CHAPTER IX: A JACOBIAN CRITERION FOR FORMAL SMOOTHNESS

Let $u:(R,m,k) \longrightarrow (S,n,\ell)$ be a local morphism of local Noetherian rings. Our goal is to prove the equivalence of the following two conditions:
(a) S is n-smooth over R.
(b) u is flat and $S \otimes_R k$ is geometrically regular over k.
In this chapter we will show that (b) $\Rightarrow$ (a). The forward direction (a) $\Rightarrow$ (b) will be proved in Chapter XI.

§1: FORMALLY LEFT INVERTIBLE MAPS

Let R be a ring, $I \subseteq R$ an ideal, and M an R-module. The I-adic topology on M is defined as follows: $\mathcal{U} \subseteq M$ is open if and only if for all $m \in \mathcal{U}$ there is an $n \in \mathbb{N}$ so that $m + I^n M \subseteq \mathcal{U}$. Note that:
(a) $I^n M$ is an open and closed subset of M.
(b) If $u: M \longrightarrow N$ is an R-linear map of R-modules then for all $n \in \mathbb{N}$ $u(I^n M) \subseteq I^n N$. u is continuous in the I-adic topologies of M and N.

(9.1) Definition: Let R be a ring, $I \subseteq R$ an ideal, and $u: M \longrightarrow N$ an R-linear map of R-modules. u is called formally left invertible with respect to the I-adic topology or I-left invertible if the following condition is satisfied:
Let $v: M \longrightarrow E$ be an R-linear map and suppose that v is continuous when M is equipped with the I-adic topology and E with the discrete topology. Then there is a continuous R-linear map $w: N \longrightarrow E$ where N carries the I-adic topology and E the discrete topology, so that $v = wu$.

(9.2) <u>Remark</u>: (a) Let M, E be R-modules, M equipped with the I-adic topology and E with the discrete topology. An R-linear map $v: M \longrightarrow E$ is continuous if and only if $v(I^n M) = 0$ for some $n \in \mathbb{N}$.
(b) Let $u: M \longrightarrow N$ be an R-linear map and $u': N \longrightarrow M$ a left inverse of u, that is, $u'u = id_M$. Then for every R-linear map $v: M \longrightarrow E$ the map $w = vu'$ satisfies $wu = vu'u = v$. Every R-linear map which has a left inverse has an I-left inverse for every ideal $I \subseteq R$.
(c) Conversely, let $u: M \longrightarrow N$ be an R-linear maps of R-modules and suppose that for every R-linear map $v: M \longrightarrow E$ there is an R-linear map $w: N \longrightarrow E$ with $wu = v$. Then u has a left inverse (set $v = id_M$).

(9.3) <u>Proposition</u>: Let R be a ring, $I \subseteq R$ an ideal, and $u: M \longrightarrow N$ an R-linear map of R-modules. The following conditions are equivalent:
(a) u is I-left invertible.
(b) For all $n \in \mathbb{N}$ the induced map $u_n: M/I^n M \longrightarrow N/I^n N$ is left invertible.

<u>Proof</u>: (a) $\Rightarrow$ (b): Let $n \in \mathbb{N}$ and set $\overline{M} = M/I^n M$, $\overline{N} = N/I^n N$. Furthermore let $\nu: M \longrightarrow \overline{M}$ and $\mu: N \longrightarrow \overline{N}$ denote the natural maps. $\nu: M \longrightarrow \overline{M}$ is continuous in the I-adic topology for M and the discrete topology for $\overline{M}$. Thus there is an R-linear map $w: N \longrightarrow \overline{M}$ with $wu = \nu$.

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "u"'] & \overline{M} \\ N \arrow[r, "\mu"'] \arrow[ur, "w"] & \overline{N} \arrow[u, "\overline{w}"'] \end{tikzcd}$$

w factors through $\overline{N}$, say $w = \overline{w}\mu$. Moreover, μu factors through $\overline{M}$ and $\mu u = u_n \nu$. Therefore $\nu = wu = \overline{w}\mu u = \overline{w} u_n \nu$ and $\overline{w} u_n = id_{\overline{M}}$.

(b) $\Rightarrow$ (a): Let $v: M \longrightarrow E$ be an R-linear map with $v(I^n M) = 0$. v factors through $M/I^n M$:

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "\nu"'] & E \\ M/I^n M \arrow[ur, "\bar{v}"'] & \end{tikzcd}$$

where ν is the natural map. Let $u_n': N/I^n N \longrightarrow M/I^n M$ be a left inverse of u_n, that is, $u_n' u_n = \mathrm{id}_{M/I^n M}$. Set $w = \bar{v}\, u_n' \mu$ where $\mu: N \longrightarrow N/I^n N$ is the natural map. Then $wu = \bar{v} u_n' \mu u = \bar{v} u_n' u_n \nu = \bar{v}\nu = v$ and u is I-left invertible.

