CHAPTER IV: HENSELIAN RINGS AND THE HENSELIZATION

§1: HENSELIAN RINGS

Let R be a ring. In the following we denote by $m\text{-Spec}(R) = \{n \subseteq R \mid n \text{ a maximal ideal of } R\}$ the set of maximal ideals of R called the <u>maximal spectrum</u> of R.

(4.1) <u>Definition:</u> Let R be a ring.

(a) R is called a <u>semilocal</u> ring if R has only finitely many maximal ideals. (Note that R may not be Noetherian.)

(b) A semilocal ring R is called <u>decomposable</u> if R is isomorphic to a direct product of quasilocal rings.

(4.2) <u>Definition:</u> Let R be a ring. An element $e \in R$ is called an <u>idempotent</u> if $e^2 = e$. The set of all idempotent elements of R is denoted by $\mathrm{idem}(R) = \{e \in R \mid e^2 = e\}$.

(4.3) <u>Remark:</u> Let R be a ring. Then:

(a) $e \in \mathrm{idem}(R) \iff 1 - e \in \mathrm{idem}(R)$.

(b) If R is semilocal, then $\mathrm{idem}(R) = \{0, 1\}$.

(4.4) <u>Proposition:</u> Let R be a semilocal ring with $m\text{-Spec}(R) = \{m_1, \ldots, m_n\}$. Suppose that there are elements $e_1, \ldots, e_n \in \mathrm{idem}(R)$ so that:

(i) $e_i \notin m_i$ and $e_i \in m_j$ for all $i \neq j$.

(ii) $\sum_{i=1}^{n} e_i = 1$

(iii) $e_i e_j = 0$ for all $i \neq j$.

Then:

(a) For all $1 \leq i \leq n$ Re_i is a ring with identity element e_i.

(b) The map $\varphi: R \longrightarrow Re_1 \times \cdots \times Re_n$ defined by $\varphi(a) = (ae_1, \ldots, ae_n)$ is an isomorphism of rings.

(c) For all $1 \leq i \leq n$ Re_i is a quasilocal ring with $Re_i \cong R_{e_i} = R_{m_i}$.

(d) The natural map $\rho : R \longrightarrow R_{m_1} \times \cdots \times R_{m_n}$ defined by $\varphi(a) = (a/1, \ldots, a/1)$ is an isomorphism.

<u>Proof</u>: (b) Obviously, $\ker(\varphi) = 0$ and φ is injective. If $(a_1e_1, \ldots, a_ne_n) \in Re_1 \times \cdots \times Re_n$, then $\varphi(\sum_{i=1}^{n} a_ie_i) = (a_1e_1, \ldots, a_ne_n)$ and φ is surjective.

(c) If $n_i \subseteq Re_i$ is a maximal ideal, then $Re_1 \times \cdots \times n_i \times \cdots \times Re_n$ is a maximal ideal of $Re_1 \times \cdots \times Re_n$. Thus $Re_1 \times \cdots \times Re_n$ has at least n maximal ideals. Since R has n maximal ideals and φ is an isomorphism, each Re_i is a quasi local ring. Note that e_i is the identity element of Re_i. Thus the map $\varphi_i : R \longrightarrow Re_i$ with $\varphi_i(a) = ae_i$ factors through R_{e_i}:

$$\begin{tikzcd} R \arrow[r, "\varphi_i"] \arrow[d, "\rho_i"'] & Re_i \\ R_{e_i} \arrow[ur, "\lambda_i"'] & \end{tikzcd}$$

Since $\ker(\rho_i) = \ker(\varphi_i)$, λ_i is an isomorphism. In particular, R_{e_i} is a quasi local ring with maximal ideal $m_i R_{e_i}$. Hence $R_{e_i} = R_{m_i} \cong Re_i$.

(d) follows from (c).

(4.4) <u>Corollary</u>: Let R be a semilocal ring with $m\text{-}\mathrm{Spec}(R) = \{m_1, \ldots, m_n\}$. If R is decomposable, then the natural map $\rho : R \longrightarrow R_{m_1} \times \cdots \times R_{m_n}$ is an isomorphism.

