CHAPTER VII : NAGATA RINGS

§1: BASIC PROPERTIES

(7.1) Definition: Let R be a Noetherian domain with field of quotients $K = Q(R)$ and S a Noetherian ring.

(a) R is called an N-1 ring (or N-1) if the integral closure of R in K is a finite R-module.

(b) R is called an N-2 ring (or N-2) if for all finite field extensions $K \subseteq L$ the integral closure of R in L is a finite R-module.

(c) The ring S is called a Nagata ring if for all prime ideals $P \subseteq S$ the residue ring S/P is N-2.

(7.2) Remark: (a) If R is a Noetherian domain with $\mathbb{Q} \subseteq K = Q(R)$ then R is N-1 if and only if R is N-2.

(b) Every field is a Nagata domain.

(c) Let R be a discrete valuation ring with $\mathbb{Q} \subseteq K = Q(R)$. Then R is a Nagata ring.

(d) $\mathbb{Z}$ is a Nagata ring.

(7.3) Remark: We will see later that every ring which is essentially of finite type over a field or over the integers is a Nagata ring. We will also construct later an example of a local Noetherian ring which is not a Nagata ring.

(7.4) Proposition: Let R be a Noetherian domain with field of quotients $K = Q(R)$. Suppose that for every finite purely inseparable field extension $K \subseteq L$ the integral closure of R in L is a finite R-module. Then R is N-2.

<u>Proof</u>: Let $K \subseteq E$ be a finite field extension. By enlarging E, if necessary, we may assume that E is normal over K. Let $L = \mathrm{Fix}(\mathrm{Aut}(E/K))$ be the fixed field of the automorphism group of E over K. Then L is purely inseparable over K and E is separable over L. By assumption the integral closure S_L of R in L is a finite R-module. In particular, S_L is a normal Noetherian domain. Since E is finite separable over L, the integral closure S of S_L in L is a finite S_L-module. S is the integral closure of R in E and a finite R-module.

Our first goal is to prove that complete local Noetherian rings are Nagata rings. We need:

(7.5) <u>Theorem:</u> (Tate) Let R be a normal Noetherian domain and $x \in R$ a nonzero element which satisfies the following conditions:

(a) The principal ideal $xR = (x)$ is a prime ideal.

(b) R is (x)-adically complete and separated, that is, $R = (R,(x))^\wedge$.

(c) $R/(x)$ is N-2.

Then R is N-2.

<u>Proof</u>: Let $K = Q(R)$ be the quotient field of R. Since R is normal, R is N-2 if $\mathrm{char}\, K = 0$. Thus we may suppose that $\mathrm{char}\, K = p > 0$. By (7.4) it suffices to show that for every finite purely inseparable field extension $K \subseteq L$ the integral closure of K in L is a finite R-module. Let $K \subseteq L$ be a finite purely inseparable field extension and $f \in \mathbb{N}$ so that with $q = p^f$: $L^q \subseteq K$.
Let S denote the integral closure of R in L.

<u>Claim 1</u>: $S = \{ s \in L \mid s^q \in R \}$

<u>Pf of Cl.1</u>: "$\supseteq$" is obvious. Let $s \in S$, then $s^q \in K$ and thus $s^q \in R$, since R is normal and therefore $K \cap S = R$.

Enlarge L, if necessary, to assume that there is an element $y \in S$ with $y^q = x$. Since $R \subseteq S$ is an integral extension and $P = xR$ a prime ideal, there is a prime ideal $Q \subseteq S$ with $Q \cap S = P$.

<u>Claim</u> 2: $Q = yS$

<u>Pf of</u> Cl. 2: For all $s \in Q$ there is an $u \in R$ with $s^q = xu$ and thus $s^q = y^q u$. Then $u = (s/y)^q \in R$ and $s/y \in S$ by Claim 1. Hence $s \in yS$ and $Q \subseteq yS$. On the other hand $x = y^q \in Q$ and $Q = yS$ follows.

By Claim 2 the local rings S_Q and R_P are discrete valuation rings with residue fields $k(Q)$ and $k(P)$. By ramification theory [Bourbaki, chap. 6, §8, Lemma 2]:

$$[k(Q) : k(P)] \leq [L : K],$$

in particular, the field extension $k(P) \subseteq k(Q)$ is finite.

Note that $k(P)$ and $k(Q)$ are the quotient fields of $R/xR = R/P$ and $S/yS = S/Q$. By assumption R/P is N-2 and the integral closure of R/P in $k(Q)$ is a finite R/P-module. Since S/Q is integral over R/P, S/Q is a finite R/P-module.

In order to finish the proof note that:

(i) $Q^q = y^q S = xS = PS$ and the Q-adic topology on S is the same as the PS-adic topology on S

(ii) S is separated in the Q-adic topology.

Pf of (ii): Since S_Q is a DVR, $\bigcap_{n \in \mathbb{N}} Q^n S_Q = 0$. Then $\bigcap_{n \in \mathbb{N}} Q^n = 0$ since S is a domain. S is separated.

(iii) Since Q is principal, Q^{n-1}/Q^n is a finitely generated S/Q-module and hence a finitely generated R/P-module. By induction on n, S/Q^n is a finitely generated R-module. In particular, S/PS is a finitely generated R/P-module.

By 911, Theorem (9.29) S is a finitely generated R-module.

(7.6) <u>Corollary</u>: Let R be a complete local Noetherian ring. Then R is a Nagata ring.

Proof: Let $P \subseteq R$ be a prime ideal. We have to show that the ring R/P is N-2. By Cohen's Structure Theorem [Matsumura, Theorem 29.4] there is a complete regular local ring R_0 and a finite injective morphism $R_0 \hookrightarrow R/P$. Thus it suffices to show that a complete regular local ring R is N-2. We proceed by induction on $n = \dim(R)$ and choose an element $x \in m - m^2$. Then:

(a) xR is a prime ideal.

(b) R/xR is a complete regular local ring of dimension $\dim(R) - 1$. By induction hypothesis R/xR is N-2.

By (7.5) R is a N-2 ring.