(9.4) Definition: Let R be a ring, $I \subseteq R$ an ideal, and M an R-module. M is called formally projective over R with respect to the I-adic topology or I-projective if for all $n \in \mathbb{N}$ the R/I^n-module $M/I^n M$ is projective.

(9.5) Lemma: Let R be a ring, $I \subseteq R$ an ideal and $u: M \longrightarrow N$ an R-linear map of R-modules. Suppose that N is I-projective. Then the following conditions are equivalent:

(a) u is I-left invertible

(b) The induced map of R/I-modules $u_1: M/IM \longrightarrow N/IN$ is left invertible.

Proof: (a) $\Rightarrow$ (b): By (9.3).

(b) $\Rightarrow$ (a): By (9.3) it suffices to show by induction on $n \in \mathbb{N}$ that the induced map of R/I^n-modules $u_n: M/I^n M \longrightarrow N/I^n N$ is left invertible. Set $M_n = M/I^n M$, $N_n = N/I^n N$ and let $\nu_n: M_n \longrightarrow M_{n-1}$, $\mu_n: N_n \longrightarrow N_{n-1}$ denote the natural maps. Suppose that $u_{n-1}: M_{n-1} \longrightarrow N_{n-1}$ is left invertible and let $v_{n-1}: N_{n-1} \longrightarrow M_{n-1}$ be an R-linear map with $v_{n-1} u_{n-1} = \mathrm{id}_{M_{n-1}}$.

We have maps:

$$\begin{tikzcd} N_n \arrow[r, "\mu_n"] & N_{n-1} \arrow[d, "v_{n-1}"] \\ M_n \arrow[r, "\nu_n"] & M_{n-1} \end{tikzcd}$$

Since N_n is R/I^n-projective there is an R-linear map $w_n : N_n \longrightarrow M_n$ with $v_n w_n = v_{n-1} \mu_n$.

We claim that $h = w_n u_n$ is bijective. First note that $h \otimes R/I^{n-1} = id_{M_{n-1}}$. Hence for all $x \in M_n$, $h(x) - x \in I^{n-1} M_n$. Write

$$h(x) - x = \sum_{i=1}^{r} a_i y_i$$

where $a_i \in I^{n-1}$ and $y_i \in M_n$. Then

$$\begin{aligned} h(h(x)-x) &= h^2(x) - h(x) \\ &= \sum_{i=1}^{r} a_i h(y_i) \\ &\overset{(*)}{=} \sum_{i=1}^{r} a_i y_i = h(x) - x. \end{aligned}$$

$(*)$ follows since $h(y_i) \in y_i + I^{n-1} M_n$ and $n \geq 2$. Therefore $x = 2h(x) - h^2(x) = h(2x - h(x))$ and h is bijective. Set $v_n = h^{-1} w_n$. Then $v_n u_n = h^{-1} w_n u_n = h^{-1} h = id_{M_n}$ and u_n is left invertible.

§2: A JACOBIAN CRITERION FOR FORMAL SMOOTHNESS

Let $u: R \longrightarrow S$ be a morphism of rings. Suppose that u factors: $u = \varepsilon\gamma : R \xrightarrow{\gamma} T \xrightarrow{\varepsilon} S$ where T is 0-smooth over R and ε is surjective. Such a factorization is always possible, for example, we can choose T to be a polynomial ring over R, $T = R[x_i]_{i\in\Lambda}$. Let $I = \ker(\varepsilon)$ and $S \cong T/I$, then by (1.12) there is an exact sequence of S-modules:
$I/I^2 \xrightarrow{\delta} \Omega_{T/R} \otimes_T S \longrightarrow \Omega_{S/R} \longrightarrow 0$ where δ is the natural map induced by the universal derivation $d: T \longrightarrow \Omega_{T/R}$. By (2.15) $\Omega_{T/R}$ is a projective T-module and $\Omega_{T/R} \otimes_T S$ is a projective S-module.

