CHAPTER VIII : FORMALLY SMOOTH MORPHISMS

§1: DEFINITIONS AND BASIC PROPERTIES

Let R be a ring and $I \subseteq R$ an ideal. The I-adic topology on R is defined by: $U \subseteq R$ is open if and only if for all $a \in U$ there is an $n \in \mathbb{N}$ with $a + I^n \subseteq U$. Equivalently, the sets $a + I^n$ where $a \in R$ and $n \in \mathbb{N}$ form a basis of the open sets of R. In particular, I^n is an open subset of R and $R - I^n$ is closed. On the other hand for all $a \in R - I^n$ we have that $a + I^n \subseteq R - I^n$ showing that $R - I^n$ is an open subset of R. Thus for all $n \in \mathbb{N}$ I^n is open and closed in R. A similar argument shows that for all $a \in R$ and all $n \in \mathbb{N}$ the set $a + I^n$ is open and closed in the I-adic topology of R.

(8.1) Remark: (a) R is separated in the I-adic topology if and only if $\bigcap_{n \in \mathbb{N}} I^n = (0)$.
(b) The (0)-adic topology of R is the discrete topology.

(8.2) Proposition: Let R and S be rings, $\varphi : R \longrightarrow S$ a homomorphism of rings, and $I \subseteq R$ an ideal. Consider the I-adic topology on R and the discrete topology on S. Then φ is continuous if and only if $\varphi(I^n) = 0$ for some $n \in \mathbb{N}$.

Proof: "$\Rightarrow$": If φ is continuous then $\varphi^{-1}(0) \subseteq R$ is open since $(0) \subseteq S$ is open. Hence $0 + I^n = I^n \subseteq \varphi^{-1}(0)$ for some $n \in \mathbb{N}$.
"$\Leftarrow$": If $\varphi(I^n) = 0$ for some $n \in \mathbb{N}$ then for all $s \in S$ either $\varphi^{-1}(s) = \emptyset$ or for all $r \in \varphi^{-1}(s)$: $r + I^n \subseteq \varphi^{-1}(s)$. Hence for all $s \in S$ $\varphi^{-1}(s)$ is open and φ is continuous.

A similar proof shows:

(8.3) <u>Proposition</u>: Let R and S be rings, $\varphi: R \to S$ a morphism of rings, and $I \subseteq R$, $J \subseteq S$ ideals. Consider the I-adic topology on R and the J-adic topology on S. Then φ is continuous if and only if $\varphi(I^n) \subseteq J$ for some $n \in \mathbb{N}$.

(8.4) <u>Definition</u>: Let R be a ring, S an R-algebra, and $I \subseteq S$ an ideal. Consider the I-adic topology on S.

(a) S is called <u>I-smooth</u> (<u>or formally smooth with respect to the I-adic topology</u>) over R if for any R-algebra C, every ideal $N \subseteq C$ with $N^2 = 0$, and every commutative diagram of R-algebra morphisms:

$$(*) \qquad \begin{tikzcd} S \arrow[r, "\bar{u}"] \arrow[dr, dashed, "u"] & C/N \\ R \arrow[u] \arrow[r] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where ν is the natural map, C/N carries the discrete topology, and $\bar{u}$ is continuous, there is a lifting $u: S \to C$ so that the diagram commutes.

(b) S is called <u>I-unramified</u> over R if under the conditions of (a) there is at most one lifting $u: S \to C$.

(c) S is called <u>I-étale</u> over R if S is I-smooth and I-unramified over R.

(8.5) <u>Remark</u>: (a) If in (8.4) $I = (0)$, then (0)-smoothness is the same as the usual smoothness from Chapter II. If in addition S is of finite presentation over R, then (0)-unramified ((0)-étale) corresponds to unramified (étale) of Chapter II.

(b) Let R be a ring, S an R-algebra, and $I, J \subseteq S$ ideals with $I \subseteq J$.

If S is I-smooth (resp. I-unramified, I-étale) over R, then $\hat{S}$ is $\hat{J}$-smooth (resp. $\hat{J}$-unramified, $\hat{J}$-étale) over R. In particular, if $\hat{S}$ is smooth over R then $\hat{S}$ is I-smooth over R for all ideals $I \subseteq \hat{S}$. In general, the converse is not true.

(c) Consider diagram $(*)$ with $\bar{u}: S \to C/N$ continuous (where S carries the I-adic topology and C/N the discrete topology). Hence $\bar{u}(I^n) = 0$ for some $n \in \mathbb{N}$. If $u: S \to C$ is a lifting of $\bar{u}$ then $u(I^n) \subseteq N$ and therefore $u(I^{2n}) = 0$. Thus u is continuous.

(18.6) <u>Theorem</u>: (Transitivity) Let $R \xrightarrow{g} S \xrightarrow{g'} S'$ be morphisms of rings and $I \subseteq S$, $I' \subseteq S'$ ideals. Consider the I-adic topology on S, the I'-adic topology on S' and assume:

(i) S is I-smooth (I-unramified) over R.

(ii) S' is I'-smooth (I'-unramified) over S.

(iii) g' is continuous, that is, $g'(I^n) \subseteq I'$ for some $n \in \mathbb{N}$.

Then S' is I'-smooth (I'-unramified) over R.

<u>Proof</u>: Let C be an R-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of R-algebra morphisms:

$$\begin{tikzcd}
S' \arrow[r, "\bar{u}"] \arrow[dr, dashed, "u"] & C/N \\
S \arrow[u, "g'"] \arrow[r, dashed, "w"'] & C \arrow[u, "v"'] \\
R \arrow[u, "g"] \arrow[ur] &
\end{tikzcd}$$

where v is the natural map, C/N carries the discrete topology, and $\bar{u}$ is continuous, that is, $\bar{u}(I'^r) = 0$ for some $r \in \mathbb{N}$. By assumption $g'(I^n) \subseteq I'$ for some $n \in \mathbb{N}$ and therefore $\bar{u} \circ g'(I^{nr}) = 0$ implying that $\bar{u} \circ g'$ is continuous. By assumption (i) there is an R-algebra morphism

$w: S \to C$ with $\bar{u}g' = \nu w$. Since S' is I'-smooth over S there is an S-algebra morphism $u: S' \to C$ with $\nu u = \bar{u}$.

Suppose that S is I-unramified over R, S' I'-unramified over S', and let $u, \tilde{u}: S' \to C$ be R-algebra morphisms with $\nu u = \nu\tilde{u} = \bar{u}$. With $w = ug'$ and $\tilde{w} = \tilde{u}g'$ we have $\nu w = \nu u g' = \bar{u}g' = \nu\tilde{u}g' = \nu\tilde{w}$ and hence $w = \tilde{w}$ since S is I-unramified over R. Then $u = \tilde{u}$ since S' is I'-unramified over S'.