(7.7) Definition: Let R be a semilocal ring, $J \subseteq R$ its Jacobson radical, and $\widehat{R}$ the J-adic completion of R. R is called analytically unramified if $\widehat{R}$ is reduced. R is called analytically unramified in $P \in \mathrm{Spec}(R)$ if R/P is analytically unramified.

(7.8) Remark: Recall from 911: Let R be a semilocal Noetherian ring, $m_1, \ldots, m_n$ the maximal ideals of R, and $J = m_1 \cap \ldots \cap m_n$ the Jacobson radical. Then

$$\begin{aligned}(R,J)^\wedge &= \varprojlim R/J^k \\ &= \varprojlim \textstyle\prod_{i=1}^n (R_{m_i}/m_i^k R_{m_i}) \\ &= \textstyle\prod_{i=1}^n \varprojlim R_{m_i}/m_i^k R_{m_i} \\ &= \textstyle\prod_{i=1}^n \widehat{R}_{m_i}.\end{aligned}$$

Note that a complete semilocal Noetherian ring is a Nagata ring.

(7.9) Proposition: (Rees) Let R be a semilocal Noetherian domain, $J \subseteq R$ its Jacobson radical, and $\widehat{R}$ the J-adic completion of R. If $\widehat{R}$ is reduced, the integral closure of R in its field of quotients is a finite R-module.

Proof: Let $\widehat{R}$ be reduced with minimal prime ideals $P_1, \ldots, P_r$. The total ring of quotients of $\widehat{R}$ is a product of fields:

$$Q(\widehat{R}) = K_1 \times \dots \times K_r$$

where $K_i = k(P_i) = Q(\widehat{R}/P_i)$. If $\widehat{R}'$ denotes the integral closure of $\widehat{R}$ in $Q(\widehat{R})$, then

$$\widehat{R}' = (\widehat{R}/P_1)' \times \dots \times (\widehat{R}/P_r)'$$

where $(\widehat{R}/P_i)'$ is the integral closure of $\widehat{R}/P_i$ in K_i. For all $1 \le i \le r$ $(\widehat{R}/P_i)'$ is a finite $\widehat{R}/P_i$-module, hence $\widehat{R}'$ is a finite $\widehat{R}$-module.

Let S denote the integral closure of R in its field of quotients $K = Q(R)$. Since $\widehat{R}$ is flat over R, $S \otimes_R \widehat{R} \subseteq Q(R) \otimes_R \widehat{R}$, and, since $\widehat{R}$ is reduced, $Q(R) \otimes_R \widehat{R} \subseteq Q(\widehat{R})$. Moreover, $S \otimes_R \widehat{R}$ is integral over $\widehat{R}$ and therefore $S \otimes_R \widehat{R} \subseteq \widehat{R}'$. $S \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module and by faithful flatness S is a finite R-module.

(7.10) <u>Lemma</u>: Let R be a semilocal Noetherian ring, J the Jacobson radical of R, and $\widehat{R}$ the J-adic completion of R. Let $P \subseteq R$ be a prime ideal which satisfies the following conditions:

(a) R_P is a discrete valuation ring.

(b) R is analytically unramified in P.

Then $\widehat{R}_{\widehat{Q}}$ is a discrete valuation ring for all $\widehat{Q} \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/P\widehat{R})$.

<u>Proof</u>: By (a) there is an element $p \in P$ with $pR_P = PR_P$. By (b) the ring $\widehat{R}/P\widehat{R}$ is reduced. Therefore for all $\widehat{Q} \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/P\widehat{R})$ the ring $(\widehat{R}/P\widehat{R})_{\widehat{Q}} = (\widehat{R}/p\widehat{R})_{\widehat{Q}}$ is a field and $\widehat{Q}\widehat{R}_{\widehat{Q}} = p\widehat{R}_{\widehat{Q}}$.

(7.11) <u>Lemma</u>: Let R be a semilocal Noetherian domain, J the Jacobson radical of R and $x \in J$ a nonzero element. Suppose that for all $P \in \mathrm{Ass}_R(R/xR)$:

(a) R_P is a discrete valuation ring.

(b) R is analytically unramified in P.

Then R is analytically unramified.

<u>Proof</u>: Suppose that

$\mathrm{Ass}_R(R/xR) = \{P_1, \ldots, P_r\}$ and $\mathrm{Ass}_{\hat{R}}(\hat{R}/P_i\hat{R}) = \{Q_{i1}, \ldots, Q_{in_i}\}$ for all $1 \le i \le r$.

By assumption $\hat{R}/P_i\hat{R}$ is reduced for all $1 \le i \le r$ and therefore

$$P_i\hat{R} = \bigcap_{j=1}^{n_i} Q_{ij}.$$

For all (i,j) consider the natural morphism

$$\varphi_{ij} : \hat{R} \longrightarrow \hat{R}_{Q_{ij}}$$

and set $W_{ij} = \ker(\varphi_{ij})$. By (7.10) $\hat{R}_{Q_{ij}}$ is a discrete valuation ring. Hence W_{ij} is a prime ideal of $\hat{R}$ and it suffices to show:

$$\bigcap_{i,j} W_{ij} = 0.$$

By 911, Theorem (9.45):

$$\mathrm{Ass}_{\hat{R}}(\hat{R}/x\hat{R}) = \bigcup_{P \in \mathrm{Ass}_R(R/xR)} \mathrm{Ass}_{\hat{R}}(\hat{R}/P\hat{R}) = \{Q_{ij}\}_{\substack{1\le i \le r \\ 1 \le j \le n_i}}$$

and therefore

$$x\hat{R} = \bigcap_{i,j} L_{ij}$$

where L_{ij} is Q_{ij}-primary. Then

$$W_{ij} \subseteq L_{ij} = \varphi_{ij}^{-1}(x\hat{R}_{Q_{ij}})$$

and hence

$$\bigcap_{i,j} W_{ij} \subseteq \bigcap_{i,j} L_{ij} = x\hat{R}.$$

For every $u \in \bigcap_{i,j} W_{ij}$ there is an element $y \in \hat{R}$ with $u = xy$. Since u is zero in $\hat{R}_{Q_{ij}}$, for all ij there is an element $t_{ij} \in \hat{R} - Q_{ij}$ with

$$t_{ij}u = t_{ij}xy = 0.$$

By assumption R is a domain and x is a regular element in R and $\hat{R}$. Thus $t_{ij}y = 0$ for all i,j and $y \in \bigcap_{i,j} W_{ij}$. This shows that $y \in x\hat{R}$ and by repeating the argument we obtain that:

$$\bigcap_{i,j} W_{ij} \subseteq \bigcap_{n \in \mathbb{N}} x^n\hat{R}.$$

Since $x \in J$, the Jacobson radical of R, $\bigcap_{n\in\mathbb{N}} x^n\hat{R} = 0$ and the lemma is proven.

