CHAPTER V: JACOBIAN CRITERIA; EQUATIONS OVER HENSELIAN RINGS

§1: A JACOBIAN CRITERION FOR ÉTALE AND UNRAMIFIED MORPHISMS

(5.1) Theorem: Let R be a Noetherian ring, $S = R[x_1, \ldots, x_n]$ the polynomial ring over R in n variables and $I \subseteq S$ an ideal. Set $T = S/I$ and let $Q \subseteq T$ be a prime ideal, $W \subseteq S$ its contraction to S.

(a) The following conditions are equivalent:

(a.1) T is unramified over R in a neighborhood of Q.

(a.2) There are polynomials $P_1, \ldots, P_n \in I$ so that $D = \det(\partial P_i/\partial x_j) \notin W$.

(a.3) There are polynomials $P_1, \ldots, P_n \in I$ such that $\bar{D}$ is invertible in T_Q where $D = \det(\partial P_i/\partial x_j)$.

(b) The following conditions are equivalent:

(b.1) T is étale over R in a neighborhood of Q.

(b.2) There are polynomials $P_1, \ldots, P_n \in I$ and $f \in S - W$ such that

(i) $(P_1, \ldots, P_n) S_f = I_f$

(ii) $D = \det(\partial P_i/\partial x_j) \notin W$.

Proof: (a) Obviously, (a.2) $\iff$ (a.3)

T is an R-algebra of finite type and $\Omega_{T/R}$ is a finitely generated T-module. Thus:

(a.1) $\iff$ T_f is unramified over R for some $f \in T - Q$

$\iff$ $\Omega_{T_f/R} = (\Omega_{T/R})_f = 0$ for some $f \in T - Q$

$\iff$ $(\Omega_{T/R})_Q = \Omega_{T_Q/R} = 0$.

Moreover,

$$\Omega_{S/R} = \bigoplus_{i=1}^{n} S\,dx_i \quad \text{and} \quad \Omega_{T/R} = \Omega_{S/R}/dI$$

where $d: S \longrightarrow \Omega_{S/R}$ is the universal derivation defined by:

$dF = \sum_{i=1}^{n} \partial F/\partial x_i \, dx_i$ for all $F \in S = R[x_1, \ldots, x_n]$. Then:

$$
\begin{aligned}
(\Omega_{T/R})_Q &= (\Omega_{S_W/R})/dI_W \\
&= (\oplus_{i=1}^{n} S_W\, dx_i)/dI_W \\
&= (\oplus_{i=1}^{n} S_W\, dx_i)/(dI_W + I_W(\oplus S_W\, dx_i)) \\
&= (\oplus_{i=1}^{n} T_Q\, dx_i)/dI_W
\end{aligned}
$$

Let $I = (P_1, \ldots, P_m)$, then

$$dI = \sum_{j=1}^{m} S\, dP_j + I\,\Omega_{S/R} \subseteq \Omega_{S/R}.$$

This implies that $(\Omega_{T/R})_Q = 0$ if and only if:

$$\oplus_{i=1}^{n} T_Q\, dx_i = \sum_{j=1}^{m} T_Q\, dP_j.$$

(a.1) $\Rightarrow$ (a.3): Assuming (a.1), then $\oplus_{i=1}^{n} T_Q\, dx_i = \sum_{j=1}^{m} T_Q\, dP_j$ and there are (after renumbering, if necessary) $P_1, \ldots, P_n \in I$ so that $dP_1, \ldots, dP_n$ are a basis of $\oplus_{i=1}^{n} T_Q\, dx_i$. Therefore the determinant of the transformation matrix $D = \det(\partial P_i/\partial x_j)$ is invertible in T_Q.

(a.3) $\Rightarrow$ (a.1): Let $P_1, \ldots, P_n \in I$ so that $D = \det(\partial P_i/\partial x_j)$ is invertible in T_Q. Then $\oplus_{i=1}^{n} T_Q\, dx_i = \sum_{j=1}^{m} T_Q\, dP_j$ and $(\Omega_{T/R})_Q = 0$.

1b) (b.2) $\Rightarrow$ (b.1): By assumption $T_f = S_f/(P_1, \ldots, P_n)$ where $f \in S - W$ and $D = \det(\partial P_i/\partial x_j) \notin W$. By (a) we know that there is an $h \in S - W$ so that T_h is unramified over R. It remains to show that there is a $g \in S - W$ so that T_g is smooth over R. Let $g = f.D$. We claim that T_g is a smooth R-algebra and T is étale over R in a neighborhood of Q.

In order to show this let C be an R-algebra, $J \subseteq C$ an ideal with $J^2 = 0$. Consider a commutative diagram of ring morphisms:

$$
\begin{tikzcd}
 & T_g \arrow[r, "\bar{u}"] & \bar{C} = C/J \\
S \arrow[ur, "\mu"] & & \\
 & R \arrow[ul] \arrow[uu] \arrow[r] & C \arrow[uu, "\nu"']
\end{tikzcd}
$$

where μ and ν are the natural maps. We need to find an R-algebra morphism $\tau: S \longrightarrow E$ with $I_g \subseteq \ker(\tau)_g$.