(8.7) <u>Theorem</u>: (Base change) Let R be a ring, S and R' R-algebras, and $I \subseteq S$ an ideal. Suppose that S is I-smooth (I-unramified) over R. Then $S' = S \otimes_R R'$ is IS'-smooth (IS'-unramified) over R'.

<u>Proof</u>: Let C be an R'-algebra, $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of R-algebra morphisms:

$$\begin{tikzcd}
S \arrow[r, "p"] \arrow[drr, dashed, "v"'] & S' \arrow[r, "\bar{u}"] \arrow[dr, dashed, "u"] & C/N \\
R \arrow[u] \arrow[r] & R' \arrow[u, "q"] \arrow[r, "\lambda"] & C \arrow[u, "\nu"']
\end{tikzcd}$$

where ν is the natural map, C/N is equipped with the discrete topology, S' with the IS'-adic topology, and $\bar{u}$ is continuous. p and q are the natural maps. Since $\bar{u}(I^n S) = 0$ for some $n \in \mathbb{N}$, $\bar{u} \circ p(I^n) = 0$ and $\bar{u} \circ p$ is continuous, where S carries the I-adic topology. Since S is I-smooth over R, there is an R-algebra morphism $v: S \to C$ with $\nu v = \bar{u}p$. Define $u = v \otimes \lambda: S' = S \otimes_R R' \to C$. u is a lifting of $\bar{u}$.
The unramified case is left as a homework.

(8.8) <u>Proposition</u>: Let $R \xrightarrow{g} S \xrightarrow{g'} S'$ be morphisms of rings and $I \subseteq S$, $I' \subseteq S'$ ideals. Consider the I-adic topology on S and the I'-adic topology on S' and suppose:

(i) S' is I'-smooth over R
(ii) S is I-unramified over R
(iii) g is continuous.
Then S' is I'-smooth over S.

Proof: Let C be an S-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram:

$$\begin{tikzcd} & S' \arrow[r, "\bar{u}"] \arrow[dr, dashed, "u"] & C/N \\ R \arrow[r, "g"] & S \arrow[u, "g'"] \arrow[r, "\lambda"] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where ν is the natural map, C/N is equipped with the discrete topology, and $\bar{u}$ is continuous. Since S' is I'-smooth over R there is an R-algebra morphism $u: S' \to C$ with $\nu u = \bar{u}$ and $u \circ g' \circ g = \lambda \circ g$. We have to show that u is an S-algebra morphism or equivalently, that $\lambda = u g$. By (iii) $g'(I^r) \subseteq I'$ for some $r \in \mathbb{N}$ and thus $\bar{u} g'(I^{rn}) = 0$ for some $n \in \mathbb{N}$ and $\bar{u} g'$ is continuous. Since S is I-unramified over R, $u g'$ is unique with $\nu u g' = \bar{u} g'$. Since $\nu \lambda = \bar{u} g'$, it follows that $u g' = \lambda$ and u is an S-algebra morphism. Hence S' is I'-smooth over S.

(8.9) Example: Let (R, m) be a local Noetherian ring and $(\hat{R}, \hat{m})$ its m-adic completion. Then $\hat{R}$ is $\hat{m}$-étale over R.

Proof: Let C be an R-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of R-algebra morphisms:

$$(*) \qquad \begin{tikzcd} \hat{R} \arrow[r, "\bar{u}"] & C/N \\ R \arrow[u] \arrow[r, "\lambda"] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where ν is the natural map, C/N carries the discrete topology and $\hat{R}$ the

$\hat{m}$-adic topology, and $\overline{u}$ is continuous. Then $\overline{u}(\hat{m}^n) = 0$ for some $n \in \mathbb{N}$ implying that $\lambda(m^n) = 0$ and $\lambda(m^{2n}) = 0$. Since $m^{2n} \subseteq m^n$, $\overline{u}$ factors through a map $\overline{u}_0 : \hat{R}/\hat{m}^{2n} = R/m^{2n} \longrightarrow C/N$ and $(*)$ induces a commutative diagram:

$$\begin{array}{ccccc} \hat{R} & \xrightarrow{\mu} & \hat{R}/\hat{m}^{2n} = R/m^{2n} & \xrightarrow{\overline{u}_0} & C/N \\ & & \uparrow \mathrm{id} & & \uparrow \nu \\ & & R/m^{2n} & \xrightarrow{\lambda_0} & C \end{array}$$

where μ is the natural map. Then $u = \lambda_0 \mu$ is a lifting of $\overline{u}$. Moreover, $u = \lambda_0 \mu$ is unique.

(8.10) <u>Remark</u>: More generally one can show:
Let R be a Noetherian ring, $I \subseteq R$ an ideal and $R^* = (R, I)^\wedge$ the I-adic completion of R. Then R^* is IR^*-étale over R.

Let $g : R \longrightarrow S$ be a morphism of Noetherian rings, $I \subseteq R$ and $J \subseteq S$ ideals. Consider the I-adic topology on R and the J-adic topology on S. and suppose that g is continuous, that is, $g(I^n) \subseteq J$ for some $n \in \mathbb{N}$. Let $R^* = (R, I)^\wedge$ $(S^* = (S, J)^\wedge)$ denote the I-adic (J-adic) completion of R (of S) and let $\alpha : R \longrightarrow R^*$, $\beta : S \longrightarrow S^*$ be the natural maps. Then there is a commutative diagram of ring morphisms:

$$\begin{array}{ccc} R^* & \xrightarrow{g^*} & S^* \\ \alpha \uparrow & & \uparrow \beta \\ R & \xrightarrow{g} & S \end{array}$$

where g^* is induced by g.

(8.11) <u>Proposition</u>: The following conditions are equivalent:
(a) S is J-smooth over R.
(b) S^* is JS^*-smooth over R.

(c) S^* is $\mathcal{J}S^*$-smooth over R^*.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (8.10) and (8.6) S^* is $\mathcal{J}S^*$-smooth over R.
(b) $\Rightarrow$ (c): S^* is $\mathcal{J}S^*$-smooth over R, by (8.10) R^* is IR^*-unramified over R, and α is continuous. By (8.8) S^* is $\mathcal{J}S^*$-smooth over R^*.
(c) $\Rightarrow$ (b): Apply (8.10) and (8.8).
(c) $\Rightarrow$ (a): Let C be an R-algebra, $N \subseteq C$ an ideal with $N^2 = (0)$.
Consider a commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\bar{u}"] & C/N \\ R \arrow[u, "g"] \arrow[r, "\lambda"] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where ν is the natural map and $\bar{u}$ is continuous (C/N is equipped with the discrete topology). Let $m \in \mathbb{N}$ with $\bar{u}(\mathcal{J}^m) = 0$. Since g is continuous, there is an $n \in \mathbb{N}$ with $\lambda(I^{nm}) = 0$. Thus λ and $\bar{u}$ factor through R^* and S^*, respectively, and we obtain a commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\beta"] & S^* \arrow[r, "\bar{v}"] \arrow[dr, dashed, "v"] & C/N \\ R \arrow[u, "g"] \arrow[r, "\alpha"] & R^* \arrow[u, "g^*"] \arrow[r, "\rho"] & C \arrow[u, "\nu"'] \end{tikzcd}$$

where $\bar{v}\beta = \bar{u}$, $\rho\alpha = \lambda$, and $\bar{v}$ continuous. Since S^* is $\mathcal{J}S^*$-smooth over R, there is a lifting $v: S^* \longrightarrow C$ of $\bar{v}$. Then $u = v\beta$ is a lifting of u.

