\varepsilon}A_{ij}\left(\frac{x}{\varepsilon}\right)v_{y_iy_j}\left(x, \frac{x}{\varepsilon}\right) \\
&+ \varepsilon^2 A_{ij}\left(\frac{x}{\varepsilon}\right)w_{x_ix_j}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon A_{ij}\left(\frac{x}{\varepsilon}\right)\left(w_{x_iy_j}w\left(x, \frac{x}{\varepsilon}\right) + w_{y_ix_j}\left(x, \frac{x}{\varepsilon}\right)\right) + A_{ij}\left(\frac{x}{\varepsilon}\right)w_{y_iy_j}\left(x, \frac{x}{\varepsilon}\right) \\
&+ \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x \bar{u}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon^2}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y \bar{u}\left(x, \frac{x}{\varepsilon}\right) \\
&+ B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x v\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y v\left(x, \frac{x}{\varepsilon}\right) \\
&+ \varepsilon B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x w\left(x, \frac{x}{\varepsilon}\right) + B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y w\left(x, \frac{x}{\varepsilon}\right) \tag{A.1}
\end{aligned}$$

In order for there to be any hope of extracting a limit from this equation, we try to collect the negative powers of ε and set them to zero. First, we have

$$\frac{1}{\varepsilon^2}A_{ij}\left(\frac{x}{\varepsilon}\right)\bar{u}_{y_iy_j}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon^2}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y \bar{u}\left(x, \frac{x}{\varepsilon}\right) = 0$$

This implies that we can choose, with an abuse of notation,

$$\bar{u}\left(x, \frac{x}{\varepsilon}\right) = \bar{u}(x).$$

Hence both $\bar{u}_{x_ix_j} = 0$ and $\bar{u}_{y_ix_j} = 0$.

The $1/\varepsilon$ terms now become

$$\frac{1}{\varepsilon}A_{ij}\left(\frac{x}{\varepsilon}\right)v_{y_iy_j}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x \bar{u}\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y v\left(x, \frac{x}{\varepsilon}\right) = 0 \tag{A.2}$$

We let $p(x) = \nabla_x \bar{u}(x)$. We need to find a v so that

$$A_{ij}(y)v_{y_iy_j}(x, y) + B(y) \cdot \nabla_y v(x, y) = -B(y) \cdot p(x). \tag{A.3}$$

Let $\chi(y) = (\chi^1(y), \dots, \chi^{d+1}(y))^T$ solve

$$A_{ij}(y)\chi_{y_jy_j}^l + B(y)\nabla_y \chi^l(y) = -B^l(y), \tag{A.4}$$

which is possible because of Assumption 2.4,

$$\int_{[0,1]^{d+1}} B^l(y)m(y)dy = 0.$$

We note that χ^l are independent of \bar{u} ! Suppose that

$$v(x, y) = \sum_l \bar{u}_{x_l}(x)\chi^l(y) + \tilde{v}(x). \tag{A.5}$$

We note that in the previous and upcoming equation, x , is just a parameter which can be considered fixed, hence the notation $p = \nabla \bar{u}(x)$. However, the x dependence is relevant

when searching for the function, $w(x, y)$. Then we have

$$\begin{aligned}
& A_{ij}(y)v_{y_i y_j}(y) + \sum_i B^i(y)v_{y_i}(y) \\
&= \sum_l p_l(x)A_{ij}(y)\chi_{y_i y_j}^l(y) + \sum_{l,i} p_l(x)B^i(y)\chi_{y_i}^l(y) \\
&= \sum_l p_l(x) \left(A_{ij}(y)\chi_{y_i y_j}^l(y) + \sum_i B^i(y)\chi_{y_i}^l(y) \right) \\
&= \sum_l p_l(x) \left(A_{ij}(y)\chi_{y_i y_j}^l(y) + B(y) \cdot \nabla_y \chi^l(y) \right) \\
&= \sum_l p_l(x) (-B^l(y)) \\
&= -B(y) \cdot p(x)
\end{aligned}$$