Set $\varphi = \bar{u}\mu$ and $\bar{e}_i = \varphi(x_i) \in \bar{C}$ and let $e_1, \ldots, e_n \in C$ be preimages of $\bar{e}_1, \ldots, \bar{e}_n$. In order to complete the proof we have to find elements $v_1, \ldots, v_n \in J$

so that $P_i(e_1+v_1, \ldots, e_n+v_n) = 0$ in C for all $1 \le i \le n$. Let $y_1, \ldots, y_n$ be variables over C, then by Taylor's formula:

$$P_i(e_1+y_1, \ldots, e_n+y_n) = P_i(e_1, \ldots, e_n) + \textstyle\sum_{j=1}^{n} \partial P_i/\partial y_j (0, \ldots, 0)\, y_j +$$
$$+ \text{ terms of total degree} \ge 2 \text{ in } y.$$

Notice that $\partial P_i/\partial y_j (0, \ldots, 0) = \partial P_i/\partial x_j (e_1, \ldots, e_n)$. Since $\mathcal{J}^2 = 0$, in order to find $v_1, \ldots, v_n \in \mathcal{J}$ with $P_i(e_1+v_1, \ldots, e_n+v_n) = 0$ for all $1 \le i \le n$ we need to solve the matrix equation:

$$(*) \qquad (\partial P_i/\partial x_j)(e_1, \ldots, e_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (-1) \begin{pmatrix} P_1(e_1, \ldots, e_n) \\ \vdots \\ P_n(e_1, \ldots, e_n) \end{pmatrix}.$$

Since $D = \det(\partial P_i/\partial x_j)$ is invertible in T_g, $\overline{u}(D) = \det(\partial P_i/\partial x_j)(\overline{e_1}, \ldots, \overline{e_n})$ is invertible in $\overline{C}$ and the matrix $(\partial P_i/\partial x_j(e_1, \ldots, e_n))$ is invertible in C^{n^2}. Thus $(*)$ is solvable and there are $v_1, \ldots, v_n \in \mathcal{J}$ with $P_i(e_1+v_1, \ldots, e_n+v_n) = 0$ in C, since $P_i(e_1, \ldots, e_n) \in \mathcal{J}$ for all $1 \le i \le n$. The R-algebra morphism $\tau : S \longrightarrow E$ defined by $\tau(x_i) = e_i + v_i$ for all $1 \le i \le n$ factors through T_g yielding a lifting of $\overline{u}$.

(b.1) $\Rightarrow$ (b.2): If T is étale over R in a neighborhood of Q, then T is unramified over R in a neighborhood of Q and by (a) there are $P_1, \ldots, P_n \in I$ so that:

$$\det(\partial P_i/\partial x_j) \notin W.$$

We want to show that there is an element $f \notin W$ so that $I_f = (P_1, \ldots, P_n)_f$. Since I is finitely generated it suffices to show that $I_W = (P_1, \ldots, P_n)_W$. Set $T' = S/(P_1, \ldots, P_n)$ and $I' = I/(P_1, \ldots, P_n)$ and let Q' be the contraction of Q to T'. Consider the exact sequence of R-modules:

$$(*) \quad 0 \longrightarrow (I'/I'^2)_{Q'} \longrightarrow (T'/I'^2)_{Q'} \xrightarrow{\mu} T_Q \longrightarrow 0$$

We claim that the sequence $(*)$ is split exact. Let $g \in T' - Q'$ so that T_g is étale over R and consider the commutative diagram of ring morphisms:

$$\begin{array}{ccc} T_g & \xrightarrow{\text{id}} & (T'/I')_g \\ \uparrow & & \uparrow \nu \\ R & \longrightarrow & (T'/I'^2)_g \end{array}$$

where ν is the natural map. Since T_g is smooth over R, the identity map lifts to an R-algebra morphism $v: T_g \longrightarrow (T'/I'^2)_g$ inducing a morphism $\varphi: T_Q \longrightarrow (T'/I'^2)_{Q'}$ with $\mu\varphi = \mathrm{id}$. The sequence $(*)$ is split exact and the sequence $0 \longrightarrow (I'/I'^2)_{Q'} \otimes_R k(P) \longrightarrow (T'/I'^2)_{Q'} \otimes_R k(P) \xrightarrow{\bar\mu} T_Q \otimes_R k(P) \longrightarrow 0$ is exact where $P = Q \cap R$. We want to show that $\bar\mu$ is an isomorphism. Notice that by (a) T' is unramified over R in a neighborhood of Q'. Hence there is an $h \in T' - Q'$ with T'_h unramified over R and $T'_h \otimes_R k(P)$ unramified over $k(P)$. By (3.3) $T'_h \otimes_R k(P)$ is a product of fields and the localized residue class ring $(T'/I'^2)_{Q'} \otimes_R k(P)$ and $T_Q \otimes_R k(P)$ are fields. $\bar\mu$ is an isomorphism implyieng that $(I'/I'^2)_Q \otimes_R k(P) = 0$. By Nakayama $(I'/I'^2)_Q = 0$ and thus $I'_{Q'} = 0$ yielding that $I_W = (P_1, \ldots, P_n)_W$.