(8.12) <u>Example</u>: Let R be a Noetherian ring, $x_1, \ldots, x_n$ variables over R. Then:
(a) $R[x_1, \ldots, x_n]$ is smooth ($\hat{=}$ (0)-smooth) over R.
(b) $R[\![x_1, \ldots, x_n]\!]$ is $(x_1, \ldots, x_n)R[\![x_1, \ldots, x_n]\!]$-smooth over R.

<u>Proof</u>: Let $I = (x_1, \ldots, x_n)R[x_1, \ldots, x_n]$, $\mathcal{J} = (x_1, \ldots, x_n)R[\![x_1, \ldots, x_n]\!]$, and consider the sequence of natural morphisms:

$$R \xrightarrow{\alpha} R[x_1, \ldots, x_n] \xrightarrow{\beta} R[\![x_1, \ldots, x_n]\!].$$

$R[x_1,\ldots,x_n]$ is (0)-smooth over R, hence by (8.5) $R[x_1,\ldots,x_n]$ is I-smooth over R. Since $R[\![x_1,\ldots,x_n]\!]$ is the I-adic completion of $R[x_1,\ldots,x_n]$, by (8.10) $R[\![x_1,\ldots,x_n]\!]$ is $IR[\![x_1,\ldots,x_n]\!]=J$-smooth over $R[x_1,\ldots,x_n]$. By (8.6) $R[\![x_1,\ldots,x_n]\!]$ is J-smooth over R.

(8.13) <u>Remark</u>: (Tanimoto) Let K be a field. $K[\![x_1,\ldots,x_n]\!]$ is (0)-smooth over K if and only if $\operatorname{char} K = p>0$ and $[K:K^p]<\infty$.

§2: COEFFICIENT FIELDS AND QUASI-COEFFICIENT FIELDS

Let (R, m, k) be a local Noetherian ring, $\varepsilon: \mathbb{Z} \longrightarrow R$ the morphism of rings defined by $\varepsilon(1) = 1_R$. If $n \in \mathbb{N}$ with $\ker(\varepsilon) = (n)$, then R is called a ring of characteristic n. Note that $n = 0$ or $n = p^m$ for some $m \in \mathbb{N}$ and $p \in \mathbb{N}$ prime, since R is local.

(8.14) Remark: Let (R, m, k) be a local Noetherian ring.

(a) If R is a domain, then $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) = p > 0$ where $p \in \mathbb{N}$ is a prime number.

(b) If $\operatorname{char}(k) = 0$, then $\operatorname{char}(R) = 0$ and R contains a field which is isomorphic to $\mathbb{Q}$.

(c) Exactly the following cases occur:

(i) $\operatorname{char}(R) = \operatorname{char}(k) = 0 \iff \mathbb{Q} \subseteq R$

(ii) $\operatorname{char}(R) = \operatorname{char}(k) = p > 0$, p prime $\iff \mathbb{Z}/(p) = \mathbb{F}_p \subseteq R$

(iii) $\operatorname{char}(R) = 0$ and $\operatorname{char}(k) = p > 0$, p prime

(iv) $\operatorname{char}(R) = p^m$, p prime, $m \in \mathbb{N}$ with $m > 1$, and $\operatorname{char}(k) = p$.

In particular, if $\operatorname{char}(R) = \operatorname{char}(k)$, then R contains a field.

(8.15) Definition: Let (R, m, k) be a local Noetherian ring.

(a) R is called of equal characteristic if $\operatorname{char}(R) = \operatorname{char}(k)$.

(b) R is called of unequal or mixed characteristic if $\operatorname{char}(R) \neq \operatorname{char}(k)$.

(8.16) Definition: Let (R, m, k) be a local Noetherian ring and $K \subseteq R$ a subfield.

(a) K is called a coefficient field of R if K maps isomorphically onto k under the natural map:

$$\begin{tikzcd} \rho: R \arrow[rr] & & R/m = k \\ K \arrow[u, phantom, "\cup"] \arrow[urr, "\cong", "\rho|_K"'] & & \end{tikzcd}$$

(b) K is a <u>quasi-coefficient</u> field if under the natural map $\rho: R \to R/m = k$ the field k is 0-étale over the image $\rho(K)$ or equivalently, if k is 0-étale over K via $\rho|_K$.

(8.17) <u>Theorem</u>: Let (R, m, k) be a local Noetherian ring of equal characteristic, $\rho: R \longrightarrow R/m = k$ the natural map. Then:

(a) Let $k_0 \subseteq R$ be a subfield so that k is separable over $\rho(k_0)$. Then there is a quasi-coefficient field $k_1 \subseteq R$ with $k_0 \subseteq k_1$.

(b) R contains a quasi-coefficient field.

(c) If $k_0 \subseteq R$ is a quasi-coefficient field, then there is a unique coefficient field K of the completion $\widehat{R}$ which contains k_0.

(d) If R is complete in the m-adic topology then R contains a coefficient field.

<u>Proof</u>: (a) Let $B = \{\lambda_i\}_{i\in I} \subseteq k$ be a differential basis of k over $\rho(k_0)$, that is, $\{d\lambda_i\}_{i\in I}$ is a k-basis of Ω_{k/k_0} where $d: k \longrightarrow \Omega_{k/k_0}$ is the universal k_0-derivation. Identify k_0 with its image $\rho(k_0)$ in k and let for all $i \in I$ $x_i \in R$ with $\rho(x_i) = \lambda_i$.

If char $k = 0$, then $B = \{\lambda_i\}_{i\in I}$ is a transcendence basis of k over k_0. If char $k = p > 0$, then by (1.42) B is a p-basis of k over k_0. Since k is separable over k_0, by (1.48) B is algebraically independent over k_0. Thus $k_0[x_i]_{i\in I} \cap m = 0$ in R and $k_1 = k_0(x_i)_{i\in I} \subseteq R$.

We claim that k is 0-étale over k_1.