Thus far we have now identified the function v , depending on \bar{u} . In what follows, we need to show that there is a particular choice of effective coefficients, \bar{a}_{ij} which will yield the existence of w . It is this compatibility condition for w , which involves the first corrector, v , that gives the effective equation for \bar{u} . After ignoring all terms with ε or ε^2 (which will be transparent from the perturbed test function proof below), what is left of

$$\text{Tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2u^\varepsilon\right) + \frac{1}{\varepsilon}B\left(\frac{x}{\varepsilon}\right) \cdot \nabla u^\varepsilon = 0,$$

now reads as

$$\begin{aligned}
& A_{ij}\left(\frac{x}{\varepsilon}\right)\bar{u}_{x_i x_j}(x) + A_{ij}\left(\frac{x}{\varepsilon}\right) \left(v_{x_i y_j}\left(x, \frac{x}{\varepsilon}\right) + v_{y_i x_j}\left(x, \frac{x}{\varepsilon}\right) \right) + A_{ij}\left(\frac{x}{\varepsilon}\right)w_{y_i y_j}\left(x, \frac{x}{\varepsilon}\right) \\
&+ B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x v\left(x, \frac{x}{\varepsilon}\right) + B\left(\frac{x}{\varepsilon}\right) \cdot \nabla_y w\left(x, \frac{x}{\varepsilon}\right) = 0.
\end{aligned}$$

Considering x fixed, setting $Q = D^2\bar{u}(x)$, and inserting v , we seek a w that satisfies

$$\begin{aligned}
& A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) \\
&= -A_{ij}(y)Q_{ij} - A_{ij}\left(\frac{x}{\varepsilon}\right) \left(v_{x_i y_j}\left(x, \frac{x}{\varepsilon}\right) + v_{y_i x_j}\left(x, \frac{x}{\varepsilon}\right) \right) - B(y) \cdot \nabla_x v(x, y),
\end{aligned}$$

or in terms of χ ,

$$\begin{aligned}
& A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) \\
&= -A_{ij}(y)Q_{ij} - A_{ij} \left(\bar{u}_{x_l x_i}(x)\chi_{y_j}^l(y) + \bar{u}_{x_l x_j}(x)\chi_{y_i}^l(y) \right) \\
&\quad - B^k(y)(\bar{u}_{x_l} \chi^l(y) + \tilde{v}(x))_{x_k}.
\end{aligned}$$

This further simplifies as

$$\begin{aligned}
& A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) \\
&= -A_{ij}(y)Q_{ij} - A_{ij}(y)\bar{u}_{x_l x_i}(x)\chi_{y_j}^l(y) - A_{ij}(y)\bar{u}_{x_l x_j}(x)\chi_{y_i}^l(y) \\
&\quad - B^k(y)\bar{u}_{x_l x_k}(x)\chi^l(y) - B^k(y)\tilde{v}_{x_k}(x).
\end{aligned}$$

Since we only have an equation in the y variable for w , in the best case scenario, we would seek a periodic w , solving

$$\begin{aligned} & A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) \\ &= -A_{ij}(y)Q_{ij} - A_{ij}Q_{li}\chi_{y_j}^l(y) - A_{ij}(y)Q_{lj}\chi_{y_i}^l(y) \\ &\quad - B^k(y)Q_{lk}\chi^l(y) - B^k(y)\tilde{v}_{x_k}(x). \end{aligned}$$

This is, of course, too strict. Thus, to relax the problem, we seek a unique choice of $\lambda(Q)$ to balance the right hand side. That is, we seek a unique choice of $\lambda(Q)$ such that there exists a w solving

$$\begin{aligned} & A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) \\ &= -A_{ij}(y)Q_{ij} - A_{ij}Q_{li}\chi_{y_j}^l(y) - A_{ij}(y)Q_{lj}\chi_{y_i}^l(y) \\ &\quad - B^k(y)Q_{lk}\chi^l(y) - B^k(y)\tilde{v}_{x_k}(x) + \lambda(Q). \end{aligned} \tag{A.6}$$

Since we already know that

$$\int_{[0,1]^{d+1}} B^k(y)m(y)dy = 0,$$

such a w can exist only if

$$\begin{aligned} & \int_{[0,1]^{d+1}} \lambda(Q)dy \\ & - \int_{[0,1]^{d+1}} \left(A_{ij}(y)Q_{ij} + A_{ij}(y)Q_{lj}\chi_{y_j}^l(y) + A_{ij}(y)Q_{li}\chi_{y_i}^l(y) + B^k(y)Q_{lk}\chi^l(y) \right) dy = 0 \end{aligned}$$