(5.2) __Corollary__: Let R, S, I and T be as in (5.2). Suppose that T is unramified over R in a neighborhood of Q and that there are elements $H_1, \ldots, H_n \in I$ with $I_W = (H_1, \ldots, H_n)_W$. Then

$$\det(\partial H_i/\partial x_j) \notin W.$$

In particular, in this case T is étale over R in a neighborhood of Q.

__Proof__: Since T is unramified over R in a neighborhood of Q, $(\Omega_{T/R}) = 0$ where $\Omega_{T/R} = (\bigoplus_{i=1}^n S\,dx_i)/dI = (\bigoplus_{i=1}^n T\,dx_i)/dI$. Hence $dI_Q = \bigoplus_{i=1}^n T_Q\,dx_i$. By assumption $dI_Q = \sum_{i=1}^n T_Q\,dH_i$ and $dH_1, \ldots, dH_n$ form a basis of $\bigoplus_{i=1}^n T_Q\,dx_i$. With $dH_i = \sum_{j=1}^n \partial H_i/\partial x_j\,dx_j$ the Jacobian determinant $\det(\partial H_i/\partial x_j)$ is invertible in T_Q. By (5.1) T is étale over R in a neighborhood of Q.

§2: A JACOBIAN CRITERION FOR SMOOTH AFFINE ALGEBRAS

(5.3) Theorem: Let (R, m) be a local Noetherian ring. Suppose that R contains a field k and that R is smooth over k. Then R is a regular local ring.

Proof: Let $x_1, -, x_d \in m$ be a minimal system of generators of m. Since R contains a field, the m-adic completion of R contains a coefficient field K and by Cohen's structure theorem:
$$\widehat{R} \cong K[\![y_1, -, y_d]\!]/I$$
where $y_1, -, y_d$ are variables over K, $I \subseteq K[\![y_1, -, y_d]\!]$ an ideal. Put $M = (y_1, -, y_d) \subseteq K[\![y_1, -, y_d]\!]$. Since $x_1, -, x_d$ is a minimal set of generators of m and $R \to \widehat{R}$ faithfully flat, we get $I \subseteq M^2$. Thus there is a surjective morphism:
$$\nu : R \longrightarrow K[\![y_1, -, y_d]\!]/M^2.$$
We claim that ν lifts to a morphism of rings:
$$\nu_0 : R \longrightarrow K[\![y_1, -, y_d]\!].$$
If $\operatorname{char} k \neq 0$, then the coefficient field K may not contain k. Therefore, let P be a prime field contained in k, then there is a canonical map $P \hookrightarrow K$. Moreover, k is separable and thus smooth over P. Thus R is smooth over P and there is a commutative diagram of ring morphisms:
$$\begin{array}{ccc} R & \xrightarrow{\nu} & K[\![y_1, -, y_d]\!]/M^2 \\ \uparrow & & \uparrow \\ P & \longrightarrow & K[\![y_1, -, y_d]\!]/M^3 \end{array}$$
By the smoothness of R over P, ν lifts to a ring morphism:
$$\nu_3 : R \longrightarrow K[\![y_1, -, y_d]\!]/M^3.$$
Using the same argument again we see that ν_3 lifts to a morphism:
$$\nu_4 : R \longrightarrow K[\![y_1, -, y_d]\!]/M^4.$$
Repeat the argument. Since the power series ring is complete, this yields a ring morphism: $\nu_0 : R \longrightarrow K[\![y_1, -, y_d]\!]$ which lifts ν.

Since ν is surjective the induced map of the complete local rings

$$\hat{\nu}_0 : \hat{R} \longrightarrow K[\![y_1, \ldots, y_d]\!]$$

is surjective [911, Theorem 9.29]. This implies that $\dim(R) = \dim(\hat{R}) \geq d = \operatorname{edim}(R)$ and R is regular.

(5.4) Theorem: Let k be a field, $S = k[x_1, \ldots, x_n]$ the polynomial ring over k and $I, P \subseteq S$ ideals with P a prime ideal and $I \subseteq P$. Set $R = S_P$, $PR = M$, $A = R/IR$, $MA = m$, $R/M = A/m = K$ and suppose that $\operatorname{ht} I = r$ and $IR = (f_1, \ldots, f_t)$ where $f_1, \ldots, f_t \in S$. Then the following conditions are equivalent:

(a) $\operatorname{rank}(\partial f_i/\partial x_j \bmod P) = r$

(b) A is smooth over k.

(c) The module of differentials $\Omega_{A/k}$ is a free A-module of rank $n-r$.