(7.12) <u>Theorem</u>: Let R be a semilocal Noetherian domain and a Nagata ring. Then R is analytically unramified.

<u>Proof</u>: The proof is by induction on $\dim(R)$. Let S be the integral closure of R in its field of quotients $Q(R)$. S is a finite R-module and for all $Q \in \operatorname{Spec}(S)$ the ring S/Q is a finite $R/Q\cap R$-module. Hence S is a Nagata ring.

If J denotes the Jacobson radical of R, then the JS-adic topology on S is the same as the topology defined by the Jacobson radical J_S of S. In particular,

$$\widehat{R} \subseteq \widehat{S} = S \otimes_R \widehat{R}$$

and it suffices to show that $\widehat{S}$ is reduced. Hence we may assume that R is a normal domain.

Let $x \in J$ be a nonzero element and $P \in \operatorname{Ass}_R(R/xR)$. Since R is normal, R satisfies Serre's conditions (R_1) and (S_2). Hence R/xR has no embedded prime ideals and R_P is a discrete valuation ring. Moreover, $\dim(R/P) < \dim(R)$ and by induction hypothesis the ring R/P is analytically unramified. By (7.11) R is analytically unramified.

(7.13) <u>Remark</u>: As example (7.16) will show Nagata rings cannot be characterized by the property that R is analytically unramified in every prime ideal $P \subseteq R$. This condition is too weak.

Next we investigate a property of Nagata rings which has been used in the last chapter on Artin approximation. This property will also help in (7.16) to construct a ring which fails to be Nagata.

We begin by recalling some facts from previous chapters. Let $K \subseteq E$ be a field extension. E is separable over K if and only if for all field extensions $K \subseteq L$ the tensor product $E \otimes_K L$ is reduced. Moreover, let $\{E_i\}_{i \in I}$ denote the set of all finitely generated K-subalgebras of E. Then

$$E = \varinjlim_{i\in I} E_i$$

and $$E\otimes_K L = \varinjlim_{i\in I} E_i \otimes_K L .$$

Moreover, by flatness of L over K

$$E_i \otimes_K L \subseteq E\otimes_K L$$

and E is separable over K if and only if E_i is separable over K for all $i\in I$. Since E_i is finitely generated over K by (1.31) E_i is separable over K if and only if $E_i \otimes_K K^{p^{-1}}$ is reduced where $p>0$ is the characteristic of K. This implies that E is separable over K if and only if $E\otimes_K K^{p^{-1}}$ is reduced. Since $K^{p^{-1}}$ is a direct limit of finite purely inseparable field extensions of K, we obtain:

(7.14) Proposition: E is separable over K if and only if $E\otimes_K L$ is reduced for all finite purely inseparable field extensions $K\subseteq L$.

(7.15) Theorem: Let R be a semilocal Nagata domain, $\hat{R}$ the completion of R with respect to its Jacobson radical, and $P_1, \dots, P_r$ the minimal prime ideals of R. For all $1\le i\le r$ the quotient field $k(P_i) = Q(\hat{R}/P_i)$ is separable over $Q(R)$, the quotient field of R.

Proof: Set $K = Q(R)$ and $E_i = Q(\hat{R}/P_i)$ for $1\le i\le r$. Since R is Nagata, by (7.12) $\hat{R}$ is reduced with total ring of quotients:

$$Q(\hat{R}) = E_1 \times \dots \times E_r .$$

By (7.14) it suffices to show that $Q(\hat{R})\otimes_K L$ is reduced for every finite field extension $K\subseteq L$. Since L is flat over K we have inclusions:

$$\hat{R}\otimes_R L \subseteq Q(\hat{R})\otimes_R L = Q(\hat{R})\otimes_K L$$

and it is enough to show that $\hat{R}\otimes_R L$ is reduced since $Q(\hat{R})\otimes_K L$ is a localization of $\hat{R}\otimes_R L$.

Let S denote the integral closure of R in L. Since $K\subseteq L$ is finite and

R is Nagata, S is a finite R-module. In particular, S is a semilocal Noetherian ring and the completion of S with respect to its Jacobson radical is given by

$$\hat{S} = \hat{R} \otimes_R S.$$

Since $\hat{R} \otimes_R L$ is a localization of $\hat{R} \otimes_R S$, it suffices to show that $\hat{S}$ is reduced. S is a finite extension of R and hence also a Nagata ring. Thus by (7.12) $\hat{S}$ is reduced.

<u>Note</u>: we will show in the next section that any algebra essentially of finite type over a field K or the integers $\mathbb{Z}$ is a Nagata ring.

(7.16) <u>Example</u>: Let K be a field of characteristic $p>0$ and x a variable over K. The discrete valuation ring $A = K[x]_{(x)}$ has completion $\hat{A} = K[\![x]\!]$. Let

$$\omega = \sum_{i=1}^{\infty} a_i x^i \in K[\![x]\!]$$

be transcendental over $K(x)$. The ring

$$R = K(x, \omega^p) \cap K[\![x]\!]$$

is a discrete valuation ring with maximal ideal $m_R = xR$ and residue field $R/m_R = K$. In particular, R has completion $\hat{R} = K[\![x]\!] = \hat{A}$.

(1) R is not a Nagata ring.

Obviously, we have that $\omega \in K[\![x]\!]$, $\omega \notin Q(R) = K(x, \omega^p)$, and ω inseparable over $Q(R)$.

(2) R is analytically unramified in every $P \in \operatorname{Spec}(R)$.