<u>Proof</u>: By assumption there is an isomorphism $\psi : R \xrightarrow{\cong} \prod_{i=1}^{t} R_i$ where (R_i, n_i) are quasi local rings. By counting maximal ideals we see that $n = t$ and that, after renumbering if necessary, $R_i \cong R_{m_i}$. For all $1 \leq j \leq n$ consider $f_j = (0, \ldots, 1, \ldots, 0) \in \prod_{i=1}^{n} R_i$ (where 1 is at the j^{th} place). Then $f_1, \ldots, f_n$ satisfy conditions (i)–(iii) of Proposition 4.3 and so do $e_1 = \psi^{-1}(f_1), \ldots, e_n = \psi^{-1}(f_n)$. The

assertion follows by (4.3).

(4.5) Definition: A quasi local ring (R, m, k) is called Henselian if every finite R-algebra is decomposable.

(4.6) Example: Every local Artinian ring is Henselian.

Let (R, m, k) be a quasi local ring and S a finite R-algebra. Then S is a semilocal ring with $m\text{-}\mathrm{Spec}(S) = \{n_1, \ldots, n_t\}$. Set $S_i = S_{n_i}$ for $1 \le i \le t$ and $\overline{S} = S/mS$, $\overline{S}_i = S_i/mS_i$. In order to investigate if S is decomposable consider the commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\rho} & \prod_{i=1}^{t} S_i \\ \nu\downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\overline{\rho}} & \prod_{i=1}^{t} \overline{S}_i \end{array}$$

where ρ and $\overline{\rho}$ are the natural maps. Since $\overline{S}$ is an Artinian ring, $\overline{\rho}$ is an isomorphism. Then S is decomposable if and only if $\overline{\rho}$ lifts to an isomorphism ρ', or in view of (4.3), if 'enough' idempotent elements of $\overline{S}$ lift to idempotent elements of S. First note that ν induces a map:

$$\nu' : \mathrm{idem}(S) \longrightarrow \mathrm{idem}(\overline{S}).$$

(4.7) Lemma: With the assumptions and notation as above, the map ν' is injective.

Proof: Let $e, f \in \mathrm{idem}(S)$ with $\nu(e) = \nu(f)$. Then $x = e - f \in mS$ and mS is contained in the Jacobson radical of S. Note that $x^3 = (e-f)^3 = x$ and hence $x(x^2-1) = 0$. Since $x^2 - 1$ is invertible in S it follows that $x = 0$.

(4.8) Proposition: With the assumptions and notations as above, the following conditions are equivalent.

(a) S is decomposable.

(b) The map $\nu': \mathrm{idem}(S) \longrightarrow \mathrm{idem}(\overline{S})$ is bijective.

Proof: Consider the commutative diagram of natural maps:

$$\begin{array}{ccc} S & \xrightarrow{\ \varrho\ } & \prod_{i=1}^{t} S_i \\ \nu \downarrow & & \downarrow \mu \\ \overline{S} & \xrightarrow{\ \overline{\varrho}\ } & \prod_{i=1}^{t} \overline{S}_i \end{array}$$

For all $1 \leq i \leq t$ S_i and $\overline{S}_i$ are quasi local rings. Since 0 and 1 are the only idempotent elements of a quasi local ring, elements of the form $(\delta_1, \dots, \delta_t)$ with $\delta_i \in \{0,1\}$ for $1 \leq i \leq t$ are the only idempotents of $\prod S_i$ and $\prod \overline{S}_i$. Hence the map $\mu': \mathrm{idem}(\prod S_i) \longrightarrow \mathrm{idem}(\prod \overline{S}_i)$ is bijective.

(a) $\Rightarrow$ (b): If ϱ is an isomorphism, then $\varrho': \mathrm{idem}(S) \longrightarrow \mathrm{idem}(\prod S_i)$ is bijective. Hence $\mu'\varrho': \mathrm{idem}(S) \longrightarrow \mathrm{idem}(\prod \overline{S}_i)$ is bijective and so is ν' since $\overline{\varrho}$ is an isomorphism.