(d) A is a domain, its field of fractions $Q(A)$ is separable over k, and the module of differentials $\Omega_{A/k}$ is a free A-module.

Proof: (a) $\Rightarrow$ (b): By assumption there are $D_1, \ldots, D_r \in \{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ and $g_1, \ldots, g_r \in \{f_1, \ldots, f_t\}$ so that $\det(D_i f_j) \notin M$. Consider the K-linear map $\varphi : M/M^2 \longrightarrow K^r$ defined by $\varphi(f) = (\overline{D_1 f}, \ldots, \overline{D_r f})$. Since $\det(D_i g_j) \notin M$, $\varphi(g_1), \ldots, \varphi(g_r)$ are linearly independent over K. Hence $g_1, \ldots, g_r$ are linearly independent in M/M^2.

Since R is a regular local ring with maximal ideal M, the elements $g_1, \ldots, g_r$ generate a prime ideal Q of height r in R. Thus $Q = IR = (g_1, \ldots, g_r)$, since $Q \subseteq IR$ and $\operatorname{ht} IR = r$. Moreover, A is a regular local ring.

Let C be a k-algebra, $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of ring morphisms:

$$\begin{tikzcd}
A \arrow[r, "\bar{u}"] & C/N \\
R \arrow[u, "\varepsilon"] \arrow[dr, "\tau"] & \\
k \arrow[u] \arrow[r] & C \arrow[uu, "\nu"']
\end{tikzcd}$$

Since $\bar{R}$ is the localization of a polynomial ring over R, $\bar{R}$ is smooth over k and there is a k-algebra morphism $\tau: \bar{R} \longrightarrow C$ with $\nu\tau = \bar{u}\varepsilon$.

Set $\tau(x_i) = u_i \in C$ and let $y_i = x_i + I\bar{R} \in A$ denote the image of x_i in A. Then $\nu(u_i) = \bar{u}(y_i)$ and we have to find elements $e_i \in N$ so that the modified k-algebra morphism $\tau': \bar{R} \longrightarrow C$ defined by $\tau'(x_i) = u_i + e_i$ factors through A, or equivalently, that $g_j(u_1+e_1, \ldots, u_n+e_n) = 0$ for all $1 \le j \le r$. Using the Taylor formula and $N^2 = 0$ we obtain:

(*) $\quad g_i(u_1+e_1, \ldots, u_n+e_n) = g_i(u_1, \ldots, u_n) + \sum_{j=1}^{n} (\partial g_i/\partial x_j)(u_1, \ldots, u_n)\, e_j$

for all $1 \le i \le r$. (*) yields a matrix equation:

$$\begin{pmatrix} g_1(u+e) \\ \vdots \\ g_r(u+e) \end{pmatrix} = \begin{pmatrix} g_1(u) \\ \vdots \\ g_r(u) \end{pmatrix} + \left(\partial g_i/\partial x_j (u) \right) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

where $u+e = (u_1+e_1, \ldots, u_n+e_n)$ and $u = (u_1, \ldots, u_n)$. By renumbering $x_1, \ldots, x_n$, if necessary, we may assume that

$$\det \left(\partial g_i/\partial x_j \right)_{1 \le i,j \le r} \notin M.$$

Hence $\det(\partial g_i/\partial x_j(u))_{1 \le i,j \le r}$ is invertible in C and there is an invertible $r \times r$ matrix $\Delta \in C^{r^2}$ so that:

$$\Delta \begin{pmatrix} g_1(u+e) \\ \vdots \\ g_r(u+e) \end{pmatrix} = \Delta \begin{pmatrix} g_1(u) \\ \vdots \\ g_r(u) \end{pmatrix} + \begin{pmatrix} 1 & 0 & \ldots & & * \\ 0 & 1 & \ldots & & * \\ & & & & \\ 0 & 0 & \ldots & 1 & * \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Therefore there are elements $e_1, \ldots, e_n \in N$ so that

$$0 = \Delta \begin{pmatrix} g_1(u) \\ \vdots \\ g_r(u) \end{pmatrix} + \begin{pmatrix} 1 & 0 & \ldots & & * \\ 0 & 1 & \ldots & & * \\ & & & & \\ 0 & 0 & \ldots & 1 & * \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

The morphism $\bar{u}$ lifts to a k-algebra morphism $u: A \longrightarrow C$ and A is smooth over k.

(b) $\Rightarrow$ (d): By (2.15) the sequence: $0 \longrightarrow I\bar{R}/I^2\bar{R} \longrightarrow \Omega_{\bar{R}/k} \otimes_{\bar{R}} A \longrightarrow \Omega_{A/k} \longrightarrow 0$

is split exact. Since $\Omega_{R/k} \otimes_R A$ is a free A-module and A is a local ring, the module of differentials $\Omega_{A/k}$ is free. Since A is smooth over k, by (5.3) A is a regular local ring and its field of quotients $Q(A)$ is smooth over k. By (2.20) $Q(A)$ is separable over k.