<u>Pf</u>: $\operatorname{Spec}(R) = \{0, m_R\}$, R is analytically unramified in 0, and $R/m_R = K$ a field.

(3) R is not essentially of finite type over K.

For all $n \in \mathbb{N}$ set
$$\tau_n = \frac{1}{x^{pn}} \sum_{i=n+1}^{\infty} a_i^p x^{pi} \in K[\![x]\!]$$

and
$$B_n = K[x, \tau_n]_{(x, \tau_n)} \subseteq K[\![x]\!].$$

B_n is the localization of a polynomial ring in 2 variables over K. Obviously, for all $n \in \mathbb{N}$: $\quad B_n \subseteq B_{n+1}$

and $\quad R = \bigcup_{n\in\mathbb{N}} B_n$.

(4) The integral closure S of R in $K(x,\omega)$ is not a finite R-module.

For all $n \in \mathbb{N}$ set
$$\omega_n = \frac{1}{x^n}\sum_{i=n+1}^{\infty} a_i x^i \in K[\![x]\!]$$
and
$$C_n = K[x,\omega_n]_{(x,\omega_n)} \subseteq K[\![x]\!].$$
For all $n \in \mathbb{N}$
$$C_n \subseteq C_{n+1}$$
and
$$S = \bigcup_{n\in\mathbb{N}} C_n = K(x,\omega) \cap K[\![x]\!].$$
Then
$$S = \left(R[\omega_n]_{n\in\mathbb{N}}\right)_{(x)}$$
and S is a discrete valuation ring. (Note that by Krull-Akizuki S is Noetherian).

(5) $\hat{R} \otimes_R K(x,\omega)$ is not reduced.

Obviously, $(\omega\otimes 1 - 1\otimes\omega)^p = 0$. It remains to show that $\omega\otimes 1 - 1\otimes\omega \neq 0$ in $\hat{R}\otimes_R K(x,\omega)$. Since
$$\hat{R}\otimes_R K(x,\omega) \subseteq Q(\hat{R})\otimes_R K(x,\omega) = Q(\hat{R})\otimes_{Q(R)} K(x,\omega)$$
it is enough to show that $\omega\otimes 1 - 1\otimes\omega \neq 0$ in $Q(\hat{R})\otimes_{Q(R)} K(x,\omega)$. This can be seen as follows: Extend $\{1,\omega\}$ to a basis $\{e_i\}_{i\in I}$ of the $Q(R)$ vector space $K((x))$ and let $\{f_k\}_{k=0,\dots,p-1} = \{1,\omega,\dots,\omega^{p-1}\}$ be the natural basis of $K(x,\omega)$ over $Q(R) = K(x,\omega^p)$. Then $\{e_i\otimes f_k\}_{\substack{i\in I\\ j=0,\dots,p-1}}$ is a basis of the $Q(R)$-vector space $Q(\hat{R})\otimes_{Q(R)} K(x,\omega)$.

Hence $\omega\otimes 1 - 1\otimes\omega \neq 0$.

§2: FORMAL FIBERS AND NAGATA RINGS

(7.17) Definition: Let (R, m, k) be a local Noetherian ring, $\widehat{R}$ the m-adic completion of R and $P \subseteq R$ a prime ideal.

(a) The formal fiber of R at P is the ring $\widehat{R} \otimes_R k(P)$ where $k(P) = (R/P)_P$.

(b) R is called a ring with geometrically reduced (geometrically regular, normal) formal fibers if for every prime ideal $P \in \mathrm{Spec}(R)$ and for every finite field extension $k(P) \subseteq L$ the ring $\widehat{R} \otimes_R L$ is reduced (regular, normal).

(7.18) Proposition: Let (R, m) be a local Noetherian ring and $\widehat{R}$ the m-adic completion of R. The following are equivalent:

(a) R has geometrically reduced (regular, normal) formal fibers.

(b) For all $P \in \mathrm{Spec}(R)$ and for all finite purely inseparable field extensions $k(P) \subseteq L$ the ring $\widehat{R} \otimes_R L$ is reduced (regular, normal).

Proof: We only need to show (b) $\Rightarrow$ (a). Let $P \in \mathrm{Spec}(R)$ and $k(P) \subseteq L$ a finite field extension. By (1.34) there is a finite purely inseparable field extension $k(P) \subseteq K'$ so that $K' \subseteq L(K')$ is separable. By assumption (b) the ring $\widehat{R} \otimes_R K'$ is reduced (regular, normal). Since $K' \subseteq L(K')$ is finite separable, $L(K')$ is étale over K'. Hence $(\widehat{R} \otimes_R K') \otimes_{K'} L(K') = \widehat{R} \otimes_R L(K')$ is étale over $\widehat{R} \otimes_R K'$ and $\widehat{R} \otimes_R L(K')$ is reduced (regular, normal) by (4.50). Since $\widehat{R} \otimes_R L \hookrightarrow \widehat{R} \otimes_R L(K')$ is faithfully flat, the assertion follows with §11, Theorem (9.60).

(7.19) Remark: Note that if K is a perfect field, then K admits exacty one purely inseparable field extension, namely $K \subseteq K$.

(7.20) <u>Theorem</u>: Let (R, m, k) be a local Noetherian ring. R is a Nagata ring if and only if R has geometrically reduced formal fibers.

<u>Proof</u>: "$\Rightarrow$": Let R be a Nagata ring, $P \subseteq R$ a prime ideal, and $k(P) \subseteq L$ a finite field extension. Then the integral closure S of R/P in L is a finite R-module and the completion of S with respect to its Jacobson radical is given by

$$\hat{S} = \hat{R} \otimes_R S.$$

Since S is a semilocal Nagata domain, by (7.12) $\hat{S}$ is reduced. The ring $\hat{R} \otimes_R L$ is a localization of $\hat{R} \otimes_R S$ and hence reduced.