(b) $\Rightarrow$ (a): If ν' is bijective, $\overline{\varrho}'\nu' = \mu'\varrho'$ is bijective. Thus, since μ' is bijective, ϱ' is bijective. Since ϱ is injective, by (4.3) ϱ is an isomorphism.

(4.9) Lemma: Let (R, m, k) be a quasi local ring with maximal ideal m and S a finite R-algebra. Suppose that there is a monic polynomial $\overline{Q} \in k[x]$ of degree n and a k-algebra isomorphism $\lambda: k[x]/(\overline{Q}) \longrightarrow \overline{S} = S/mS$. Then there is a monic polynomial $Q \in R[x]$ of degree n such that:

(a) $Q + mR[x] = \overline{Q}$ in $k[x]$

(b) There is a surjective R-algebra morphism $\varphi: R[x]/(Q) \longrightarrow S$.

In particular, the R-algebra S is generated by one element if and only if the k-algebra $\overline{S}$ is generated by one element.

Proof: Set $\overline{x} = x + (\overline{Q}) \in k[x]/(Q)$ and let $t \in S$ be so that $\lambda(\overline{x}) = t + mS$. Then $S = \sum_{i=0}^{n-1} Rt^i + mS$ and hence by Nakayama, $S = \sum_{i=0}^{n-1} Rt^i$, since S is a finitely generated R-module. In particular, $S = R[t]$, where t is root of a monic polynomial $Q \in R[x]$ of degree n. This yields a surjective R-algebra **morphism** $\varphi: R[x]/(Q) \longrightarrow S$ with $\varphi(x+(Q)) = t$. Consider the k-algebra morphism $\overline{\varphi}: k[x]/(Q + mR[x]) \longrightarrow \overline{S}$ induced by φ. Since Q is monic with $\deg Q = \deg \overline{Q}$ and since $\overline{\varphi}(x+(Q)+mR[x]) = t + mS$, $\overline{\varphi} = \lambda$ and hence $Q + mR[x] = \overline{Q}$.

(4.10) <u>Lemma</u>: Let (R, m, k) be a quasi local ring and $P \in R[x]$ a monic polynomial. Suppose that there are monic polynomials $Q, F \in R[x]$ so that $P = QF$ and in $k[x]$ the polynomials $\overline{Q} = Q + mR[x]$ and $\overline{F} = F + mR[x]$ are relatively prime. Then the natural map $\varphi: R[x]/(P) \longrightarrow R[x]/(Q) \times R[x]/(F)$ is an isomorphism.

<u>Proof</u>: First note that if $G \in R[x]$ is a monic polynomial of degree m then $R[x]/(G)$ is a finite free R-module of rank m. In particular,

$$\mathrm{rk}_R\left(R[x]/(P)\right) = \deg P = \deg Q + \deg F = \mathrm{rk}_R\left(R[x]/(Q)\right) + \mathrm{rk}_R\left(R[x]/(F)\right) = \mathrm{rk}_R\left(R[x]/(Q) \times R[x]/(F)\right).$$

Consider φ as an R-linear map between finite free R-modules of the same rank. Hence $\det(\varphi)$ is defined and φ is an isomorphism if and only if $\det(\varphi)$ invertible in R. Modulo m we obtain a k-linear map

$$\overline{\varphi}: k[x]/(\overline{P}) \longrightarrow k[x]/(\overline{Q}) \times k[x]/(\overline{F}).$$

Since $\overline{Q}$ and $\overline{F}$ are relatively prime, $\overline{\varphi}$ is an isomorphism and $\det(\overline{\varphi}) \neq 0$. Thus $\det(\varphi) \notin m$ is a unit in R and φ is an isomorphism.

(4.11) <u>Proposition</u>: Let (R, m, k) be a quasi local ring. Then the following conditions are equivalent:

(a) R is Henselian.

(b) Every finite R-algebra, which is free as an R-module, is decomposable.

(c) For all monic polynomials $P \in R[x]$ the R-algebra $R[x]/(P)$ is decomposable.