(d) $\Rightarrow$ (c): By assumption $Q(A) = L$ is separable and finitely generated over k. By (1.31) L is separably generated over k. Since $\operatorname{trdeg}_k L = \dim S/I = n-r$ there is a separating transcendence basis $y_1, \ldots, y_{n-r}$ of L over k. We claim that $dy_1, \ldots, dy_{n-r}$ is a basis of $\Omega_{L/k}$. In order to prove this let $E = k(y_1, \ldots, y_{n-r}) \subseteq L$. L is separable algebraic over E and by (2.14) the sequence of L-vector spaces:

$$0 \longrightarrow \Omega_{E/k} \otimes_E L \longrightarrow \Omega_{L/k} \longrightarrow \Omega_{L/E} \longrightarrow 0$$

is exact. Since L is separable algebraic over E, $\Omega_{L/E} = 0$ and $dy_1, \ldots, dy_{n-r}$ is a basis of $\Omega_{L/k}$. Therefore:

$$\operatorname{rk}(\Omega_{A/k}) = \dim_L(\Omega_{L/k}) = \operatorname{trdeg}_k L = n - r = \dim(S/I).$$

(c) $\Rightarrow$ (a): Consider the exact sequence of A-modules (see (1.12)):

$$IR/I^2R \xrightarrow{\overline{\delta}} \Omega_{R/k} \otimes_R A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

where $\overline{\delta}(a + I^2R) = \delta(a) + I\,\Omega_{R/k}$. Let $N = \operatorname{im}(\overline{\delta})$. Since A is local and $\Omega_{A/k}$ is a free A-module, N is a free A-module of rank r. N is generated by $(\partial f_i/\partial x_1, \ldots, \partial f_i/\partial x_n) \in \Omega_{R/k} \otimes_R A$ for $1 \le i \le t$. Since any basis of N is part of a basis of $\Omega_{R/k} \otimes_R A$, assertion (a) follows.

§3: EQUATIONS OVER HENSELIAN RINGS

(5.5) Lemma: Let (R, m, k) be a quasi local Henselian ring, $f \in R[x]$ a polynomial, and set $\bar{f} = f + mR[x] \in k[x]$. Suppose that there is an $\bar{a} \in k$ with $f(\bar{a}) = 0$ and $f'(\bar{a}) \neq 0$. Then there is an element $a \in R$ with

$$f(a) = 0 \quad \text{and} \quad a + m = \bar{a} \in k.$$

Proof: Consider the natural R-algebra morphism:

$$\varphi: \left(R[x]/(f)\right)_{f'} \longrightarrow k$$

defined by $\varphi(x) = \bar{a}$. By (5.1) $T = \left(R[x]/(f)\right)_{f'}$ is étale over R in the neighborhood of every $Q \in \operatorname{Spec}(T)$ with $Q \cap R = m$. In particular, with $n = \ker(\varphi)$, T_n is locally étale over R. Since $T_n/nT_n = k$ and R Henselian, $T_n \cong R$. Thus there is an element $a \in R$ with $f(a) = 0$ and $a + m = \bar{a} \in k$.

(5.6) Theorem (Implicit Function Theorem) Let (R, m, k) be a quasi local Henselian ring, $S = R[x_1, \dots, x_n]$ the polynomial ring over R, and $I = (f_1, \dots, f_n) \subseteq S$ an ideal. Let $J \subseteq m$ be an ideal in R and $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n) \in (R/J)^n$ an element so that:

(a) $f_i(\bar{a}_1, \dots, \bar{a}_n) = 0$ in R/J for all $1 \leq i \leq n$.

(b) The Jacobian determinant $\det(\partial f_i/\partial x_j)(\bar{a})$ is invertible in R/J.

Then there is an $a = (a_1, \dots, a_n) \in R^n$ so that for all $1 \leq i \leq n$ $f_i(a) = 0$ and $a_i + J = \bar{a}_i$.

Proof: Consider the R-algebra morphism:

$$\varphi: T = R[x_1, \dots, x_n]/I \longrightarrow R/J$$

defined by $\varphi(x_i) = \bar{a}_i$ and let $Q = \varphi^{-1}(m)$ and $W \subseteq S$ the preimage of Q in S. Since $\det(\partial f_i/\partial x_j)(\bar{a})$ invertible in R/J, the Jacobian $\det(\partial f_i/\partial x_j) \notin W$ and T_Q is locally étale over R by (5.1). Moreover, $T_Q/QT_Q = k$ and thus $T_Q \cong R$ (as R-algebras), since R is Henselian. Hence there are elements $a_i \in R$ with

$a_i + J = \overline{a_i}$ and $f_i(a_1, \ldots, a_n) = 0$ for all $1 \le i \le n$.