Let $K \subseteq S$ be an ideal and set $J = \varepsilon^{-1}(K)$. The T-module I/I^2 carries two (possibly different) topologies induced by J. The first is the J-adic topology with fundamental open sets $J^n(I/I^2)$ for $n \in \mathbb{N}$. The second is the topology induced by the J-adic topology of T/I^2 onto the ideal I/I^2. We call this topology the (J)-topology of I/I^2. A fundamental system of open subsets of I/I^2 in the (J)-topology is given by:
For all $r \in \mathbb{N}$:

$$I \cap J^r + I^2/I^2 \cong I \cap J^r / I^2 \cap J.$$

Then a subset $U \subseteq I/I^2$ is open if and only if for all $a \in U$ there is an $r \in \mathbb{N}$ so that $a + I\cap J^r + I^2/I^2 \subseteq U$. Note that the natural map $\delta: I/I^2 \longrightarrow \Omega_{T/R} \otimes_T S$ is continuous if I/I^2 carries the (J)-topology and $\Omega_{T/R} \otimes_T S$ the K-adic topology.

(9.6) Remark: If R is Noetherian and S an R-algebra of finite type then one can choose for T a polynomial ring in finitely many variables over R. In this case T is Noetherian and by Artin-Rees the (J)-topology and the J-adic topology on I/I^2 are the same.

Similar to definition (9.1) we say that the S-linear map $\delta : I/I^2 \longrightarrow \Omega_{T/R} \otimes_T S$ is $(\mathcal{J})$-<u>left invertible</u> if for every S-module E and every continuous map $v : I/I^2 \longrightarrow E$ where I/I^2 carries the $(\mathcal{J})$-topology and E the discrete topology, there is an S-linear map $w : \Omega_{T/R} \otimes_T S \longrightarrow E$ so that $w\delta = v$.

(9.7) <u>Lemma:</u> Assumptions as above. For all $r \in \mathbb{N}$ let

$$\delta_r : I/I^2 + I \cap \mathcal{J}^r \longrightarrow \Omega_{T/R} \otimes_T S/K^{r-1}$$

denote the S-linear map induced by δ. Then the following conditions are equivalent:

(a) δ is $(\mathcal{J})$-left invertible.

(b) For all $r \in \mathbb{N}$, $r > 1$, there is an S-linear map

$$\gamma_{r-1} : \Omega_{T/R} \otimes_T S/K^{r-1} \longrightarrow I/I^2 + I \cap \mathcal{J}^{r-1}$$

so that

$$\sigma_r = \gamma_{r-1}\delta_r : I/I^2 + I \cap \mathcal{J}^r \longrightarrow I/I^2 + I \cap \mathcal{J}^{r-1}$$

is the natural map.

<u>Proof:</u> (a) $\Rightarrow$ (b): Let $\nu_{r-1} : I/I^2 \longrightarrow I/I^2 + I \cap \mathcal{J}^{r-1}$ be the natural map. ν_{r-1} is continuous in the $(\mathcal{J})$-topology of I/I^2 and the discrete topology of $I/I^2 + I \cap \mathcal{J}^{r-1}$. By (a) there is an S-linear map: $w : \Omega_{T/R} \otimes_T S \longrightarrow I/I^2 + I \cap \mathcal{J}^{r-1}$ so that $w\delta = \nu_{r-1}$. w factors through $\Omega_{T/R} \otimes_T S/K^{r-1}$ and we obtain a commutative diagram:

$$\begin{tikzcd}
I/I^2 \arrow[r, "\nu_{r-1}"] \arrow[d, "\delta"'] & I/I^2 + I \cap \mathcal{J}^{r-1} \\
\Omega_{T/R} \otimes_T S \arrow[ur, "w"] \arrow[r, "\mu_{r-1}"'] & \Omega_{T/R} \otimes_T S/K^{r-1} \arrow[u, "\gamma_{r-1}"']
\end{tikzcd}$$

Moreover, $\mu_{r-1}\delta$ factors through $I/I^2 + I \cap \mathcal{J}^r$, that is, $\mu_{r-1}\delta = \delta_r \nu_r$ where ν_r is the natural map. Then $\nu_{r-1} = w\delta = \gamma_{r-1}\mu_{r-1}\delta = \gamma_{r-1}\delta_r\nu_r$ and

$\gamma_r \delta_{r-1}$ is the natural map $\sigma_r : I/I^2 + I\cap J^r \longrightarrow I/I^2 + I \cap J^{r-1}$.