(d) For all monic polynomials $P \in R[x]$ which decompose modulo m into a product of monic polynomials $\overline{P} = \overline{Q}\,\overline{F}$ where $\overline{Q}, \overline{F} \in k[x]$ are relatively prime, there are monic polynomials $Q, F \in R[x]$ so that $Q + \mathfrak{m}R[x] = \overline{Q}$, $F + \mathfrak{m}R[x] = \overline{F}$ and $P = QF$.

<u>Proof</u>: (a) $\Rightarrow$ (b) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (d): Let $P \in R[x]$ be a monic polynomial and $\overline{Q}, \overline{F} \in k[x]$ relatively prime monic polynomials with $\overline{P} = P + \mathfrak{m}R[x] = \overline{Q}\,\overline{F}$. Hence the natural map

$$\overline{\varphi}: k[x]/(\overline{P}) \longrightarrow k[x]/(\overline{Q}) \times k[x]/(\overline{F})$$

is an isomorphism. Set $S = R[x]/(P)$. By assumption (c) S is decomposable and by (4.4) the natural map $\varphi: S \longrightarrow S_{n_1} \times \ldots \times S_{n_t}$ is an isomorphism where $\mathfrak{m}\text{-Spec}(S) = \{n_1, \ldots, n_t\}$. Since $\overline{\varphi}$ is an isomorphism we can rearrange and collect factors S_{n_i} so that $S = S_1 \times S_2$ with $\overline{S_1} = S_1/\mathfrak{m}S_1 = k[x]/(\overline{Q})$ and $\overline{S}_2 = k[x]/(\overline{F}) = S_2/\mathfrak{m}S_2$. S is a free R-module, thus the direct summands S_1 and S_2 are projective R-modules. Since projective modules over quasi local rings are free, S_1 and S_2 are free R-modules. By Nakayama, the R-modules S_i are generated by $1, x, \ldots, x^{m_i - 1}$ where $m_1 = \deg \overline{Q}$, $m_2 = \deg \overline{F}$, and hence $\operatorname{rk}_R S_i = m_i$. Thus there are monic polynomials $Q, F \in R[x]$ with $\deg Q = m_1$, $\deg F = m_2$, and surjective R-algebra morphisms $\gamma_1: R[x]/(Q) \longrightarrow S_1$ and $\gamma_2: R[x]/(F) \longrightarrow S_2$. By rank reasons the maps γ_i are isomorphisms. Moreover, $Q + \mathfrak{m}R[x] = \overline{Q}$ and $F + \mathfrak{m}R[x] = \overline{F}$. Since $QF \in \operatorname{ann}_{R[x]}(S)$ it follows that $P = QF$ (since $\deg P = \deg Q + \deg F$).

(d) $\Rightarrow$ (c): Write $\overline{P} = \prod_{i=1}^{t} \overline{P_i} \in k[x]$ with $\overline{P_i} \in k[x]$ the power of an irreducible monic polynomial and $(\overline{P_i}, \overline{P_j}) = 1$ for $i \neq j$. By (d) $P = \prod_{i=1}^{t} P_i$ with $P_i \in R[x]$ a monic polynomial and $P_i + \mathfrak{m}R[x] = \overline{P_i}$. By (4.10) the natural morphism

$$\varphi: R[x]/(P) \longrightarrow \prod_{i=1}^{t} R[x]/(P_i)$$

is an isomorphism. Since for all $1 \leq i \leq t$ $k[x]/(\overline{P_i})$ is a local Artinian ring, the rings $R[x]/(P_i)$ are quasi local.