(5.7) <u>Theorem</u>: (Newton Lemma) Let (R, m, k) be a quasi local Henselian ring. Suppose that $f \in R[x]$ is a polynomial and $a \in R$ an element with $f(a) \equiv 0 \bmod (f'(a))^2 m$. Then there is an element $b \in R$ with $f(b) = 0$ and $a \equiv b \bmod f'(a) m$.

<u>Proof</u>: Let y be another variable over R. Then by Taylor's formula:

$$f(a + f'(a)y) = f(a) + f'(a)^2 y + f'(a)^2 y^2 g(y)$$

where $g(y) \in R[y]$. By assumption there is an element $c \in m$ with $f(a) = c\, f'(a)^2$. Hence:

$$\begin{aligned} f(a + f'(a)y) &= c\, f'(a)^2 + f'(a)^2 y + f'(a)^2 y^2 g(y) \\ &= f'(a)^2 h(y) \end{aligned}$$

where $h(y) = c + y + y^2 g(y) \in R[y]$. Setting $y = 0$ we obtain in k:

$$\overline{h}(0) = 0 \quad \text{and} \quad \overline{h'}(0) \neq 0.$$

By (5.5) there is an element $d \in m$ so that $h(d) = 0$. Hence $f(a + f'(a)d) = f'(a)^2 h(d) = 0$. $b = a + f'(a) d \in R$ is a root of f with $b \equiv a \bmod f'(a) m$.

(5.8) <u>Theorem</u>: Let (R, m, k) be a quasi local Henselian ring and $F = (f_1, \ldots, f_r) \subseteq R[x_1, \ldots, x_n]$ an ideal in the polynomial ring over R. Let

$$\Delta = \left(\partial f_i / \partial x_j \right)_{1 \le i \le r,\, 1 \le j \le n}$$

denote the Jacobian matrix of $f_1, \ldots, f_r$ and let D be an $r \times r$-minor of Δ. Suppose that there is an element $a = (a_1, \ldots, a_n) \in R^n$ so that

$$F(a) \equiv 0 \bmod (D(a))^2 m.$$

Then there is a $b = (b_1, \ldots, b_n) \in R^n$ with

$$F(b) = 0 \quad \text{and} \quad b_i \equiv a_i \bmod D(a) m \quad \text{for all } 1 \le i \le n.$$

<u>Proof</u>: For $r > n$ the statement is trivial. Hence we may assume that $r \le n$ and that:

$$D = \det \left(\partial f_i / \partial x_j \right)_{1 \le i, j \le r}.$$

If $r<n$ we complete $f_1,\dots,f_r$ to a system of n polynomials by setting $f_{r+1}=x_{r+1}-a_{r+1},\dots,f_n=x_n-a_n$. Hence we may assume $r=n$, $F=(f_1,\dots,f_n)$, and $D=\det(\partial f_i/\partial x_j)_{1\le i,j\le n}$. Obviously, the assumptions of the theorem are still valid.

Let $y_1,\dots,y_n$ be variables over R. Then by Taylor's formula for all $1\le i\le n$:

$$
\begin{aligned}
f_i(a+D(a)y) &= f_i(a_1+D(a)y_1,\dots,a_n+D(a)y_n)\\
&= f_i(a)+\textstyle\sum_{j=1}^{n}(\partial f_i/\partial x_j)(a)\,D(a)\,y_j + D(a)^2 g(a)
\end{aligned}
$$

where $a=(a_1,\dots,a_n)$, $y=(y_1,\dots,y_n)$, and $g_i(y)\in R[x_1,\dots,x_n]$ a polynomial in which every term has total degree at least 2. Hence:

$$
\begin{bmatrix} f_1(a+D(a)y)\\ \vdots\\ f_n(a+D(a)y)\end{bmatrix} = \begin{bmatrix} f_1(a)\\ \vdots\\ f_n(a)\end{bmatrix} + D(a)\,\Delta(a)\begin{bmatrix} y_1\\ \vdots\\ y_n\end{bmatrix} + D(a)^2\begin{bmatrix} g_1(y)\\ \vdots\\ g_n(y)\end{bmatrix}
$$

By assumption there are elements $c_i\in m$ so that $f_i(a)=D(a)^2c_i$ for all $1\le i\le n$ and therefore

$$
\begin{bmatrix} f_1(a)\\ \vdots\\ f_n(a)\end{bmatrix} = D(a)^2\begin{bmatrix} c_1\\ \vdots\\ c_n\end{bmatrix}
$$