"$\Leftarrow$": Let $P \subseteq R$ be a prime ideal, $k(P) \subseteq L$ a finite field extension and S the integral closure of R/P in S. We have to show that S is a finite R-module. Let $S_0 \subseteq S$ be an R-subalgebra of S with S_0 a finite R/P-module and $Q(S_0) = L$. S_0 is a semi local domain with completion

$$\hat{S}_0 = \hat{R} \otimes_R S_0 .$$

By flatness of $\hat{R}$ over R the ring $\hat{R} \otimes_R S_0$ is contained in $\hat{R} \otimes_R L$. Hence $\hat{S}_0$ is reduced. By Rees' theorem (7.9) the integral closure S of S_0 in L is a finite S_0-module. Thus S is a finite R-module.

Theorem (7.20) characterizes local Nagata rings completely. A characterization of arbitrary Nagata rings also involves geometric reducedness of formal fibers. For non semilocal rings this condition is not sufficient. Additionally we need conditions on the normal locus of the ring R and some of its extensions.

(7.21) <u>Definition</u>: Let R be a Noetherian ring. R is said to have <u>geometrically reduced</u> (<u>regular</u>, <u>normal</u>) <u>formal fibers</u> if for all maximal ideals $m \subseteq R$ the localization R_m has geometrically reduced (regular, normal) formal fibers.

(7.22) <u>Exercise</u>: Let R be a semilocal Noetherian ring. Show that R is a

Nagata ring if and only if the formal fibers of R are geometrically reduced.

(7.23) <u>Definition</u>: Let R be a Noetherian ring. The <u>normal</u> <u>locus</u> of R is the set of prime ideals:

$$\mathrm{Nor}(R) = \{ p \in \mathrm{Spec}(R) \mid R_p \text{ is normal} \}.$$

The complement of $\mathrm{Nor}(R)$ in $\mathrm{Spec}(R)$ is denoted by $\mathrm{NNor}(R) = \mathrm{Spec}(R) - \mathrm{Nor}(R)$.

(7.24) <u>Lemma</u>: Let R be a Noetherian domain. Then $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$ if and only if there is a nonzero element $f \in R$ with R_f a normal domain.

<u>Proof</u>: "$\Longrightarrow$": If $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$ then $\mathrm{NNor}(R)$ is closed and there is an ideal $I \subseteq R$ with $\mathrm{NNor}(R) = V(I)$. Since R is a domain, $I \neq 0$, and there is an $f \in I - (0)$. Then R_f is a normal domain.

"$\Longleftarrow$": Let $f \in R - (0)$ with R_f normal. Then there are only finitely many height one primes $P_1, \ldots, P_s$ with $f \in P_i$. Hence there are at most finitely many (if any) height one primes Q so that R_Q is not a regular ring. After renumbering — if necessary — let $P_1, \ldots, P_r$ be the height one prime ideals of R with R_{P_i} not regular. Then

$$V_1 = V(P_1 \cap \cdots \cap P_r) \subseteq \mathrm{NNOR}(R)$$

If R_Q is regular for all height one primes $Q \subseteq R$ set $V_1 = \emptyset$.

Since R_f is a normal domain, for all primes $P \in \mathrm{Spec}(R)$ with $f \notin P$ and $\mathrm{ht}\, P \geq 2$:

$$\mathrm{depth}\,(R_P) \geq 2.$$

Let $P \in \mathrm{Spec}(R)$ be a prime ideal with $f \in P$ and $\mathrm{depth}\,(R_P) = 1$. Then $P \in \mathrm{Ass}_R(R/fR)$. Let

$$\{Q_1, \ldots, Q_t\} = \{ Q \in \mathrm{Ass}_R(R/fR) \mid \mathrm{ht}\, Q \geq 2 \}$$

and $V_2 = V(Q_1 \cap \cdots \cap Q_t)$ where $V_2 = \emptyset$ if R/fR has no embedded primes.

Then $$\mathrm{NNor}(R) = V_1 \cup V_2$$
$$= V(P_1 \cap \dots \cap P_r) \cup V(Q_1 \cap \dots \cap Q_t)$$
$$= V(P_1 \cap \dots \cap P_r \cap Q_1 \cap \dots \cap Q_t)$$
is a closed subset of $\mathrm{Spec}(R)$ and $\mathrm{Nor}(R)$ is open.

(7.25) <u>Proposition</u>: Let R be a Noetherian domain and S the integral closure of R in its field of quotients $Q(R)$. If S is a finite R-module then $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$.

<u>Proof</u>: Let
$$I = \{ r \in R \mid rS \subseteq R \} = R :_R S$$
denote the conductor of S. Since S is a finitely generated R-module and R is a domain, there is a nonzero element $f \in I$. We claim that R_f is normal. First note that S_f is the integral closure of R_f in $Q(R)$. Let $t \in S$, then $tf \in R$ and hence $t = tf/f \in R_f$. Therefore $S_f = R_f$ and R_f is normal. By Lemma (7.24) $\mathrm{Nor}(R)$ is an open subset of $\mathrm{Spec}(R)$.

(7.26) <u>Remark</u>: Note that under the assumptions of (7.25) $\mathrm{NNor}(R) = V(I)$ where I is the conductor of S.

(7.27) <u>Theorem:</u> Let R be a Noetherian domain and $K = Q(R)$ the quotient field of R. Suppose that:

(a) $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$.

(b) R_m is N-1 for all $m \in m\text{-}\mathrm{Spec}(R)$.

Then R is a N-1 ring.

<u>Proof</u>: We have to show that the integral closure S of R in $K = Q(R)$ is a

finitely generated R-module. Since the integral closure commutes with localization, S_m is the integral closure of R_m in K where m is a maximal ideal of R and S_m the localization of S at $R-m$.

For all $m \in m\text{-}\mathrm{Spec}(R)$ there are elements $\omega_1, \ldots, \omega_n \in S$ with
$$S_m = \sum_{i=1}^{n} R_m \omega_i.$$
Note that n and ω_i depend on $m \in m\text{-}\mathrm{Spec}(R)$ and set
$$S(m) = R[\omega_1, \ldots, \omega_n] \subseteq S.$$
Since R is a domain and $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$, by (7.24) there is a nonzero element $f \in R$ with R_f a normal domain. In particular, $R_f = S_f = S(m)_f$.