(c) $\Rightarrow$ (a): Let S be a finite R-algebra. Set $\overline{S} = S/\mathfrak{m}S$. Let $\mathfrak{m}\text{-Spec}(S) = \{n_1, \ldots, n_t\}$ and consider the natural surjective map: $S \longrightarrow \overline{S} \cong \prod_{i=1}^{t} \overline{S}_{n_i}$. In the following we identify $\overline{S}$ and $\prod_{i=1}^{t} \overline{S}_{n_i}$. For all $1 \leq i \leq n$ consider the idempotent elements

$f_i = (1,\dots,1,0,1,\dots,1)$ where 0 is at the i^{th} place. Since every idempotent in $\bar{S}$ (except 1) can be factored into a product of certain f_i it suffices to show that every f_i lifts to an idempotent of S. Fix an $i \in \{1,\dots,t\}$ and choose $y \in S$ with $\nu(y) = f_i$. Since S is finite over R, there is a monic polynomial $P \in R[x]$ with $P(y) = 0$ in S. Let $u \colon R[x]/(P) \longrightarrow S$ denote the R-algebra morphism defined by $u(x) = y$ and consider the commutative diagram:

$$\begin{tikzcd} R[x]/(P) \arrow[r, "u"] \arrow[d, "\lambda"'] & S \arrow[d, "\nu"] \\ k[x]/(\bar{P}) \arrow[r, "\bar{u}"] & \bar{S} = \prod \bar{S}_{n_i} \end{tikzcd}$$

where λ and ν are the natural maps. It suffices to show that there is an idempotent $e \in R[x]/(P)$ so that $\bar{u}\lambda(e) = f_i$. Set $u^{-1}(n_j) = \mathfrak{N}_j$. Since S is integral over $R[x]/(P)$, $\mathfrak{N}_j$ is a maximal ideal of $R[x]/(P)$. Moreover, since $y = u(x) \in n_i$ and $y \notin n_j$ for all $i \neq j$, it follows that $\mathfrak{N}_i \neq \mathfrak{N}_j$ for all $i \neq j$ (but possibly $\mathfrak{N}_j = \mathfrak{N}_k$ for $j \neq k$ and $j \neq i$, $k \neq i$). Since $R[x]/(P)$ is decomposable there is an idempotent $e \in R[x]/(P)$ with $e \in \mathfrak{N}_i$ and $e \notin \mathfrak{N}_j$ for all $i \neq j$. Note that f_i is the unique idempotent of $\bar{S}$ with $f_i \in n_i$ and $f_i \notin n_j$ for all $i \neq j$, thus $\bar{u}\lambda(e) = f_i$. Hence $u(e)$ is an idempotent of S which lifts f_i.

(4.12) <u>Example</u>: Let $R = \hat{R}$ be a complete local Noetherian ring. Then R is Henselian.

<u>Proof</u>: Let S be a finite R-algebra and $m \subseteq R$ the maximal ideal of R. Then S is complete with respect to the mS-adic topology, that is, $S = \varprojlim S/m^n S$. Let $m\text{-Spec}(S) = \{n_1,\dots,n_t\}$. For all $n \in \mathbb{N}$ the ring S/m^nS is Artinian and decomposes: $S/m^nS \cong \prod_{i=1}^t (S/m^nS)_{n_i} = \prod_{i=1}^t S_{n_i}/m^n S_{n_i}$. The inverse limit commutes with direct products hence $S = \prod_{i=1}^t S_i$ where $S_i = \varprojlim S_{n_i}/m^t S_{n_i}$ is a (quasi) local ring. Thus R is Henselian. Also note that with the above assumptions $S \cong \prod S_{n_i}$ by (4.4). Since each S_{n_i} is a homomorphic image of S, each localization S_{n_i} is complete.

§2: A STRUCTURE THEOREM FOR HENSELIAN RINGS

Let R be a ring and $e \in R$ an idempotent element. For all maximal ideals $\mathfrak{m} \in \text{m-Spec}(R)$ let $i_{\mathfrak{m}} : R \longrightarrow R_{\mathfrak{m}}$ denote the natural map. Then

$$i_{\mathfrak{m}}(e) = \begin{cases} 0 & \text{if } e \in \mathfrak{m} \\ 1 & \text{if } e \notin \mathfrak{m}. \end{cases}$$

Moreover, note that the ring morphism $i : R \longrightarrow \prod_{\mathfrak{m} \in \text{m-Spec}(R)} R_{\mathfrak{m}}$ defined by $i(a) = (i_{\mathfrak{m}}(a))_{\mathfrak{m} \in \text{m-Spec}(R)}$ is injective. (If $i(a) = 0$, then $\text{ann}(a) \notin \mathfrak{m}$ for all $\mathfrak{m} \in \text{m-Spec}(R)$ and $a = 0$.) This implies that two idempotent elements $e, f \in R$ are identical if and only if they are contained in the same maximal ideals.