We note that this can be re-written in new index variables as

$$\begin{aligned} & \int_{[0,1]^{d+1}} \lambda(Q)m(y)dy \\ & - \int_{[0,1]^{d+1}} A_{mn}(y)Q_{mn} + A_{pm}(y)Q_{mn}\chi_{y_p}^n(y) + A_{pn}(y)Q_{mn}\chi_{y_p}^m(y) \\ & \quad + \frac{1}{2}(B^m(y)Q_{mn}\chi^n(y) + B^n(y)Q_{mn}\chi^m(y))m(y)dy = 0 \end{aligned}$$

We know that $\lambda(Q)$ will be linear in Q , so we will use the form

$$\lambda(Q) = \bar{a}_{mn}Q_{mn},$$

and hence we need

$$\begin{aligned} & Q_{mn} \int_{[0,1]^{d+1}} \bar{a}_{mn}m(y)dy \\ & - Q_{mn} \int_{[0,1]^{d+1}} A_{mn}(y) + A_{pm}(y)\chi_{y_p}^n(y) + A_{pn}(y)\chi_{y_p}^m(y) \\ & \quad + \frac{1}{2}(B^m(y)\chi^n(y) + B^n(y)\chi^m(y))m(y)dy = 0. \end{aligned}$$

We see that \bar{a}_{mn} must be uniquely chosen as

$$\begin{aligned} \bar{a}_{mn} = & \int_{[0,1]^{d+1}} A_{mn}(y) + A_{pm}(y)\chi_{y_p}^n(y) + A_{pn}(y)\chi_{y_p}^m(y) \\ & + \frac{1}{2}(B^m(y)\chi^n(y) + B^n(y)\chi^m(y))m(y)dy. \end{aligned} \quad (\text{A.7})$$

All of this work can be summarized in the following proposition.

Proposition A.5. *Assume that $\phi \in C^2$, $p(x) = \nabla\phi(x)$, and that $Q \in \mathcal{S}(d+1)$ is fixed. Define χ^l and v respectively by using ϕ_{x_i} in (A.4) and (A.5). There exists a unique choice of $\lambda(Q) \in \mathbb{R}$ such that (A.6) admits a (classical) periodic solution, w , as a function of y . In this instance, $\lambda(Q)$ is computed explicitly in terms of χ via (A.7), with*

$$\lambda(Q) = \bar{a}_{mn}Q_{mn}.$$

Remark A.6. *The previous statement is just a summary of the steps that culminate on [6, p. 416].*

Remark A.7. *We note that (A.6) indicates that \tilde{v} can be any reasonable function of x , and w can still be determined. Thus we are free to set, a posteriori, $\tilde{v}(x) \equiv 0$, hence making w a function of only y .*

Now we move onto implementing the perturbed test function method for this equation. It is just a rewriting of the details in [13] in our context. We claim that

Proposition A.8. *Assume that \bar{u} is any local uniform limit of u^ε in Σ^1 . Then \bar{u} must be a solution of*

$$\bar{a}_{ij}\bar{u}_{x_i x_j} = 0 \text{ in } \Sigma^1, \quad (\text{A.8})$$

with \bar{a}_{ij} defined in (A.7).

Proof. We will prove that \bar{u} is a (viscosity) subsolution of (A.8). Similarly one establishes that \bar{u} is a (viscosity) supersolution. We will proceed by contradiction and assume that \bar{u} is not a viscosity subsolution. That is, we assume that ϕ is smooth and bounded and that $\bar{u} - \phi$ attains a strict local max in $B_r(x_0)$ at x_0 , but for some $\delta > 0$

$$\bar{a}_{ij}\phi_{x_i x_j}(x_0) \leq -\delta < 0.$$

Let us take

$$p(x) = \nabla\phi(x) \text{ and } Q = D^2\phi(x_0),$$

and let χ^l , v , and w be as in Proposition A.5. We claim that

$$\psi^\varepsilon(x) = \phi(x) + \varepsilon v(x, \frac{x}{\varepsilon}) + \varepsilon^2 w^\varepsilon(\frac{x}{\varepsilon})$$