If char $k = 0$, then B is a transcendence basis of k over k_0 and k is separable algebraic over $\rho(k_1) = k_0(B)$. Thus $\Omega_{k/k_1} = 0$ and k is smooth and formally unramified over k_1.

If char $k = p > 0$, then by (1.48) k is separable over $k_0(B)$. Thus k is 0-smooth over k_1. By (1.17) there is an exact sequence of k-vector

spaces: $\Omega_{k_0(B)/k_0} \otimes_{k_0(B)} k \xrightarrow{\alpha} \Omega_{k/k_0} \longrightarrow \Omega_{k/k_0(B)} \longrightarrow 0$.
Since α is surjective, $\Omega_{k/k_0(B)} = 0$, and k is formally unramified over $k_0(B)$. k_1 is a quasi-coefficient field of R containing k_0.

(b) Since $\mathrm{char}(R) = \mathrm{char}(k)$, R contains a subfield. Hence R contains a perfect subfield k_0 and k is separable over k_0. By (a) R contains a quasi-coefficient field.

(c) Consider the following diagram:

$$\begin{tikzcd}
k \arrow[r, "u_1 = id"] \arrow[dr, "u_2"] \arrow[ddr, "u_3"] & \hat{R}/\hat{m} \\
& \hat{R}/\hat{m}^2 \arrow[u] \\
& \hat{R}/\hat{m}^3 \arrow[u] \\
& \vdots \arrow[u] \\
k_0 \arrow[uuuu] \arrow[r] & \hat{R} \arrow[u]
\end{tikzcd}$$

Since k is 0-étale over k_0, there is a unique lifting $u_2 : k \longrightarrow \hat{R}/\hat{m}^2$ of the identity map $u_1 : k \longrightarrow \hat{R}/\hat{m}$. The same argument implies that there is a unique lifting $u_3 : k \longrightarrow \hat{R}/\hat{m}^3$ of u_2. Thus for all $i \in \mathbb{N}$ we obtain a k_0-algebra morphism $u_i : k \longrightarrow \hat{R}/\hat{m}^i$ so that the diagram

$$\begin{tikzcd}
k \arrow[r, "u_{i-1}"] \arrow[dr, "u_i"'] & \hat{R}/\hat{m}^{i-1} \\
& \hat{R}/\hat{m}^i \arrow[u]
\end{tikzcd}$$

commutes. Since $\hat{R}$ is $\hat{m}$-adically complete, by the universal property of the inverse limit there is a k_0-algebra morphism $u : k \longrightarrow \hat{R}$ which lifts u_1. Uniqueness follows from the uniqueness of the maps u_i.

(d) follows from (b) and (c).

(8.18) <u>Theorem</u>: (Cohen's structure theorem in the equal characteristic case)
Let (R, m, k) be a complete local Noetherian ring.

(a) There is an $n \in \mathbb{N}$ and a surjective morphism of rings:
$\varphi : k[\![x_1, \ldots, x_n]\!] \longrightarrow R$, where $x_1, \ldots, x_n$ are variables over k.

(b) If $\dim R = r$, then there is an injective finite morphism of rings:

$\psi: k[[y_1, \dots, y_r]] \longrightarrow R$, where $y_1, \dots, y_r$ are variables over k.

(8.19) <u>Remark</u>: If (R, m, k) is a complete local Noetherian ring of equal characteristic, then
(a) R is a homomorphic image of a power series ring over k and
(b) R is a finite extension of a power series ring over k.

<u>Proof of (8.18)</u>: By (8.17) R contains a coefficient field $k' \subseteq R$. Consider the natural maps $k' \overset{j}{\hookrightarrow} R \xrightarrow{p} k$. Then pj is an isomorphism of fields.
(a) Let $n_1, \dots, n_n$ be a system of generators of m and $x_1, \dots, x_n$ variables over k. By 911, Proposition (9.32) the morphism $j(pj)^{-1}: k \longrightarrow R$ extends to a morphism $\varphi: k[[x_1, \dots, x_n]] \longrightarrow R$ with $\varphi(x_i) = n_i$ for $1 \le i \le n$. Moreover, the composition $p \circ \varphi$ is surjective, thus by 911, Proposition (9.32)(b) φ is surjective.
(b) Let $\dim R = r$ and $t_1, \dots, t_r \in m$ a system of parameters of m. Again by 911, Proposition (9.32) there is a morphism $\psi: k[[y_1, \dots, y_r]] \longrightarrow R$ with $\psi|_k = j(pj)^{-1}$ and $\psi(y_i) = t_i$ for all $1 \le i \le r$. Since $t_1, \dots, t_r$ is a system of parameters of R and the ring $R/(t_1, \dots, t_r) = R/\psi((y_1, \dots, y_r))R$ is a finite k-vector space, by 911, Theorem (9.29) R is a finite $k[[y_1, \dots, y_r]]$-module.

(8.20) <u>Proposition</u>: Let k_0 be a field and (R, m, k) a local Noetherian k_0-algebra with $k_0 \subseteq R$.
(a) If R is m-smooth over k_0, then R is regular.
(b) If R is regular and k is separable over k_0, then R is m-smooth over k_0.

<u>Proof</u>: (a) Let $\mathbb{F}$ be the prime field of k_0. $\mathbb{F}$ is perfect, thus k_0 is separable

and 0-smooth over $\mathbb{F}$. Hence R is m-smooth over $\mathbb{F}$ and we may assume that k_0 is perfect. Since $\widehat{R}$ is $m\widehat{R}$-smooth over R by (8.9), we may assume that R is complete. Let $x_1, —, x_n$ be a minimal system of generators of m and $t_1, —, t_n$ variables over k. Put $S = k[[t_1, —, t_n]]$ and $J = (t_1, —, t_n) \subseteq S$. By (8.17) R has a coefficient field $\tilde{k}$ with $k_0 \subseteq \tilde{k}$. Thus there is an k_0-algebra isomorphism $\bar{v}: R/m^2 \xrightarrow{\cong} S/J^2$. Let $\bar{u}: R \longrightarrow S/J^2$ be the composition $R \xrightarrow{p} R/m^2 \xrightarrow{\bar{v}} S/J^2$ where p is the natural map. Since R is m-smooth over k_0 there is a sequence of liftings:

$$(*)\qquad \begin{tikzcd} R \arrow[r, "\bar{u}"] \arrow[dr, "u_3"] \arrow[ddr, "u_4"'] & S/J^2 \\ & S/J^3 \arrow[u] \\ & S/J^4 \arrow[u] \\ & \vdots \arrow[u] \\ k_0 \arrow[uuuu] \arrow[r] & S \end{tikzcd}$$

Since S is J-adically complet and since diagram $(*)$ commutes, there is a k_0-algebra morphism $u: R \longrightarrow S = k[[t_1, —, t_n]]$ which lifts u. By 911, Theorem (9.29) u is surjective. Thus $\dim R \geq \dim S = n$. On the other hand $\dim R \leq \operatorname{edim} R = n$ and $\dim R = \operatorname{edim} R = n$. R is a regular local ring.