For all $m \in m\text{-}\mathrm{Spec}(R)$ set
$$X(m) = \mathrm{Spec}(S(m))$$
$$F(m) = X(m) - \mathrm{Nor}(S(m)) = \mathrm{NNor}(S(m)).$$
Since $S(m)_f$ is normal, by (7.24) $\mathrm{Nor}(S(m))$ is an open subset of $X(m)$ and $F(m)$ is closed in $X(m)$. The canonical morphism $R \longrightarrow S(m)$ induces a continuous map:
$$\tau_m : \mathrm{Spec}(S(m)) \longrightarrow \mathrm{Spec}(R).$$
Moreover, since $S(m)$ is integral over R, the map τ_m is closed (Homework) and
$$\tau_m(F(m)) \subseteq \mathrm{Spec}(R)$$
is a closed subset of $\mathrm{Spec}(R)$. Let $\mathfrak{M} \subseteq S(m)$ be a maximal ideal with $\mathfrak{M} \cap R = m$. Since $S(m)_m$ is normal and $S(m)_{\mathfrak{M}}$ is a localization of $S(m)_m$, $\mathfrak{M} \notin F(m)$ and hence $m \notin \tau_m(F(m))$. The set
$$V = \bigcap_{m \in m\text{-}\mathrm{Spec}(R)} \tau_m(F(m))$$
is a closed subset of $\mathrm{Spec}(R)$ which fails to contain any maximal ideal of R. Thus $V = \emptyset$. Since $\mathrm{Spec}(R)$ is quasi compact, there are finitely many maximal ideals $m_1, \ldots, m_r \in m\text{-}\mathrm{Spec}(R)$ with:
$$\bigcap_{i=1}^{r} \tau_{m_i}(F(m_i)) = \emptyset.$$
Set $T = R[S(m_1), \ldots, S(m_r)] \subseteq S$. T is a finite R-module and we claim

that $T = S$.

It suffices to show that T is normal. The normal property is local and we have to show that T_P is normal for all $P \in \mathrm{Spec}(T)$. Let $P \subseteq T$ be a prime ideal. Since

$$\bigcap_{i=1}^{r} \tau_{m_i}(F(m_i)) = \emptyset$$

there is an $1 \leq i \leq r$ with $P \cap R \notin \tau_{m_i}(F(m_i))$. Notice that $S(m_i) \subseteq T$ and set $P_i = P \cap S(m_i)$. Then $P_i \notin F(m_i)$ implying that $P_i \in \mathrm{Nor}(S(m_i))$. Thus $S(m_i)_{P_i} = T_P$ is a normal domain.

(7.28) <u>Theorem</u>: Let R be a Noetherian ring. The following conditions are equivalent:

(a) R is a Nagata ring.

(b) R satisfies the following conditions:

(i) The formal fibers of R are geometrically reduced.

(ii) For every finite R-algebra S, which is a domain, $\mathrm{Nor}(S)$ is open in $\mathrm{Spec}(S)$.

<u>Proof</u>: (a) $\Rightarrow$ (b): Let R be a Nagata ring. Then R_P is a Nagata ring for all $P \in \mathrm{Spec}(R)$ and by (7.20) the formal fibers of R_P are geometrically reduced. If S is a finite R-algebra and a domain, then S is a Nagata ring and its integral closure T in $Q(S)$, the quotient field of S, is a finite S-module. By (7.25) $\mathrm{Nor}(S)$ is open in $\mathrm{Spec}(S)$.

(b) $\Rightarrow$ (a): We have to show that the ring R/P is N-2 for all $P \in \mathrm{Spec}(R)$. Note that if R satisfies assumptions (i) and (ii) so does R/P. Thus we may assume that R is a domain satisfying (i) and (ii) and have to show that R is N-2.

Let $K = Q(R)$, $K \subseteq L$ a finite field extension, and S the integral closure of R in L. Then there is a finite R-subalgebra T of S with quotient field $Q(T) = L$. By assumption (ii) $\mathrm{Nor}(T)$ is open in $\mathrm{Spec}(T)$. Let

$\mathfrak{N} \in \text{m-Spec}(T)$ be a maximal ideal, $\mathfrak{m} = \mathfrak{N} \cap R$ its contraction to R, and $\mathfrak{N} = \mathfrak{N}_1, \ldots, \mathfrak{N}_r$ the maximal ideals of T which lie over $\mathfrak{m}$. $T_\mathfrak{m}$ is a semilocal ring and its completion with respect to the Jacobson radical is given by:

$$\hat{T}_\mathfrak{m} = T_\mathfrak{m} \otimes_{R_\mathfrak{m}} \hat{R}_\mathfrak{m} \cong \prod_{i=1}^{r} \hat{T}_{\mathfrak{N}_i} .$$

Since $\hat{R}_\mathfrak{m}$ is flat over $R_\mathfrak{m}$, $\hat{T}_\mathfrak{m}$ is isomorphic to a subring of $L \otimes_{R_\mathfrak{m}} \hat{R}_\mathfrak{m}$. By assumption (i) $L \otimes_{R_\mathfrak{m}} \hat{R}_\mathfrak{m}$ is reduced and hence $\hat{T}_\mathfrak{m}$ and $\widehat{T_{\mathfrak{N}_i}}$ are reduced. By Rees (7.9) the integral closure of $T_\mathfrak{N}$ in $Q(T_\mathfrak{N}) = L$ is a finite $T_\mathfrak{N}$-module and $T_\mathfrak{N}$ is N-1. By (7.27) T is a N-1 ring and S is a finite R-module.

(7.29) <u>Theorem</u>: Let R be a Nagata ring and S an R-algebra of finite type. Then S is a Nagata ring.

<u>Proof</u>: Let S be an R-algebra of finite type. A homomorphic image of a Nagata ring is a Nagata ring and we may assume that $R \subseteq S$ and $S = R[x_1, \ldots, x_n]$. We proceed by induction on n, the number of generators of S as an R-algebra. Thus we can assume that $S = R[x]$ and have to show that the ring $S/\mathfrak{p} = (R/\mathfrak{p} \cap R)[x]$ is N-2 for all $\mathfrak{p} \in \text{Spec}(S)$. It suffices to show:

($*_1$) Let R be a Nagata domain and $S = R[x]$ a domain with $R \subseteq S$. Then S is a N-2 ring.