(b) $\Rightarrow$ (a): Let $v : I/I^2 \longrightarrow E$ be a continuous S-linear map where I/I^2 carries the (J)-topology and E the discrete topology. Then there is an $r \in \mathbb{N}$ so that $v(I^2 + I\cap J^{r-1}/I^2) = 0$ and v factors through $I/I^2 + I\cap J^{r-1}$, say $v = \bar{v}\, \nu_{r-1}$ where $\nu_{r-1} : I/I^2 \longrightarrow I/I^2 + I\cap J^{r-1}$ is the natural map. We obtain a commutative diagram:

$$\begin{tikzcd}
I/I^2 \arrow[r, "\nu_{r-1}"] \arrow[d, "\nu_r"'] & I/I^2 + I\cap J^{r-1} \arrow[r, "\bar{v}"] & E \\
I/I^2 + I\cap J^r \arrow[ur, "\sigma_r"] \arrow[r, "\delta_r"'] & \Omega_{T/R} \otimes_T S/K^{r-1} \arrow[u, "\gamma_{r-1}"'] & \Omega_{T/R} \otimes_T S \arrow[l, "\mu_{r-1}"']
\end{tikzcd}$$

where σ_r, μ_{r-1} are the natural maps. Set $W = \bar{v}\, \gamma_{r-1} \mu_{r-1}$. Then $W\delta = \bar{v}\, \gamma_{r-1} \mu_{r-1} \delta$ $= \bar{v}\, \gamma_{r-1} \delta_r \nu_r = \bar{v}\, \sigma_r \nu_r = \bar{v}\, \nu_{r-1} = v$ and δ is (J)-left invertible.

(9.8) <u>Theorem</u>: (Jacobian criterion for formal smoothness) With notations as above the following conditions are equivalent:

(a) S is K-smooth over R.

(b) δ is (J)-left invertible.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (9.7) it suffices to show that for all $r \in \mathbb{N}$, $r > 1$, there is an S-linear map $\gamma_{r-1} : \Omega_{T/R} \otimes_T S/K^{r-1} \longrightarrow I/I^2 + I\cap J^{r-1}$ so that the composition $\gamma_{r-1}\delta_r = \sigma$ is the natural map $\sigma : I/I^2 + I\cap J^r \longrightarrow I/I^2 + I\cap J^{r-1}$ and δ_r is the map $\delta_r : I/I^2 + I\cap J^r \longrightarrow \Omega_{T/R} \otimes_T S/K^{r-1}$ induced by δ. Consider the commutative diagram of R-algebra morphisms:

$$\begin{tikzcd}
S \arrow[r, "\gamma"] & S/K^{r-1} \\
R \arrow[u] \arrow[r, "\tau"] & T/I^2 + J^{r-1} \arrow[u, "\nu"']
\end{tikzcd}$$

where γ, τ, ν are the natural maps. The ideal $\ker \nu = I + J^{r-1}/I^2 + J^{r-1}$ is nilpotent. Moreover, γ is continuous when S carries the K-adic topology and S/K^{r-1} the discrete topology. Since S is K-smooth over R, there is an R-algebra morphism $w: S \longrightarrow T/I^2+J^{r-1}$ which lifts γ. Consider the R-algebra morphism $v = w\varepsilon : T \xrightarrow{\varepsilon} S \xrightarrow{w} T/I^2+J^{r-1}$ where $\varepsilon : T \longrightarrow S$ is the natural map. Let $\rho : T \longrightarrow T/I^2+J^{r-1}$ be the natural map and consider $D = \rho - v : T \longrightarrow T/I^2+J^{r-1}$.

Obviously, D is R-linear with $D|_R = 0$ and $\operatorname{im}(D) \subseteq I+J^{r-1}/I^2+I\cap J^{r-1}$. (since $\nu D = \nu\rho - \nu v = \nu\rho - \nu w\varepsilon = \nu\rho - \gamma\varepsilon = 0$).

Let $y \in I/I^2+I\cap J^{r-1}$ and $x \in T$. Since $(\rho - v)(x) \in I/I^2+I\cap J^{r-1}$ and $(I/I^2+I\cap J^{r-1})^2 = 0$, we have that $\rho(x)y = v(x)y$. Hence for $x,z \in T$:

$$
\begin{aligned}
D(xz) &= \rho(xz) - v(xz) \\
&= \rho(x)\rho(z) - v(x)v(z) \\
&= \rho(x)(\rho(z) - v(z)) + v(z)(\rho(x) - v(x)) \\
&= \rho(x)D(z) + \rho(z)D(x) \\
&= xD(z) + zD(x)
\end{aligned}
$$

and D is an R-derivation from T into $I/I^2+I\cap J^{r-1}$. Hence D corresponds to a T-linear map: $h : \Omega_{T/R} \longrightarrow I/I^2+I\cap J^{r-1}$ so that the diagram:

$$
\begin{tikzcd}
T \arrow[r, "d"] \arrow[d, "D"'] & \Omega_{T/R} \arrow[dl, "h"] \\
I/I^2+I\cap J^{r-1} &
\end{tikzcd}
$$

commutes where d is the universal R-derivation of T. Since $h(I\Omega_{T/R}) = 0$ and $h(J^{r-1}\Omega_{T/R}) = 0$, h factors through $\Omega_{T/R} \otimes_T S/K^{r-1}$:

$$
\begin{tikzcd}
\Omega_{T/R} \arrow[r, "\lambda"] \arrow[d, "h"'] & \Omega_{T/R} \otimes_T S/K^{r-1} \arrow[dl, "\gamma_{r-1}"] \\
I/I^2+I\cap J^{r-1} &
\end{tikzcd}
$$

where λ is the natural map. It remains to show that

$$\gamma_{r-1}\delta_r = \sigma : I/I^2+I\cap J^r \longrightarrow I/I^2+I\cap J^{r-1}$$

is the natural map. Let $x \in I$ and $\overline{x} = x + I^2 + I\cap J^r \in I/I^2+I\cap J^r$. Then $\delta_r(\overline{x}) = d(x) + (I+J^{r-1})\Omega_{T/R}$ and (by diagram chasing)

$$\begin{aligned}\gamma_{r-1}\delta_r(\bar{x}) &= D(x)\\ &= \rho(x) - v(x)\\ &= \rho(x)\\ &= \sigma(\bar{x}).\end{aligned}$$

$\gamma_{r-1}\delta_r$ is the natural map.

(b) $\Rightarrow$ (a): Let C be an R-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of morphisms of rings:

$$\begin{tikzcd} S \arrow[r, "\bar{v}"] & C/N \\ T \arrow[u, "\varepsilon"] \arrow[dr, "w"] & \\ R \arrow[u] \arrow[r] & C \arrow[uu, "\lambda"'] \end{tikzcd}$$

where λ is the natural map. Suppose that $\bar{v}$ is continuous in the K-adic topology of S and the discrete topology of C/N, that is, $\bar{v}(K^r) = 0$ for some $r \in \mathbb{N}$. Since T is smooth over R, $\bar{v}\varepsilon$ lifts to an R-algebra morphism w, i.e. $\lambda w = \bar{v}\varepsilon$. Hence $\lambda(w(I)) = 0$ and $w(I) \subseteq N$. Since $w(I^2) \subseteq N^2 = 0$, w induces an R-linear map: $g: I/I^2 \longrightarrow N$, defined by $g(y + I^2) = w(y)$. Consider N as a T-module via w. Since $w(I)N \subseteq N^2 = 0$, N is an S-module and g is S-linear. Moreover, since $\bar{v}(K^r) = 0$ for some $r > 0$, we have that $w(J^r) \subseteq N$ and therefore $w(J^{2r}) = 0$. Hence g factors through $I/I^2 + I \cap J^{2r}$, say:

$$\begin{tikzcd} I/I^2 \arrow[r, "g"] \arrow[d, "\tau"] & N \\ I/I^2 + I \cap J^{2r} \arrow[ur, "h"'] & \end{tikzcd}$$

where τ is the natural map and $g = h\tau$.

Let $\delta_{2r+1} : I/I^2 + I \cap J^{2r+1} \longrightarrow \Omega_{T/R} \otimes_T S/K^{2r}$ be the S-linear map which is induced by the universal R-derivation $d: T \longrightarrow \Omega_{T/R}$. By assumption (b) there is an S-linear map:

$$\gamma_{2r} : \Omega_{T/R} \otimes_T S/K^{2r} \longrightarrow I/I^2 + I \cap J^{2r}$$

so that $\gamma_{2r}\delta_{2r+1} = \sigma_{2r+1} : I/I^2 + I\cap J^{2r+1} \longrightarrow I/I^2 + I \cap J^{2r}$ is the natural map. Consider the composition $f = h\gamma_{2r} : \Omega_{T/R} \otimes_T S/K^{2r} \longrightarrow N$ and let $\rho : \Omega_{T/R} \longrightarrow \Omega_{T/R} \otimes_T S/K^{2r}$ be the natural map. Set $D = f\rho d : T \longrightarrow N$ where $d : T \longrightarrow \Omega_{T/R}$ is the universal R-derivation. In particular, D is an R-derivation.