(4.13) <u>Proposition</u>: Let R be a ring, $I \subseteq R$ an ideal with $I \subseteq \text{nil}(R)$ and $\nu : R \to R/I$ the natural map. The induced map $\nu' : \text{idem}(R) \longrightarrow \text{idem}(R/I)$ is bijective.

<u>Proof</u>: Let $e, f \in \text{idem}(R)$ with $\nu(e) = \nu(f)$. Hence $e - f \in I$. Since I is contained in every maximal ideal, e and f are contained in exactly the same maximal ideals. Thus $e = f$.

In order to show surjectivity let $\bar{e} \in \text{idem}(R/I)$. Then the canonical morphism $\varphi : R/I \longrightarrow (R/I)\bar{e} \times (R/I)(\bar{e}-1)$ is bijective and $\text{Spec}(R/I)$ is disconnected, i.e. $\text{Spec}(R/I) = V(\bar{e}) \cup V(\bar{e}-1)$, $V(\bar{e}) \cap V(\bar{e}-1) = \emptyset$, and $V(\bar{e}) \neq \emptyset$, $V(\bar{e}-1) \neq \emptyset$ (assuming $\bar{e} \neq 0, 1$). Since $\text{Spec}(R)$ and $\text{Spec}(R/I)$ are homeomorphic, $\text{Spec}(R)$ is disconnected, say $\text{Spec}(R) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ where $\mathfrak{a}, \mathfrak{b} \subseteq R$ are proper ideals with $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$ and $V(\mathfrak{a}) \cong V(\bar{e})$, $V(\mathfrak{b}) \cong V(\bar{e}-1)$. In particular, $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal and there are $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ with $a + b = 1$. Since $\mathfrak{a} \cap \mathfrak{b} \subseteq \text{nil}(R)$ there is an $n \in \mathbb{N}$ with $(ab)^n = 0$. Let $\alpha, \beta \in R$ with $\alpha a^n + \beta b^n = 1$. Since $(\alpha a^n)(1 - \alpha a^n) = (\alpha a^n)(\beta b^n) = 0$, the element αa^n is idempotent. Moreover, αa^n is contained in all maximal ideals of $V(\mathfrak{a}) \cong V(\bar{e})$ and not contained in any maximal ideal of $V(\mathfrak{b}) = V(\bar{e}-1)$. Therefore $\nu(\alpha a^n) = \bar{e}$.

(4.14) Theorem: Let (R,m,k) be a quasi local ring. The following conditions are equivalent.

(a) R is Henselian.

(b) Every monic polynomial $P \in R[x]$, whose image $\overline{P}$ in $k[x]$ has a simple root in k, has a root in R.

(c) For every étale algebra S over R and every prime ideal $n \in \mathrm{Spec}(S)$ with $n \cap R = m$ and $k(n) = (S/n)_n = k$, the morphism $R \longrightarrow S_n$ is an isomorphism.

Proof: (a) $\Rightarrow$ (b): Let $\overline{a} \in k$ be a simple root of $\overline{P} \in k[x]$. Then $\overline{P} = (x-\overline{a})\overline{Q}$ where $\overline{a} \in k$, $\overline{Q} \in k[x]$ monic, and $(x-\overline{a}), \overline{Q}$ relatively prime. By (4.11) there are an $a \in R$ and a $Q \in R[x]$ with $P = (x-a)Q$.