Let $\Delta'\in R^{n\times n}$ be the adjoint matrix of $\Delta(a)$. Then $\Delta(a)\Delta'=D(a)E$ where E is the $n\times n$ identity matrix. In particular, $D(a)^2E=D(a)\,\Delta(a)\,\Delta' E$ and

$$
\begin{aligned}
\begin{bmatrix} f_1(a+D(a)y)\\ \vdots\\ f_n(a+D(a)y)\end{bmatrix} &= D(a)\,\Delta(a)\left[\Delta'\begin{bmatrix} c_1\\ \vdots\\ c_n\end{bmatrix} + \begin{bmatrix} y_1\\ \vdots\\ y_n\end{bmatrix} + \Delta'\begin{bmatrix} g_1(y)\\ \vdots\\ g_n(y)\end{bmatrix}\right]\\
&= D(a)\,\Delta(a)\left(\begin{bmatrix} d_1\\ \vdots\\ d_n\end{bmatrix} + \begin{bmatrix} y_1\\ \vdots\\ y_n\end{bmatrix} + \begin{bmatrix} l_1(y)\\ \vdots\\ l_n(y)\end{bmatrix}\right)
\end{aligned}
$$

for some $d_i\in m$ and $l_i(y)\in R[y_1,\dots,y_n]$. For $1\le i\le n$ consider the polynomial

$$h_i(y) = d+y_i+l_i(y)\in R[y_1,\dots,y_n].$$

Since each term of $l_i(y)$ has total degree at least 2 we have:

$$\left(\partial h_i/\partial y_j\right)_{1\le i,j\le n} = \begin{bmatrix} 1+q_1(y) & & * \\ & \ddots & \\ * & & 1+q_n(y) \end{bmatrix}$$

where $q_i(y) \in R[y_1, \dots, y_n]$ are polynomials with $q_i(0) = 0$. The entries outside the diagonal are also polynomials with constant term 0. Hence $h_i(0) \in m$ and $\det(\partial h_i/\partial y_j)(0) \equiv 1 \bmod m$. By the implicit function theorem (5.6) there are elements $t_1, \dots, t_n \in R$ with $t_i \in m$ and $h_j(t_1, \dots, t_n) = 0$ for all $1 \le i, j \le n$. Then with $t = (t_1, \dots, t_n) \in R^n$:

$$\begin{bmatrix} f_1(a+D(a)t) \\ \vdots \\ f_n(a+D(a)t) \end{bmatrix} = D(a)\Delta(a) \begin{bmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{bmatrix} = 0$$

$a + D(a)t \in R^n$ is a solution of $F = 0$ with $a_i + D(a)t_i \equiv a_i \bmod D(a)m$.

Let (R, m, k) be a local Noetherian Henselian ring and $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ polynomials over R. Let Δ denote the $m \times n$ Jacobian matrix

$$\Delta = \left(\partial f_i/\partial x_j\right)_{\substack{1\le i\le m \\ 1\le j\le n}} = \begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_m/\partial x_1 & \dots & \partial f_m/\partial x_n \end{bmatrix}.$$

For every $a = (a_1, \dots, a_n) \in R^n$ the matrix $\Delta(a)$ defines an R-linear transformation:

$$R^n \xrightarrow{\Delta(a)} R^m.$$

We denote by $C(a) = R^m/\mathrm{im}(\Delta(a))$ the cokernel of $\Delta(a)$ and by $I(a) = \mathrm{ann}_R(C(a))$ the annihilator of the R-module $C(a)$.

(5.9) <u>Theorem</u>: (Tougeron) Under assumptions and notations as above, let $d \in \mathbb{N}$ and $a = (a_1, \dots, a_n) \in R^n$ so that

$$f_i(a) \equiv 0 \bmod I(a)^2 m^d \quad \text{for all } 1 \le i \le m.$$

Then there is an element $b = (b_1, \dots, b_n) \in R^n$ with

$$f_i(b) = 0 \quad \text{for all } 1 \le i \le m \text{ and}$$

$$b_j \equiv a_j \bmod I(a) m^d \quad \text{for all } 1 \le j \le n.$$

Proof: Set $I = I(a)$ and let $d_1, \dots, d_r \in R$ be a system of generators of I; $e_1 = (1, \dots, 0), \dots, e_m = (0, \dots, 1)$ the canonical basis of R^m. Then for all $1 \le j \le r$, $1 \le i \le m$: $d_j e_i \in \operatorname{im}(\Delta(a))$. By definition of $\Delta(a)$ we have that:

$$\operatorname{im}(\Delta(a)) = \sum_{k=1}^{n} R \begin{bmatrix} \partial f_1/\partial x_k(a) \\ \vdots \\ \partial f_m/\partial x_k(a) \end{bmatrix}.$$

Thus for all $1 \le j \le r$ there is an $n \times m$-matrix $N_j \in R^{n \times m}$ so that:

$$d_j E = \Delta(a) N_j$$

where E denotes the $m \times m$ identity matrix. By assumption for all $1 \le k \le m$:

$$f_k(a) = \sum_{i,j=1}^{r} d_i d_j c_{kij}$$

where $c_{kij} \in m^d$. Let $t_1, \dots, t_n$ be variables and consider the system:

$$\begin{aligned} f_1(a_1+t_1, \dots, a_n+t_n) &= 0 \\ &\vdots \\ f_m(a_1+t_1, \dots, a_n+t_n) &= 0. \end{aligned}$$