(b) Suppose that R is regular and that k is separable over k_0. By (8.17) the completion $\widehat{R}$ of R is regular and contains a coefficient field $\tilde{k}$ with $k_0 \subseteq \tilde{k}$. $\tilde{k}$ is separable over k_0. By the proof of (8.18) $\widehat{R} \cong \tilde{k}[[x_1, —, x_n]] = S$ where $x_1, —, x_n$ are variables over $\tilde{k}$. By (8.12) S is $(x_1, —, x_n)S$-smooth over $\tilde{k}$. Since $\tilde{k}$ is separable over k_0, S is $(x_1, —, x_n)S$-smooth over k_0 and R is m-smooth over k_0.

§3: SOME DIFFERENTIAL CRITERIA FOR I-SMOOTHNESS

(8.21) Definition: Let $k \longrightarrow R \longrightarrow S$ be morphisms of rings, $I \subseteq S$ an ideal, and suppose that S is equipped with the I-adic topology. S is called I-smooth over R relative to k if for every R-algebra C, every ideal $N \subseteq C$ with $N^2 = 0$, and every commutative diagram of ring morphisms:

$$\begin{tikzcd} & S \arrow[r, "\overline{u}"] \arrow[dr, dashed, "u"] \arrow[dr, dashed, "v"'] & C/N \\ k \arrow[r] & R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

where C/N carries the discrete topology and $\overline{u}$ is continuous, satisfies the following condition: whenever there is a k-algebra morphism $u: S \longrightarrow C$ lifting $\overline{u}$, then there is an R-algebra morphism $v: S \longrightarrow C$ lifting $\overline{u}$.

(8.22) Remark: Let $k \longrightarrow R \longrightarrow S$ be a morphism of rings and $I \subseteq S$ an ideal. If S is I-smooth over k and I-smooth over R relative to k, then S is I-smooth over R.

(8.23) Definition: Let R be a ring, M, M' R-modules, and $\varphi: M \longrightarrow M'$ an R-linear map. φ has a left inverse if there is an R-linear map $\psi: M' \longrightarrow M$ with $\psi \circ \varphi = \mathrm{id}_M$.

(8.24) Lemma: Let R be a ring, M, M' R-modules and $\varphi: M \longrightarrow M'$ an R-linear map. φ has a left inverse if and only if for every R-module N the induced map $\varphi^*: \mathrm{Hom}_R(M', N) \longrightarrow \mathrm{Hom}_R(M, N)$ is surjective.

Proof: Homework

(8.25) Theorem: Let $k \xrightarrow{f} R \xrightarrow{g} S$ be morphisms of rings and $I \subseteq S$

an ideal. The following conditions are equivalent:

(a) S is I-smooth over R relative to k.

(b) If M is an S-module with $I^n M = 0$ for some $n \in \mathbb{N}$, then the natural map $\tau: \mathrm{Der}_k(S,M) \longrightarrow \mathrm{Der}_k(R,M)$ is surjective, that is, every k-derivation $d: R \longrightarrow M$ lifts to a k-derivation $D: S \longrightarrow M$.

(c) For all $n \in \mathbb{N}$ the natural map $\varphi_n: \Omega_{R/k} \otimes_R (S/I^n) \longrightarrow \Omega_{S/k} \otimes_S (S/I^n)$ has a left inverse.

Proof: (a) $\Rightarrow$ (b): Let M be an S-module with $I^n M = 0$ for some $n \in \mathbb{N}$ and let $C = (S/I^n) * M$ be the trivial extension of M. Recall that as an S-module $C = (S/I^n) \oplus M$ and that C is a ring under the multiplication $(s,m)(s',m') = (ss', sm' + s'm)$. Consider C as an S-algebra via the ring morphism $h: S \longrightarrow C$ defined by $h(s) = (s + I^n, 0)$ for all $s \in S$ and identify M with the set $\{(0,m) \mid m \in M\} \subseteq C$. Obviously, M is an ideal of C with $M^2 = 0$. Let $\bar{u}: S \longrightarrow S/I^n = C/M$ denote the natural map. For $d \in \mathrm{Der}_k(R,M)$ define a morphism of rings $\lambda: R \longrightarrow C$ by $\lambda(a) = (\bar{u}(g(a)), d(a))$ for $a \in R$ and verify that the diagram:

$$\begin{tikzcd} & S \arrow[r, "\bar{u}"] & S/I^n = C/M \\ k \arrow[r, "f"] & R \arrow[u, "g"] \arrow[r, "\lambda"] & C \arrow[u, "\mu"'] \end{tikzcd}$$

commutes, where $\mu: C \longrightarrow C/M$ is the natural map. Let $u: S \longrightarrow C$ be defined by $u(s) = (\bar{u}(s), 0)$ and note that for all $a \in k$: $ugf(a) = (\bar{u}g(f(a)), 0) = \lambda f(a)$. Thus u is a k-algebra morphism which lifts $\bar{u}$. By assumption (a) there is an R-algebra morphism $v: S \longrightarrow C$ which lifts $\bar{u}$. For all $s \in S$ $v(s) = (\bar{u}(s), D(s))$ for some map $D: S \longrightarrow M$. We claim that $D \in \mathrm{Der}_k(S,M)$.

Let $s, t \in S$, then:

$$v(st) = v(s)\, v(t)$$

$$= (\bar{u}(s), D(s))(\bar{u}(t), D(t))$$
$$= (\bar{u}(s)\bar{u}(t), \bar{u}(s)D(t) + \bar{u}(t)D(s))$$
$$= (\bar{u}(st), D(st)).$$

Hence $D(st) = \bar{u}(s)D(t) + \bar{u}(t)D(s) = sD(t) + tD(s)$ and D is a derivation from S to M. Moreover, $vg = \lambda$, therefore $Dg = d$ or equivalently, $D|_R = d$. In particular, D is a k-derivation from S to M which extends d.