is in fact a viscosity supersolution of

$$\text{Tr}(A(\frac{x}{\varepsilon})D^2\psi^\varepsilon) + \frac{1}{\varepsilon}B(\frac{x}{\varepsilon}) \cdot \nabla\psi^\varepsilon = 0 \text{ in } B_\rho(x_0),$$

for some $\rho > 0$ appropriately small, depending on $\|\phi\|_{C^{2,\gamma}(B_r(x_0))}$, $\|v\|_{C^{1,1}}$, and $\|w\|_{C^{1,1}}$ (in fact, ψ^ε is a classical solution, but we only care about a class of solutions that satisfies a comparison theorem). We note that the contradiction assumption for ϕ can be restated as

$$\lambda(D^2\phi(x_0)) \leq -\delta.$$

We also note by the boundedness of A , B , χ^l , and $\nabla\chi^l$, that for all $x \in B_\rho(x_0)$, we can effectively localize the equation at x_0 because

$$\begin{aligned} & |A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x) + A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_i}(x)\chi_{y_j}^l(\frac{x}{\varepsilon}) + A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x)\chi_{y_i}^l(\frac{x}{\varepsilon}) + B^k(\frac{x}{\varepsilon})\phi_{x_i x_k}(x)\chi^l(\frac{x}{\varepsilon}) \\ & - \left(A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x_0) + A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_i}(x_0)\chi_{y_j}^l(\frac{x}{\varepsilon}) + A_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x_0)\chi_{y_i}^l(\frac{x}{\varepsilon}) + B^k(\frac{x}{\varepsilon})\phi_{x_i x_k}(x_0)\chi^l(\frac{x}{\varepsilon}) \right) | \\ & \leq \frac{\delta}{4}. \end{aligned}$$

Furthermore, we can possibly restrict ρ to be smaller, depending upon the $C^{1,1}$ norms of v and w so that

$$\begin{aligned} & |\varepsilon A_{ij}(\frac{x}{\varepsilon})v_{x_i x_j}(\frac{x}{\varepsilon}) + \varepsilon^2 A_{ij}(\frac{x}{\varepsilon})w_{x_i x_j}(\frac{x}{\varepsilon}) + \varepsilon A_{ij}(\frac{x}{\varepsilon}) \left(w_{x_i y_j} w(\frac{x}{\varepsilon}) + w_{y_i x_j}(\frac{x}{\varepsilon}) \right) \\ & + \varepsilon B(\frac{x}{\varepsilon}) \cdot \nabla_x w(\frac{x}{\varepsilon})| \leq \frac{\delta}{4} \end{aligned}$$

Hence, plugging ψ^ε into (A.1), using the particular choices of v and w , and inspecting, we see that in $B_\rho(x_0)$

$$\begin{aligned} & \text{Tr}(A(\frac{x}{\varepsilon})D^2\psi^\varepsilon) + \frac{1}{\varepsilon}B(\frac{x}{\varepsilon}) \cdot \nabla\psi^\varepsilon \\ & \leq A_{ij}(y)w_{y_i y_j}(x, y) + B(y) \cdot \nabla_y w(x, y) + A_{ij}(y)Q_{ij} + A_{ij}Q_{li}\chi_{y_j}^l(y) + A_{ij}(y)Q_{lj}\chi_{y_i}^l(y) \\ & \quad + B^k(y)Q_{lk}\chi^l(y) + B^k(y)\tilde{v}_{x_k}(x) + \frac{\delta}{4} + \frac{\delta}{4} \\ & \leq \lambda(Q) + \frac{\delta}{2} < 0. \end{aligned}$$

We importantly note that by the construction of v , we had exact equality in the equation (A.3), thus canceling these $1/\varepsilon$ terms above, which are the same as in (A.2). Thus we conclude by the comparison of sub and super solutions that

$$u^\varepsilon(x_0) - \phi^\varepsilon(x_0) \leq \max_{B_\rho(x_0)} u^\varepsilon - \psi^\varepsilon \leq \max_{\partial B_\rho(x_0)} u^\varepsilon - \psi^\varepsilon.$$

Now we note that $u^\varepsilon \rightarrow \bar{u}$ and $\psi^\varepsilon \rightarrow \phi$ uniformly in $\overline{B_\rho(x_0)}$. Thus

$$\bar{u}(x_0) - \phi(x_0) \leq \max_{\partial B_\rho(x_0)} \bar{u} - \phi,$$

which is a contradiction to the strict max at x_0 . Thus, we see that in fact \bar{u} is a subsolution of (A.8). This concludes the proof of proposition A.8. \square

A.4. The rate of convergence for the regular homogenization. Here we mention how some minor modifications to the arguments in [6, Chp. 3, Sec 5] yield the rate of convergence under our assumptions on A and B .