Let $K = Q(R)$ be the field of quotients of R and $\tilde{R}$ the integral closure of R in K. Note that $\tilde{R}$ is a finite R-module and a Nagata ring. Moreover, we have inclusions

$$S = R[x] \subseteq \tilde{R}[x] \subseteq Q(S)$$

where $\tilde{R}[x]$ is a finite S-module. It suffices to show that $\tilde{R}[x]$ is N-2. We have to show:

($*_2$) Let R be a normal Nagata domain and $S = R[x]$ a domain with $R \subseteq S$.

Then $\tilde{S}$ is a N-2 ring.

We distinguish two cases:

Case 1: x is transcendental over R.

Pf of 1: In this case $S = R[x]$ is a normal domain. Let $K = Q(R)$. We first show that if $K(x) \subseteq L$ is a finite purely inseparable field extension then the integral closure of S in L is a finite S-module. Write $L = K(x, \alpha_1, \dots, \alpha_n)$ and let $e \in \mathbb{N}$ be an integer with $\alpha_i^q \in K(x)$ where $q = p^e$ and $\operatorname{char} K = p > 0$. Then there is a finite purely inseparable field extension $K \subseteq K'$ so that $\alpha_i \in K'(x^{1/q})$ for all $1 \leq i \leq r$. Let R' denote the integral closure of R in K' and S' the integral closure of S in L. Since R is a Nagata domain, R' is a finite R-module, and $R'[x^{1/q}]$ is a finite $S = R[x]$-module. Then $S' \subseteq R'[x^{1/q}]$, since $R'[x^{1/q}]$ is normal, and S' is a finite S-module.

If $K(x) \subseteq L$ is an arbitrary finite field extension, by (1.34) there is a finite purely inseparable field extension $K(x) \subseteq E$ with $E \subseteq L(E)$ a separable extension. By the previous argument the integral closure T_1 of S in E is a finite S-module. Then T_1 is a normal Noetherian domain and by 910, Corollary (5.62) the integral closure T of T_1 in $L(E)$ is a finite T_1-module. Hence T is a finite S-module and so is the integral closure of S in L.

Case 2: x is algebraic over R

Pf of 2: In this case $K = Q(R) \subseteq Q(S) = K(x)$ is a finite field extension and for every finite field extension $K(x) \subseteq L$, L is finite over K. Let $\tilde{R}$ be the integral closure of R in L and $\tilde{S}$ the integral closure of S in L. Then there are inclusions:

$$S = R[x] \subseteq \tilde{R}[x] \subseteq \tilde{S}$$

where $\tilde{R}[x]$ is a finite S-module. Thus it suffices to show:

($*_3$) Let R be a normal Nagata ring with field of quotients $K = Q(R)$ and $x \in K$ an element. Then the ring $S = R[x]$ is N-1.

Write $x = b/a$ for some $a, b \in R$ with $a \neq 0$. Then $S_a = S[1/a] = R[1/a] = R_a$ is normal and by (7.24) $\mathrm{Nor}(S)$ is open in $\mathrm{Spec}(S)$. By (7.27) it suffices to show:

$(*_4)$ Assumptions as in $(*_3)$. With $S = R[x]$ the ring S_P is N-1 for all maximal ideals $P \subseteq S$.

Obviously, if $a \notin P$, then S_P is a normal ring. Fix a maximal ideal $P \subseteq S$ with $a \in P$ and let $Q = P \cap R$. The ring $S_Q = R_Q[x]$ contains a maximal ideal $P' = PS_Q$ with $(S_Q)_{P'} = S_P$ and we have to show:

$(*_5)$ Let (R, m) be a normal local Nagata ring, $K = Q(R)$ its quotient field, and $x = b/a \in K$ where $a, b \in R$ and $a \in m - (0)$. Let $S = R[x]$. Then for every maximal ideal $P \subseteq S$ with $P \cap R = m$ the ring S_P is N-1.

We want to show that we can assume that $x \in P$. Suppose that $a \in P$ and $x = b/a \notin P$. Then there are elements $\alpha \in S$ and $p_0 \in P$ with

$$\alpha x + p_0 = 1.$$

Since $\alpha \in S = R[x]$ we may write:

$$\alpha = \sum_{i=0}^{t} r_i x^i$$

where $r_i \in R$ for all $0 \leq i \leq t$. This implies

$$\sum_{i=0}^{t} r_i x^{i+1} + p_0 = 1$$

and $x + P \in S/P$ is algebraic over the residue field $k = R/m$. Let t be a variable over R. Then there is a monic polynomial $f \in R[t]$ with $f(x) \in P$.

Let $K \subseteq L$ be a finite field extension so that $f(t) \in K[t]$ splits completely into linear factors over L. Let R' be the integral closure of R in L and set $S' = R'[x]$. Since R is a Nagata ring, R' is a finite R-module and S' is a finite S-module. Hence S'_P is a finite S_P-module and a semilocal ring. There is a one-to-one correspondence between the maximal ideals of S'_P and the maximal ideals of S' which lie over P.

Let $P_1, \ldots, P_s$ be the maximal ideals of S' which lie over P and T the integral closure of S'_P in L. Since S'_P is a finite S_P-module it suffices to show that T is a finite S'_P-module or equivalently that S'_P is N-1. Note that $(S'_P)_a = R'_a$ is a normal ring. Thus $\mathrm{Nor}(S'_P)$ is open by (7.24). By (7.27) it suffices to show that S'_{P_i} is N-1 for all $1 \le i \le r$ or equivalently:

($*_i$) For all maximal ideals $P_i \in m-\mathrm{Spec}(S'_P)$ the integral closure of S'_{P_i} in $L = Q(S')$ is a finite S'_{P_i}-module.