(b) $\Rightarrow$ (a): Consider a commutative diagram of ring morphisms:

$$\begin{array}{ccccc} & & S & \xrightarrow{\bar{u}} & C/N \\ & & g\uparrow & & \uparrow j \\ k & \xrightarrow{f} & R & \xrightarrow{\lambda} & C \end{array}$$

where C is an R-algebra via λ, $N \subseteq C$ an ideal with $N^2 = 0$ and $\bar{u}$ a continuous map where C/N carries the discrete topology. Let $u: S \longrightarrow C$ be a k-algebra morphism which lifts $\bar{u}$, i.e. $\bar{u} = ju$ and $\lambda f = ugf$. Set $d = \lambda - ug$ and note that $d \in \mathrm{Der}_k(R, N)$. The ideal N of C carries an R-module structure via λ and an S-module structure via u, i.e. if $r \in R, s \in S, n \in N$, then $rn = \lambda(r)n$ and $sn = u(s)n$. For all $r \in R, n \in N$: $g(r)n = u(g(r))n = \lambda(r)n$ since $\lambda(r) - ug(r) = d(r) \in N$ and $N^2 = 0$. Hence the S-module structure on N extends the R-module structure on N. Since $\bar{u}(I^n) = 0$ for some $n \in \mathbb{N}$, $u(I^n) \subseteq N$ and thus $I^n N = 0$. By assumption (b) there is a $D \in \mathrm{Der}_k(S, N)$ extending d, i.e. $Dg = d$. Set $v = u + D: S \rightarrow C$. Then for all $s, t \in S$:

$$v(st) = u(st) + D(st)$$
$$= u(st) + sD(t) + tD(t)$$
$$= u(s)u(t) + u(s)D(t) + u(t)D(s)$$
$$= (u(s) + D(s))(u(t) + D(t)) \qquad \text{since } D(s)D(t) = 0$$
$$= v(s)v(t)$$

and v is a morphism of rings. Moreover,

$vg = ug + Dg = ug + d = ug + \lambda - ug = \lambda$ and

$jg = ju + jD = ju = \overline{u}$.

This shows that v is an R-algebra morphism which lifts u.

(b)$\Leftrightarrow$(c): Apply Lemma (8.24). For all (S/I^n)-modules N consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_S(\Omega_{S/k} \otimes_S S/I^n, N) & \xrightarrow{\varphi_n^*} & \mathrm{Hom}_S(\Omega_{R/k} \otimes_R S/I^n, N) \\ \wr\| & & \wr\| \\ \mathrm{Der}_k(S,N) & \xrightarrow{\tau} & \mathrm{Der}_k(R,N). \end{array}$$

φ_n^* is surjective if and only if τ is.

(8.26) Theorem: Let R be a ring, S an R-algebra, and $I \subseteq S$ an ideal. If S is I-smooth over R, then $\Omega_{S/R} \otimes_S S/I$ is a projective S/I-module.

Proof: Set $\overline{S} = S/I$, then it suffices to show: For every exact sequence of $\overline{S}$-modules $L \xrightarrow{\varphi} M \longrightarrow 0$ the induced sequence:

$$\mathrm{Hom}_{\overline{S}}(\Omega_{S/R} \otimes_S \overline{S}, L) \xrightarrow{\varphi_*} \mathrm{Hom}_{\overline{S}}(\Omega_{S/R} \otimes_S \overline{S}, M) \longrightarrow 0$$

is exact. Hence it suffices to show that the sequence

$$\mathrm{Der}_R(S,L) \longrightarrow \mathrm{Der}_R(S,M) \longrightarrow 0$$

is exact. Consider the trivial extension of L: $C = \overline{S} * L$. Then L and $N = \ker\varphi$ are ideals of C with $L^2 = N^2 = 0$. Moreover, C/N is isomorphic to the trivial extension of M: $C/N \cong \overline{S} * M$. Consider C and C/N as S-algebras via $\lambda: S \longrightarrow C$ with $\lambda(a) = (a+I, 0)$ for all $a \in S$.

Let $d \in \mathrm{Der}_R(S,M)$ be an R-derivation. d induces an R-algebra morphism $\overline{u}: S \longrightarrow C/N \cong \overline{S} * M$ defined by $\overline{u}(a) = (a+I, d(a))$. Since S is I-smooth over R, $\overline{u}$ lifts to an R-algebra morphism $u: S \longrightarrow C$. For all $a \in S$ let $u(a) = (a+I, D(a))$. Then $D: S \longrightarrow L$ is an R-derivation which lifts d.

(8.27) <u>Remark</u>: Let S be I-smooth over R. Then S is J-smooth over R for all ideals $J \subseteq S$ with $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$. Hence for all ideals $J \subseteq S$ with $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$, the S/J-module $\Omega_{S/R} \otimes_S S/J$ is projective.

(8.28) <u>Lemma</u>: Let S be a ring, $I \subseteq S$ an ideal, and $u: L \longrightarrow M$ an S-linear map of S-modules. Suppose that M is projective and that one of the following conditions is satisfied:

(α) I is nilpotent

(β) L is a finitely generated S-module and $I \subseteq \operatorname{rad}(S)$.

Then u has a left inverse if and only if the induced map $\bar{u}: L/IL \longrightarrow M/IM$ has a left inverse.

<u>Proof</u>: The forward direction is trivial. For the backward direction let $\bar{v}: M/IM \longrightarrow L/IL$ be a left inverse of $\bar{u}$. Since M is projective, there is an S-linear map $v: M \longrightarrow L$ so that the diagram:

$$\begin{array}{ccc} M & \xrightarrow{v} & L \\ \nu \downarrow & & \downarrow \mu \\ M/IM & \xrightarrow{\bar{v}} & L/IL \end{array}$$

commutes where ν, μ are the natural maps.

Set $w = vu: L \longrightarrow L$. We want to show that w is bijective. Since w induces the identity on L/IL, it follows that $L = w(L) + IL$. This implies that $L = w(L)$, i.e. w is surjective, under assumption (α) or (β):

(α) If $I^t = 0$ for some $t \in \mathbb{N}$, then $L = w(L) + IL = w(L) + Iw(L) + I^2L = \dots = w(L) + I^tL = w(L)$.

(β) If L is finitely generated and $I \subseteq \operatorname{rad}(S)$, then $L = w(L)$ by Nakayama. Moreover, by Matsumura, Commutative ring theory, Theorem 2.4, w is injective.

It remains to show that in case (α) w is injective. Let $x \in \ker(w)$.

Since $\overline{w} = \overline{v}\,\overline{u} = \mathrm{id}_{L/IL}$, $x \equiv w(x) \bmod IL$ and therefore $x \in IL$ since $w(x) = 0$. Write $x = \sum_{i=1}^{n} a_i y_i$ with $a_i \in I$ and $y_i \in L$. Then $0 = w(x) = \sum_{i=1}^{n} a_i w(y_i) \equiv \sum a_i y_i \bmod I^2 L$ and $x \in I^2 L$. Continuing like this we obtain that $x \in I^t L$ and thus $x = 0$.
Hence w is an isomorphism and $w^{-1}v$ is a left inverse of u.