First, we note that in the proof of Proposition A.8, the expansion

$$\psi^\varepsilon(x) = \phi(x) + \varepsilon v(x, \frac{x}{\varepsilon}) + \varepsilon^2 w^\varepsilon(\frac{x}{\varepsilon})$$

is only used locally, and we did not utilize x dependence for w or the values of \bar{u} (but \bar{u} implicitly plays a role through the ϕ). However, as pointed out on [6, p.418-419], you can get much more information out of this expansion by using a better choice for w .

In all that follows, [6, p.418-419] have more coefficients in L (they call their operator A) than we do. In the context of their notation, we have for A and B given in this work, and a, b, c, a_0 in [6],

$$a_{ij}(y) = A_{ij}(y), \quad b_i(y) = B_i(y), \quad c_i(y) = a_0 \equiv 0.$$

Assume that w^ε and \bar{w} are as in the statement of Proposition A.3; here w^ε and \bar{w} play the role of respectively u^ε and u in [6, p.418-419]. We note that \bar{A} is a constant coefficient and uniformly elliptic matrix, and thus \bar{w} is locally smooth and globally as smooth as is f (the Dirichlet data), in particular $\|\bar{w}\|_{C^{4,\gamma}(\bar{\Sigma}^1)} \leq C\|f\|_{C^{4,\gamma}(\Sigma_0)}$. The good expansion is

$$\tilde{w}^\varepsilon(x) = \bar{w}(x) + \varepsilon v(x, \frac{x}{\varepsilon}) + \varepsilon^2 w_2(x, \frac{x}{\varepsilon}),$$

where v is defined in (A.3) using $p(x) = \nabla \bar{w}(x)$, and w_2 is defined as

$$w_2(x, y) = \bar{w}_{x_i x_j}(x) \chi^{ij}(y),$$

and χ^{ij} (note these are different than χ^i with one upper index) are chosen to solve

$$\begin{aligned} \text{Tr}(A(y)D^2\chi^{ij}(y)) + B(y) \cdot \nabla \chi^{ij}(y) = \\ \bar{a}_{ij} - \{A_{ij}(y) + A_{kj}(y)\chi_{y_k}^i(y) + A_{ki}(y)\chi_{y_k}^j(y) + \frac{1}{2}(B^i(y)\chi^j(y) + B^j(y)\chi^i(y))\}. \end{aligned}$$

Now, we note that A and B periodic and in C^γ implies that χ^l , and as a result also χ^{ij} , are all periodic and $C^{2,\gamma}(\Sigma^1)$.

The final step is to compute the equation for

$$z^\varepsilon = w^\varepsilon - \tilde{w}^\varepsilon,$$

which can be followed directly in [6, p. 418, eq. (5.23)], for

$$L(w^\varepsilon - \tilde{w}^\varepsilon) = \varepsilon g^\varepsilon.$$

We use the same definition of g^ε as in [6, p.418] (note here it is significantly simpler due to the absence of c_i and a_0). The important thing to observe about g^ε is that it involves: the coefficients, A and B ; the function χ^l ; the function and up to one derivative of χ^{ij} ; third and fourth derivatives of \bar{w} . Thus by the regularity that is noted above, $\|g^\varepsilon\|_{L^\infty} \leq C$, independently of ε . Furthermore, $(w^\varepsilon - \tilde{w}^\varepsilon)|_{\Sigma_0 \cup \Sigma_1} \leq C\varepsilon$ (which can be checked by a simple calculation). Thus we conclude

$$\|z^\varepsilon\|_{L^\infty(\Sigma^1)} \leq \varepsilon C,$$

and this implies

$$\|w^\varepsilon - \bar{w}\|_{L^\infty(\Sigma^1)} \leq \varepsilon C.$$

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