(b) $\Rightarrow$ (c): Let S be an étale R-algebra, $n \in \mathrm{Spec}(S)$ with $n \cap R = m$ and $k(n) = k$. By (3.25) we may assume that S is a standard étale R-algebra, that is, $S = (R[x]/(f))_g$ where $f, g \in R[x]$, f monic, and f' invertible in $(R[x]/(f))_g$. Since $f' \notin n$ we may assume that $f' = g$. Then by (2.5) $\overline{S} = S/mS$ is étale over $k = R/m$ and hence by (3.3):

$$\overline{S} = (k[x]/(\overline{f}))_{\overline{f}'} = \prod_{i=1}^{n} K_i$$

where K_i are finite separable field extensions of k. Moreover, $k(n) = (S/n)_n = K_i = k$ for some $1 \le i \le n$. Hence $\overline{f} = (x-\overline{a})\overline{Q}$ for some $\overline{a} \in k$ and some monic polynomial $\overline{Q} \in k[x]$. Since $\overline{f}'$ is invertible in $\overline{S}$, $x-\overline{a}$ and $\overline{Q}$ are relatively prime. By (b) there is an $a \in R$ and a monic polynomial $Q \in R[x]$ with $f = (x-a)Q$ in $R[x]$. Therefore by (4.10):

$$R[x]/(f) \cong R[x]/(x-a) \times R[x]/(Q).$$

The ring S_n corresponds to $R[x]/(x-a) \cong R$.

(c) $\Rightarrow$ (a): In order to show that R is Henselian we use criterion (4.11). Let S be a finite R-algebra which is free as an R-module. We need to show that S is decomposable. Let $\overline{S} = S/mS$ and $\nu: S \longrightarrow \overline{S}$ the natural map. By (4.8) it suffices to show that $\nu': \mathrm{idem}(S) \longrightarrow \mathrm{idem}(\overline{S})$ is bijective. We start the proof with the construction of a special étale extension E of R.

<u>Step 1</u>: Construction of E

Since S is a free R-module we may write $S = \bigoplus_{i=1}^{r} Re_i$ and suppose that
$$e_i e_j = \sum_{k=1}^{r} a_{ijk} e_k$$
where $a_{ijk} \in R$ for all $1 \le i,j,k \le r$. For all $1 \le i \le r$ let x_i be variables. Consider the following equation in $S[x_1, \ldots, x_r]$:
$$\left(\sum_{i=1}^{r} x_i e_i\right)^2 = \sum_{k=1}^{r} Q_k e_k$$
where $Q_k \in R[x_1, \ldots, x_r]$ are defined by:
$$Q_k = \sum_{i,j=1}^{r} a_{ijk} x_i x_j .$$

For all $1 \le i \le r$ set: $P_i = Q_i - x_i \in R[x_1, \ldots, x_n]$.

Let $f = \sum_{i=1}^{r} c_i e_i \in S$. The element f is idempotent if and only if the R-algebra morphism: $\varphi: R[x_1, \ldots, x_r] \longrightarrow R$ given by $\varphi(x_i) = c_i$ factors through
$$\varphi: R[x_1, \ldots, x_r]/(P_1, \ldots, P_r) \longrightarrow R.$$
(that is, $f \in \mathrm{idem}(R) \Longleftrightarrow (P_1, \ldots, P_r) \subseteq \ker(\varphi)$). Define
$$E = R[x_1, \ldots, x_r]/(P_1, \ldots, P_r) = R[y_1, \ldots, y_r]$$
where $y_i = x_i + (P_1, \ldots, P_r)$ for all $1 \le i \le r$.

<u>Step 2</u>: Functorial properties of E.

For any R-algebra the tensor product $S \otimes_R T$ is a finite free T-module with basis $\{e_i \otimes 1 \mid 1 \le i \le n\}$.

<u>Claim</u>: Let $t_1, \ldots, t_r \in T$. There is an R-algebra morphism $u: E \longrightarrow T$ with $u(y_i) = t_i$ if and only if the element $\sum_{i=1}^{r} e_i \otimes t_i$ is a idempotent of $S \otimes_R T$.