(8.29) Theorem: Let $k \longrightarrow R \longrightarrow S$ be morphisms of rings and $I \subseteq S$ an ideal. Suppose that S is I-smooth over k. Then the following conditions are equivalent:
(a) S is I-smooth over R
(b) The natural map $\Omega_{R/k} \otimes_R S/I \longrightarrow \Omega_{S/k} \otimes_S S/I$ has an S/I-linear left inverse.

Proof: (a) $\Rightarrow$ (b): If S is I-smooth over R, S is I-smooth over R relative to k. (b) follows by (8.25).
(b) $\Rightarrow$ (a): By (8.22) it suffices to show that S is I-smooth over R relative to k. By (8.25) it suffices to show that for all $n \in \mathbb{N}$ the natural map $\varphi_n : \Omega_{R/k} \otimes_R S/I^n \longrightarrow \Omega_{S/k} \otimes_S S/I^n$ has a left inverse.
S is I-smooth over k. Hence for all $n \in \mathbb{N}$ S is I^n-smooth over k and by (8.26) $\Omega_{S/k} \otimes_S S/I^n$ is a projective S/I^n-module. We proceed by induction on n. By assumption φ_1 has a left inverse. Suppose that φ_n has a left inverse. Since $(I^n S/I^{n+1})^2 = 0$ and $\Omega_{S/k} \otimes_S S/I^n$ a projective S/I^{n+1}-module, by (8.28) φ_{n+1} has a left inverse.

(8.30) Corollary: Let (R, m, k) be a regular local ring which contains a field k_0. The following conditions are equivalent:
(a) R is m-smooth over k_0.

(b) The natural map $\Omega_{k_0} \otimes_{k_0} k \longrightarrow \Omega_R \otimes_R k$ is injective.

<u>Proof</u>: Let $P \subseteq k_0$ be the prime field of k_0. By (8.20) R is m-smooth over P. Apply (8.29) to the ring morphisms $P \longrightarrow k_0 \longrightarrow R$. Then R is m-smooth over $k_0 \iff \Omega_{k_0} \otimes_{k_0} k \longrightarrow \Omega_R \otimes_R k$ has a left inverse $\iff \Omega_{k_0} \otimes_{k_0} k \longrightarrow \Omega_R \otimes_R k$ is injective.

§4: FORMALLY SMOOTH MORPHISMS OVER A FIELD

(8.31) Definition: Let (R, m, k) be a local Noetherian ring, $k_0 \subseteq R$ a subfield. R is called geometrically regular over k_0 if $R \otimes_{k_0} L$ is a regular ring for every finite extension field L of k_0.

(8.32) Remark: Exactly the same proof as in (7.18) shows that R is geometrically regular over k_0 if and only if for every finite purely inseparable field extension $k_0 \subseteq L$ the ring $R \otimes_{k_0} L$ is regular.

(8.33) Theorem: Let (R, m, k) be a local Noetherian ring and $k_0 \subseteq R$ a subfield. The following conditions are equivalent:

(a) R is m-smooth over k_0

(b) R is geometrically regular over k_0.

Proof: (a) $\Rightarrow$ (b): Let $k_0 \subseteq L$ be a finite field extension. Since R is m-smooth over k_0, by (8.7) $R' = R \otimes_{k_0} L$ is mR'-smooth over L. Since R' is finite over R, every maximal ideal $n \subseteq R'$ contains mR' and is minimal over mR'. Let $n \subseteq R'$ be a maximal ideal and set $S = R'_n$. The natural map $R' \longrightarrow S = R'_n$ is continuous if R' is equipped with the mR'-adic topology and S with the nS-adic topology. Since S is 0-étale over R', by (8.6) S is nS-smooth over L. By (8.20) S is regular.

(b) $\Rightarrow$ (a): If char $k_0 = 0$, then (a) follows by (8.20). Hence we may assume that char $k_0 = p > 0$. By (8.30) it suffices to show that the natural map $\Omega_{k_0} \otimes_{k_0} k \longrightarrow \Omega_R \otimes_R k$ is injective. If $B = \{y_i\}_{i \in I}$ is a p-basis of k_0 over its prime field P, then by (1.41) the set $\{dy_i\}_{i \in I}$ is a basis of the k_0-vector space Ω_{k_0}, where $d: k_0 \longrightarrow \Omega_{k_0}$

is the universal derivation. Thus it suffices to show:

(Δ) If $x_1, \dots, x_r \in k_o$ are p-independent over P, then $\delta x_1 \otimes 1, \dots, \delta x_r \otimes 1$ are linearly independent over k in $\Omega_R \otimes_R k$ where $\delta : R \longrightarrow \Omega_R$ is the universal derivation.

Let $\alpha_i = x_i^{1/p} \in \overline{k_o}$ and $k_o' = k_o(\alpha_1, \dots, \alpha_r)$. Since $x_1, \dots, x_r \in k_o$ are p-independent over P, for all $1 \le i \le r$: $\alpha_i \notin k_o(\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_r)$. In particular, $k_o' = k_o[t_1, \dots, t_r]/(t_1^p - x_1, \dots, t_r^p - x_r)$ where $t_1, \dots, t_r$ are variables. Consider

$$S = R \otimes_{k_o} k_o' = R[t_1, \dots, t_r]/(t_1^p - x_1, \dots, t_r^p - x_r).$$

By assumption (b) S is a regular. Moreover, since for all $b \in S$, $b^p \in R$, S is a regular local ring with maximal ideal $n \subseteq S$ and residue field $\ell = S/n$. Consider the following natural maps:

$$P \subseteq R \subseteq k = R/m \qquad \text{and} \qquad P \subseteq S \subseteq \ell = S/n.$$

Since P is perfect, k and ℓ are smooth over P. By (2.15) the sequences:

$$0 \longrightarrow m/m^2 \longrightarrow \Omega_R \otimes_R k \longrightarrow \Omega_k \longrightarrow 0$$
and
$$0 \longrightarrow n/n^2 \longrightarrow \Omega_S \otimes_S \ell \longrightarrow \Omega_\ell \longrightarrow 0$$

are (split) exact. This induces a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & m/m^2 \otimes_k \ell & \longrightarrow & \Omega_R \otimes_R \ell & \longrightarrow & \Omega_k \otimes_k \ell & \longrightarrow & 0 \\
 & & \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \\
0 & \longrightarrow & n/n^2 & \longrightarrow & \Omega_S \otimes_S \ell & \longrightarrow & \Omega_\ell & \longrightarrow & 0
\end{array}$$

where φ_i are the natural maps. The snake lemma induces a long exact sequence of ℓ-vector spaces:

$$0 \to \ker \varphi_1 \to \ker \varphi_2 \to \ker \varphi_3 \to \operatorname{coker} \varphi_1 \to \operatorname{coker} \varphi_2 \to$$
$$\to \operatorname{coker} \varphi_3 \to 0.$$

Since R and S are regular local rings with $\dim R = \dim S$, $\operatorname{rk} n/n^2 = \operatorname{rk} m/m^2$ and hence $\operatorname{rk} \ker \varphi_1 = \operatorname{rk} \operatorname{coker} \varphi_1$.