Let $x \in I$ and $\bar{x} = x + I^2 + I \cap J^{2r+1} \in I/I^2 + I\cap J^{2r+1}$. Then $\rho d(x) = \delta_{2r+1}(\bar{x})$ and $D(x) = f\delta_{2r+1}(\bar{x}) = h\gamma_{2r}\delta_{2r+1}(\bar{x}) = h\sigma_{2r+1}(\bar{x}) = w(x+I^2)$. Hence $(w-D)|_I = 0$. Moreover, since $\mathrm{im}(D) \subseteq N$ and $N^2 = 0$, $w - D$ is an R-algebra morphism. Thus $w-D$ factors through S:

$$\begin{tikzcd} T \arrow[r, "w-D"] \arrow[d, "\varepsilon"'] & C \\ S \arrow[ur] & \end{tikzcd}$$

that is, $v\varepsilon = w - D$.

We claim that v lifts $\bar{v}$. Let $y \in S$ and $y_0 \in T$ with $\varepsilon(y_0) = y$. Then

$$\begin{aligned} \lambda v(y) &= \lambda(w-D)(y_0) \\ &= \lambda w(y_0) \qquad\qquad \text{since } D(y_0) \in N \\ &= \bar{v}\varepsilon(y_0) \\ &= \bar{v}(y). \end{aligned}$$

This proves the theorem.

(9.9) <u>Remark</u>: If S is 0-smooth over R then by (9.8) δ is left invertible and the sequence: $0 \rightarrow I/I^2 \longrightarrow \Omega_{T/R} \otimes_T S \longrightarrow \Omega_{S/R} \longrightarrow 0$ is split exact. In particular, $\Omega_{S/R}$ is a projective S-module. The forward direction (a) $\Rightarrow$ (b) of Theorem (9.8) is a generalization of (2.15) to formal smoothness.

§3: APPLICATIONS

(9.10) <u>Lemma</u>: Let R be a ring and $I, J \subseteq R$ ideals with R/I Noetherian and J finitely generated. Then for all $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ so that $J^m \cap I \subseteq J^n I$.

<u>Proof</u>: First note that for every ideal $K \subseteq R$ with $J \subseteq \mathrm{rad}(K)$ there is an $n \in \mathbb{N}$ with $J^n \subseteq K$ since J is finitely generated. Hence it suffices to show for all $n \in \mathbb{N}$ there is an ideal $K \subseteq R$ with $J \subseteq \mathrm{rad}(K)$ and $K \cap I \subseteq J^n I$.
Fix $n \in \mathbb{N}$ and consider the set:
$\mathcal{M} = \{ K \subseteq R \mid K \text{ an ideal with } J^n I \subseteq K \text{ and } K \cap I \subseteq J^n I \}$.
Since $J^n I \in \mathcal{M}$, the set $\mathcal{M}$ is not empty. $\mathcal{M}$ is partially ordered by inclusion and let $\mathcal{K} = \{K_i\}_{i \in \Lambda}$ be a well-ordered subset of $\mathcal{M}$. Then $K_0 = \bigcup_{i \in \Lambda} K_i$ is an ideal of R with $K_0 \in \mathcal{M}$. By Zorn's Lemma $\mathcal{M}$ contains a maximal element $K \in \mathcal{M}$.

Suppose that $J \not\subseteq \mathrm{rad}(K)$ and let $x \in J - \mathrm{rad}(K)$. Since $I \subseteq K : x^n$ and since R/I is Noetherian there is an $i \geq n$ so that

$$(*) \quad K : x^i = K : x^{i+j} \quad \text{for all } j \in \mathbb{N}.$$

<u>Claim</u>: $K = (K + Rx^i) \cap (K : x^i)$ where $i \in \mathbb{N}$ as in $(*)$.
<u>Pf of claim</u>: Obviously, $K \subseteq (K + Rx^i) \cap (K : x^i)$. Let $y \in (K + Rx^i) \cap (K : x^i)$. Then there are a $z \in K$ and an $a \in R$ so that $y = z + ax^i$. Since $y \in K : x^i$ it follows that $ax^i \in K : x^i$ and $ax^{2i} \in K$. Thus $a \in K : x^{2i} = K : x^i$ and $y = z + ax^i \in K$. This proves the claim.

Therefore
$$K \cap I = (K + Rx^i) \cap (K : x^i) \cap I$$
$$= (K + Rx^i) \cap I \subseteq J^n I$$
and $K + Rx^i \in \mathcal{M}$. By the maximality of K in $\mathcal{M}$, $K = K + Rx_i$ and $x^i \in K$, contradicting $x \notin \mathrm{rad}\, K$.