Fix a maximal prime $P_i \in m-\mathrm{Spec}(S'_P)$. Since R' is normal and $f(t)$ splits into linear factors over $L = Q(R')$ we have that

$$f(t) = \prod_{i=1}^{m} (t - a_i)$$

where $a_i \in R'$. Assume that $x \equiv a_1 \pmod P$. Then by replacing R, S and P by $R'_{P_i \cap R'}$, S', P_i and x by $x - a_1$ we have to show:

($*_1$) Let (R, m) be a normal local Nagata ring with field of quotients $K = Q(R)$, $x \in K$, and $S = R[x]$. Let $P \subseteq S$ be a maximal ideal with $P \cap R = m$ and $x \in P$. Then S_P is N-1.

Let t be a variable over R. Consider the R-algebra morphism

$$\varphi : R[t] \longrightarrow S$$

defined by $\varphi(t) = x$. Set $I = \ker(\varphi)$. We claim that

$$(\Delta) \qquad I = \sum_{\substack{a, b \in R \text{ with} \\ x = b/a}} (at - b) R[t] = \tilde{I}.$$

<u>Pf of</u> (Δ): Obviously, $\tilde{I} \subseteq I$. Suppose that $\tilde{I} \neq I$ and let $F \in I - \tilde{I}$ of minimal degree. Then

$$F = a_n t^n + a_{n-1} t^{n-1} + \text{——} + a_1 t + a_0$$

where $a_i \in R$. Then $a_n x$ is integral over R and hence $a_n x = b_n \in R$ since R is normal. This yields that

$$F(t) - (a_n t - b_n) t^{n-1} = G(t) \in \tilde{I}$$

since $\deg G < \deg F$ and therefore $F \in \tilde{I}$, a contradiction.

Next consider the R-ideal:

$$J = xR \cap R = \sum_{\substack{b \in R \text{ and } at - b \in I \\ \text{for some } a \in R}} Rb.$$

Then:

$$S/xS = R[t]/(tR[t]+I) = R[t]/(tR[t]+J) \cong R/J.$$

Our goal is to show that the ring S_p is analytically unramified. Then S_p is N-1 by Rees theorem (7.9). By (7.11) we have to show for all $Q \in \mathrm{Ass}_{S_p}(S_p/xS_p)$:

(a) $(S_p)_Q$ is a DVR

(b) S_p is analytically unramified in Q.

Pf (a): Since R is a normal Noetherian ring by 910, Theorem (5.44):

$$R = \bigcap_{\substack{W \in \mathrm{Spec}(R) \\ \mathrm{ht} W = 1}} R_W.$$

Then in $K = Q(R)$:

$$xR = \bigcap_{\substack{W \in \mathrm{Spec}(R) \\ \mathrm{ht} W = 1}} xR_W$$

and

$$J = xR \cap R = \bigcap_{\substack{W \in \mathrm{Spec}(R) \\ \mathrm{ht} W = 1}} (xR_W \cap R).$$

R_W is a DVR and $x \notin WR_W$ implies that $x^{-1} \in R_W$. Thus $R_W = xR_W$ if $x \notin WR_W$. Since R is a Krull ring there are only finitely many $W_1, \ldots, W_t \in \mathrm{Spec}(R)$ with $\mathrm{ht} W_i = 1$ and $x \in W_i R_{W_i}$. This implies that

$$J = \bigcap_{i=1}^{t} (xR_{W_i} \cap R).$$

Each ideal $xR_{W_i} \cap R$ is W_i-primary showing that the ring R/J has no embedded prime ideals.

Let $Q \in \mathrm{Ass}_{S_p}((S/xS)_p)$, then $W = Q \cap R \in \mathrm{Ass}_R(R/J)$ and we have ring extensions:

$$R_W \subseteq (S_p)_W \subseteq K = Q(R) = Q(S).$$

Since R_W is a DVR, $R_W = (S_p)_W$ and $(S_p)_W = (S_p)_Q$ is a DVR.

Pf (b): Let $Q \in \mathrm{Ass}_{S_p}((S/xS)_p)$ and $W = Q \cap R \in \mathrm{Ass}_R(R/J)$. Then $S_p/Q = R/W$ and $\widehat{S_p/Q} = \widehat{R/W}$ is reduced since R is a Nagata ring.

(7.30) <u>Exercise</u>: (a) Let R be a Nagata ring and $S \subseteq R$ a multiplicative set. Show that $S^{-1}R$ is a Nagata ring.

(b) Let R be a Noetherian ring and S a finite R-algebra with $R \subseteq S$. Show that if S is a Nagata ring, then R is a Nagata ring.

(7.31) <u>Examples:</u> (a) Every field K is a Nagata ring. By (7.29) and (7.30)(a) a localization of a K-algebra of finite type $S^{-1}(K[x_1, -, x_n]/I)$ is a Nagata ring.

(b) Every discrete valuation ring (R, m) with $\mathbb{Z} \subseteq R$ is a Nagata ring.

<u>Pf</u>: Since R is local, we have to show that the formal fibers of R are geometrically reduced. Since $\dim(R) = 1$, we only have to show that the formal fiber at $P = (0)$ is geometrically reduced. $\widehat{R} \otimes_R Q(R)$ is a field with char $Q(R) = 0$.

(c) $\mathbb{Z}$ is a Nagata ring.

<u>Pf</u>: By (b) for every prime element $p \in \mathbb{Z}$ the formal fibers of $\mathbb{Z}_{(p)}$ are geometrically reduced. By (7.28) it remains to show that for every finite $\mathbb{Z}$-algebra S which is a domain, $\mathrm{Nor}(S)$ is open in $\mathrm{Spec}(S)$. If $\mathbb{Z} \nsubseteq S$, S is a field. Hence assume that $\mathbb{Z} \subseteq S$. Then $\mathbb{Q} \subseteq Q(S)$ is a finite separable field extension and by 910, Theorem (5.44) the integral closure T of $\mathbb{Z}$ in $Q(S)$ is a finite $\mathbb{Z}$-module. Thus T is a finite S-module and by (7.25) $\mathrm{Nor}(S)$ is open.

(d) By (7.29) and (7.30)(a) every localization of a $\mathbb{Z}$-algebra of finite type $S^{-1}(\mathbb{Z}[x_1, -, x_n]/I)$ is a Nagata ring.