We want to find solutions for t_i in the ideal $I m^d$. Let w_{ij}, $1 \le i \le n$, $1 \le j \le r$, be variables and write:

$$t_i = \sum_{j=1}^{r} d_j w_{ij}$$

Then by Taylor's formula:

$$f_i(a_1 + \Sigma d_j w_{1j}, \dots, a_n + \Sigma d_j w_{nj}) =$$
$$f_i(a) + \sum_{k=1}^{n} \partial f_i/\partial x_k(a) \sum_{j=1}^{r} d_j w_{kj} + \sum_{j,k} d_j d_k q_{ijk}$$

where $q_{ijk} \in R[w_{ij}]_{1 \le i \le n, 1 \le j \le m}$ and every term of q_{ijk} has total degree at least 2. This yields a matrix equation:

$$\begin{bmatrix} f_1(a + \Sigma d_j w_j) \\ \vdots \\ f_m(a + \Sigma d_j w_j) \end{bmatrix} = \begin{bmatrix} f_1(a) \\ \vdots \\ f_m(a) \end{bmatrix} + \Delta(a) \begin{bmatrix} \Sigma_j d_j w_{1j} \\ \vdots \\ \Sigma_j d_j w_{nj} \end{bmatrix} + \sum_{j,k} d_j d_k \begin{bmatrix} q_{1jk} \\ \vdots \\ q_{mjk} \end{bmatrix}$$

Since

$$\begin{bmatrix} f_1(a) \\ \vdots \\ f_m(a) \end{bmatrix} = \sum_{j,k} d_j d_k \begin{bmatrix} c_{1jk} \\ \vdots \\ c_{mjk} \end{bmatrix}$$

we obtain:

$$\begin{bmatrix} f_1(a+\sum d_j w_j) \\ \vdots \\ f_m(a+\sum d_j w_j) \end{bmatrix} = \Delta(a) \begin{bmatrix} \sum d_j w_{1j} \\ \vdots \\ \sum d_j w_{nj} \end{bmatrix} + \sum_{j,k} d_j d_k \begin{bmatrix} q_{1jk} + c_{1jk} \\ \vdots \\ q_{mjk} + c_{mjk} \end{bmatrix}.$$

Write

$$\Delta(a) \begin{bmatrix} \sum d_j w_{1j} \\ \vdots \\ \sum d_j w_{nj} \end{bmatrix} = \sum_{j=1}^{r} d_j \, \Delta(a) \begin{bmatrix} w_{1j} \\ \vdots \\ w_{nj} \end{bmatrix}$$

and use $d_j E = \Delta(a) N_j$ and $d_j d_k E = d_j \Delta(a) N_k$ to obtain:

$$\sum_{j,k} d_j d_k \begin{bmatrix} q_{1jk} + c_{1jk} \\ \vdots \\ q_{mjk} + c_{mjk} \end{bmatrix} = \sum_{j=1}^{r} d_j \Delta(a) \sum_{k=1}^{r} N_k \begin{bmatrix} q_{1jk} + c_{1jk} \\ \vdots \\ q_{mjk} + c_{mjk} \end{bmatrix}$$

yielding

$$\begin{bmatrix} f_1(a+\sum d_j w_j) \\ \vdots \\ f_m(a+\sum d_j w_j) \end{bmatrix} = \sum_{j=1}^{r} d_j \Delta(a) \left[\begin{bmatrix} w_{1j} \\ \vdots \\ w_{nj} \end{bmatrix} + \sum_{k=1}^{r} N_k \begin{bmatrix} q_{1jk} + c_{1jk} \\ \vdots \\ q_{mjk} + c_{mjk} \end{bmatrix} \right]$$

We want to solve the system of equations which is given by setting the inside of the brackets equal to zero, i.e. consider

$$h_{ij} = w_{ij} + g_{ij}(w)$$

where $g_{ij}(w)$ is the ij-entry of the matrix

$$\sum_{k=1}^{r} N_k \begin{bmatrix} q_{1jk} + c_{1jk} \\ \vdots \\ q_{mjk} + c_{mjk} \end{bmatrix}.$$

Every term of the polynomial $q_{\ell jk}$ has total degree at least 2, $c_{\ell jk} \in m^d$, and hence:

$$\left(\partial h_{ij} / \partial w_{\ell k}\right) = \begin{bmatrix} 1+p_1 & & * \\ & \ddots & \\ * & & 1+p_t \end{bmatrix}$$

with $t = rn$. Moreover, each p_i has constant term 0 and so has every entry outside the diagonal. Hence $h_{ij}(0) \equiv 0 \bmod m^d$ and $(\partial h_{ij}/\partial w_{ek})(0)$ invertible in R/m^d. By the implicit function theorem (5.6) the system of equations $h_{ij} = 0$ has a solution $(u_{ek}) \in R^{rn}$ with $u_{ek} \in m^d$. Then

$$b_i = a_i + \sum_{j=1}^{r} d_j u_{ij}$$

is a solution of the system $f_i = 0$ with $b_i \equiv a_i \bmod I m^d$.