Let R and S be Noetherian rings, $u: R \longrightarrow S$ a morphism of rings, and $K \subseteq S$ an ideal. Suppose that u factors $u = \varepsilon\gamma : R \xrightarrow{\gamma} T \xrightarrow{\varepsilon} S$ where T is smooth over R and ε is surjective. (For example, T could be a polynomial ring over R). Let $I = \ker(\varepsilon)$ and $J = \varepsilon^{-1}(K)$. Since $T/I \cong S$ is Noetherian and $I \subseteq J$ there is a finitely generated ideal $J_0 \subseteq J$ so that

(a) $I + J_0 = J$

Since R is Noetherian, the ideal $u^{-1}(K)$ is finitely generated and we may require additionally that:

(b) $\gamma(u^{-1}(K)) \subseteq J_0$.

(Condition (b) may be used later.)

By (9.10) for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ so that $J_0^m \cap I \subseteq J_0^n I$. Moreover,

(9.11) <u>Lemma</u>: On T/I^2 the J-adic topology and the J_0-adic topology are identical.

<u>Proof</u>: For all $n \in \mathbb{N}$ with $n > 1$:

$$\begin{aligned} J_0^n + I^2/I^2 &\subseteq J^n + I^2/I^2 \\ &= (J_0 + I)^n + I^2/I^2 \\ &= J_0^n + J_0^{n-1} I + I^2/I^2 \\ &\subseteq J_0^{n-1} + I^2/I^2 . \end{aligned}$$

(9.12) <u>Corollary</u>: The J-adic topology and the J_0-adic topology of T induce the same topology on I/I^2.

Hence we may choose the set $\{ I \cap J_0^r + I^2/I^2 \}_{r \in \mathbb{N}}$ as a basis for the (J)-topology on I/I^2. Moreover, by (9.10) the sets

$$\{\mathfrak{J}_0^r I + I^2/I^2\}_{r\in\mathbb{N}} \quad \text{and} \quad \{\mathfrak{J}^r I + I^2/I^2\}_{r\in\mathbb{N}}$$

are bases for the $(\mathfrak{J})$-topology of I/I^2. In particular, the $\mathfrak{J}$-adic topology and the $(\mathfrak{J})$-topology of I/I^2 are identical.

(9.13) Corollary: Under assumptions as above the following conditions are equivalent:

(a) S is K-smooth over R.

(b) The natural map $\delta: I/I^2 \longrightarrow \Omega_{T/R} \otimes_T S$ is $\mathfrak{J}$-left invertible.

(c) For all $r \in \mathbb{N}$ the induced map

$$\delta_r : I/I^2 + \mathfrak{J}^r I \longrightarrow \Omega_{T/R} \otimes_T S/K^r$$

is left invertible.

(d) The induced map

$$\delta_1 : I/I^2 + \mathfrak{J} I \longrightarrow \Omega_{T/R} \otimes_T S/K$$

is left invertible.

Proof: (a) $\Rightarrow$ (b): Since the $(\mathfrak{J})$-topology and the $\mathfrak{J}$-adic topology of I/I^2 are the same, the statement follows from (9.8).

(b) $\Leftrightarrow$ (c): By (9.3)

(a) $\Leftrightarrow$ (d): By (9.5), since $\Omega_{T/R}$ is R-projective.

(9.14) Theorem: Let $u: (R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, \ell)$ be a local morphism of local Noetherian rings. Suppose that S is flat over R and that $S \otimes_R k$ is geometrically regular over k. Then S is $\mathfrak{n}$-smooth over R.

Proof: Let T be a polynomial ring over R so that u factors: $u = \varepsilon\gamma : R \xrightarrow{\gamma} T \xrightarrow{\varepsilon} S$ where γ is the natural map and ε is surjective. Let $I = \ker(\varepsilon)$ and $\mathfrak{J} = \varepsilon^{-1}(\mathfrak{n})$. Note that $\mathfrak{J} \subseteq T$ is a maximal ideal. By (9.13) we have to show that the induced map $\delta_1 : I/I^2 + \mathfrak{J} I \longrightarrow \Omega_{T/R} \otimes_T \ell$

is left invertible. δ_1 is a linear map of ℓ-vector spaces. Hence it suffices to show that δ_1 is injective.