Note that $\operatorname{coker} \varphi_3 = \Omega_{\ell/k}$ and $\ker \varphi_3 = \Gamma_{\ell/k/P}$. Since ℓ is algebraic over k, by the Cartier equality (2.23) $\operatorname{rk} \ker \varphi_3 = \operatorname{rk} \operatorname{coker} \varphi_3$.

This implies that $\operatorname{rk} \ker \varphi_2 = \operatorname{rk} \operatorname{coker} \varphi_2$.

Since $\operatorname{coker} \varphi_2 = \Omega_{S/R} \otimes_S \ell$ and S is an R-algebra of finite type, it follows that $\operatorname{rk} \operatorname{coker} \varphi_2 < \infty$. Moreover, $S = R[t_1, \ldots, t_r]/(t_1^p - x_1, \ldots, t_r^p - x_r)$ and hence by (1.10):

$$\Omega_{S/R} = \Omega_{R[t_1,\ldots,t_r]/R} / R[t_1,\ldots,t_r]\, \delta((t_1^p - x_1, \ldots, t_r^p - x_r))$$

where $\delta : R[t_1, \ldots, t_r] \longrightarrow \Omega_{R[t_1,\ldots,t_r]/R} = \bigoplus_{i=1}^r R[t_1,\ldots,t_r]\, \delta t_i$ is the universal R-derivation of $R[t_1, \ldots, t_r]$. Since $\delta(t_i^p - x_i) = 0$ it follows that:

$$\begin{aligned} \Omega_{S/R} &= \Omega_{R[t_1,\ldots,t_r]/R} / (t_1^p - x_1, \ldots, t_r^p - x_r)\, \Omega_{R[t_1,\ldots,t_r]/R} \\ &= \bigoplus_{i=1}^r S\, dt_i \end{aligned}$$

where $d : S \longrightarrow \Omega_{S/R}$ is the universal R-derivation of S. This shows that $\operatorname{rk} \ker \varphi_2 = \operatorname{rk} \operatorname{coker} \varphi_2 = r$.

By (1.12) there is an exact sequence of S-modules

$$(*) \qquad J/J^2 \xrightarrow{\delta} \Omega_{R[t_1,\ldots,t_r]} \otimes S \longrightarrow \Omega_S \longrightarrow 0$$

where $J = (t_1^p - x_1, \ldots, t_r^p - x_r) \subseteq R[t_1, \ldots, t_r]$ and $\delta : R[t_1, \ldots, t_r] \longrightarrow \Omega_{R[t_1,\ldots,t_r]}$ the universal derivation of $R[t_1, \ldots, t_r]$.

Consider the sequence of morphisms of rings: $P \longrightarrow R \longrightarrow R[t_1, \ldots, t_r]$. Since $R[t_1, \ldots, t_r]$ is smooth over R, by (2.14) there is an exact sequence: $0 \longrightarrow \Omega_R \otimes_R R[t_1, \ldots, t_r] \longrightarrow \Omega_{R[t_1,\ldots,t_r]} \longrightarrow \Omega_{R[t_1,\ldots,t_r]/R} \longrightarrow 0$.

Moreover, $\Omega_{R[t_1,\ldots,t_r]/R} = \bigoplus_{i=1}^r R[t_1,\ldots,t_r]\, \delta t_i$ and by (2.16)

$$\Omega_{R[t_1,\ldots,t_r]} = (\Omega_R \otimes_R R[t_1,\ldots,t_r]) \oplus \bigoplus_{i=1}^r R[t_1,\ldots,t_r]\, \delta t_i .$$

Therefore:

$$\Omega_{R[t_1,\ldots,t_r]} \otimes S = (\Omega_R \otimes_R S) \oplus \bigoplus_{i=1}^r S\, \delta t_i .$$

Tensoring $(*)$ with ℓ over S yields an exact sequence:

$$J/J^2 \otimes_S \ell \xrightarrow{\delta \otimes 1} (\Omega_R \otimes_R \ell) \oplus \bigoplus_{i=1}^r \ell\, \delta t_i \xrightarrow{\tau} \Omega_S \otimes_S \ell \longrightarrow 0 .$$

Note that $(\delta \otimes 1)((t_i^p - x_i) \otimes 1) = -\delta x_i \otimes 1$ and thus $\operatorname{im}(\delta \otimes 1) \subseteq \Omega_R \otimes_R \ell$.

Moreover, $\tau|_{\Omega_R \otimes \ell} = \varphi_2$ and therefore

$$\ker \tau = (\operatorname{im}(\delta \otimes 1)) \cap (\Omega_R \otimes_R \ell) = \ker \varphi_2 .$$

This implies:

(i) $\delta x_1 \otimes 1, \ldots, \delta x_r \otimes 1$ generate $\ker \varphi_2$.

(ii) Since $\operatorname{rk} \ker \varphi_2 = \operatorname{rk} \operatorname{coker} \varphi_2 = \operatorname{rk} \Omega_{S/R} \otimes \ell = r$, $\delta x_1 \otimes 1, \ldots, \delta x_r \otimes 1$ are linearly independent in $\Omega_R \otimes_R \ell$. Thus $\delta x_1 \otimes 1, \ldots, \delta x_r \otimes 1$ are linearly independent in $\Omega_R \otimes_R k$ and (Δ) is proven.

(8.34) Corollary: Let k_0 be a field, (R, m, k) a local Noetherian ring and a k_0-algebra. Suppose that R is m-smooth over k_0. If $P \subseteq R$ is a prime ideal, then R_P is PR_P-smooth over k_0.

Proof: By (8.33) it suffices to show that R_P is geometrically regular over k_0. Let $k_0 \subseteq L$ be a finite field extension. By assumption $R \otimes_{k_0} L$ is a regular ring. Since $R_P \otimes_{k_0} L$ is a localization of $R \otimes_{k_0} L$, the ring $R_P \otimes_{k_0} L$ is regular.

(8.35) Remark: Let $\varphi: (R, m, k) \longrightarrow (S, n, \ell)$ be a local morphism of local Noetherian rings. Suppose that S is n-smooth over R and let $Q \subseteq S$ be a prime ideal, $P = \varphi^{-1}(Q)$ its contraction to R. Then, in general, under the induced morphism $\varphi_Q : R_P \longrightarrow S_Q$ the ring S_Q is not QS_Q-smooth over R_P. For example, if $S = (R, m)^\wedge$, the completion of R, in order for formal smoothness to localize, the ring R has to be 'almost' excellent. (See (7.16), the example of a local Noetherian ring which is not Nagata).