Let $\mu: I \longrightarrow T$ be the embedding and consider the exact sequence: $0 \longrightarrow I \xrightarrow{\mu} T \xrightarrow{\varepsilon} S \longrightarrow 0$. Tensoring with k over R yields a long exact sequence: $\dots \longrightarrow \mathrm{Tor}_1^R(S,k) \longrightarrow I \otimes_R k \xrightarrow{\mu\otimes 1} T \otimes_R k \xrightarrow{\varepsilon \otimes 1} S \otimes_R k \longrightarrow 0$.

Since S is R-flat, $\mathrm{Tor}_1^R(S,k) = 0$ and $\mu \otimes 1$ is injective. Hence $\ker(\varepsilon\otimes 1) \cong I \otimes_R k$. This implies that the kernel of the induced map $\bar{\varepsilon}: T/mT \longrightarrow S/mS$ is given by $\ker(\bar{\varepsilon}) = I/mI$. By (8.33) $S/mS = \bar{S}$ is $n/mS = \bar{n}$-smooth over k. Hence by (9.13) the induced map

$$\bar{\delta}_1 : \bar{I}/\bar{I}^2 + \bar{\mathfrak{J}}\bar{I} \longrightarrow \Omega_{\bar{T}/k} \otimes_{\bar{T}} \ell$$

is injective, where $\bar{T} = T/mT$, $\bar{I} = \ker(\bar{\varepsilon}) = I/mI$, and $\bar{\mathfrak{J}} = \mathfrak{J}/mT$, since $\mathfrak{J} \subseteq T$ is maximal. Then:

(i) $\bar{I}/\bar{I}^2 + \bar{\mathfrak{J}}\bar{I} \cong I/I^2 + \mathfrak{J}I$

(ii)
$$\begin{aligned} \Omega_{T/R} \otimes_T \bar{T} &= \Omega_{T/R} / m\Omega_{T/R} \\ &= \Omega_{T/R} / m\Omega_{T/R} + d(mT) \qquad \text{since } m \subseteq R \text{ and } d|_R = 0 \\ &= \Omega_{\bar{T}/R} = \Omega_{\bar{T}/k} \end{aligned}$$

and therefore $\Omega_{\bar{T}/k} \otimes_{\bar{T}} \ell = \Omega_{T/R} \otimes_T \ell$.

δ_1 and $\bar{\delta}_1$ are ℓ-linear maps which are induced by the universal derivations $d: T \longrightarrow \Omega_{T/R}$ and $\bar{d}: \bar{T} \longrightarrow \Omega_{\bar{T}/k}$. Since $\bar{d}$ is induced by d it follows that $\delta_1 = \bar{\delta}_1$ (up to isomorphisms). Hence δ_1 is injective and by (9.13) S is n-smooth over R.

(9.15) <u>Theorem</u>: Let (R, m, k) be a local Noetherian ring and S a Noetherian flat R-algebra. If $\bar{S} = S/mS$ is 0-smooth over k, then S is mS-smooth over R.

<u>Proof</u>: The proof is similar to the proof of (9.14). Let T be a polynomial ring over R and suppose that the morphism of rings $u: R \longrightarrow S$

factors : $u = \varepsilon\gamma : R \xrightarrow{\gamma} T \xrightarrow{\varepsilon} S$ where γ is the natural map and ε is surjective. As before set $I = \ker(\varepsilon)$ and $J = \varepsilon^{-1}(mS) \subseteq T$. By (9.13) we have to show that the induced map:

$$\delta_1 : I/I^2 + JI \longrightarrow \Omega_{T/R} \otimes_T S/mS$$

is left invertible. Set $\overline{T} = T/mT$, $\overline{S} = S/mS$, and let $\overline{\varepsilon} : \overline{T} \longrightarrow \overline{S}$ be the morphism induced by ε. Since S is flat over R, the same argument as in the proof of (9.14) shows that $\ker(\overline{\varepsilon}) = I/mI = \overline{I}$.

Since $\overline{S}$ is 0-smooth over k, by (9.13) the induced map

$$\overline{\delta} : \overline{I}/\overline{I}^2 \longrightarrow \Omega_{\overline{T}/k} \otimes_{\overline{T}} \overline{S}$$

is left invertible. Since

$$\begin{aligned} \overline{I}/\overline{I}^2 &\cong (I/mI)/((I^2+mI)/mI) \\ &\cong I/I^2+mI \\ &= I/I^2+JI \quad (\text{since } I+mT=J) \end{aligned}$$

and $\Omega_{\overline{T}/k} = \Omega_{T/R} \otimes_T \overline{T}$ we obtain that $\overline{\delta} = \delta_1$. δ_1 is left invertible and S is mS-smooth over R by (9.13).