<u>Pf of Cl.</u> It is easy to verify that if such an R-algebra morphism exists, then $\sum_{i=1}^{r} e_i \otimes t_i$ is idempotent. Conversely, suppose that $\left(\sum_{i=1}^{r} e_i \otimes t_i\right)^2 = \sum_{i=1}^{r} e_i \otimes t_i$. Since $\{e_i \otimes 1\}_{1 \le i \le r}$ is a basis of the free T-module $S \otimes_R T$, we obtain that $P_i(t_1, \ldots, t_r) = 0$ for all $1 \le i \le r$. Hence there is an R-algebra morphism $u: E \longrightarrow T$ with $u(y_i) = t_i$ for all $1 \le i \le r$ and the claim is proven.

By the claim there is a bijective map:
$$\Delta_T : \mathrm{Hom}_{R\text{-alg}}(E, T) \xrightarrow{\cong} \mathrm{idem}(S \otimes_R T)$$

defined by $\Delta_T(u) = \sum_{i=1}^{r} e_i \otimes u(y_i)$.

Let $\mathfrak{A}_R$ denote the category of R-algebras and R-algebra morphisms and $\mathcal{S}$ the category of sets and maps. We are interested in two functors from $\mathfrak{A}_R$ to $\mathcal{S}$:

The functor $\mathcal{F}: \mathfrak{A}_R \longrightarrow \mathcal{S}$.

If T is an R-algebra then $\mathcal{F}(T) = \mathrm{idem}(S \otimes_R T)$ and if $\varphi: T_1 \longrightarrow T_2$ is an R-algebra morphism then $\mathcal{F}(\varphi) = 1 \otimes \varphi|_{\mathrm{idem}} : \mathrm{idem}(S \otimes_R T_1) \longrightarrow \mathrm{idem}(S \otimes_R T_2)$.

The functor $\mathcal{G}: \mathfrak{A}_R \longrightarrow \mathcal{S}$

If T is an R-algebra then $\mathcal{G}(T) = \mathrm{Hom}_{R\text{-alg}}(E, T)$ and if $\varphi: T_1 \longrightarrow T_2$ is an R-algebra morphism then $\mathcal{G}(\varphi) = \varphi_* : \mathrm{Hom}_{R\text{-alg}}(E, T_1) \longrightarrow \mathrm{Hom}_{R\text{-alg}}(E, T_2)$ where $\varphi_*(u) = \varphi u$ for all $u \in \mathrm{Hom}_{R\text{-alg}}(E, T_1)$.

The claim shows that the functors $\mathcal{F}$ and $\mathcal{G}$ are equivalent. That is, for all $T_1, T_2 \in \mathfrak{A}_R$ and $\varphi \in \mathrm{Hom}_{R\text{-alg}}(T_1, T_2)$ there is a commutative diagram with bijective horizontal maps:

$$\begin{tikzcd}
\mathcal{G}(T_1) = \mathrm{Hom}_{R\text{-alg}}(E, T_1) \arrow[r, "\Delta_{T_1}", "\cong"'] \arrow[d, "\mathcal{G}(\varphi) = \varphi_*"'] & \mathrm{idem}(S \otimes_R T_1) = \mathcal{F}(T_1) \arrow[d, "\mathcal{F}(\varphi) = 1 \otimes \varphi"] \\
\mathcal{G}(T_2) = \mathrm{Hom}_{R\text{-alg}}(E, T_2) \arrow[r, "\Delta_{T_2}", "\cong"'] & \mathrm{idem}(S \otimes_R T_2) = \mathcal{F}(T_2)
\end{tikzcd}$$

<u>Step</u> 3: E is an étale R-algebra

Let C be an R-algebra, $I \subseteq C$ an ideal with $I^2 = 0$, and $\mu: C \longrightarrow C/I$ the natural map. Consider a commutative diagram of R-algebra morphisms:

$$\begin{tikzcd}
E \arrow[r, "\overline{u}"] & C/I \\
R \arrow[u] \arrow[r] & C \arrow[u, "\mu"']
\end{tikzcd}$$

This induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{R\text{-alg}}(E, C/I) \arrow[r, "\Delta_{C/I}", "\cong"'] & \mathrm{idem}(S \otimes_R C/I) \\
\mathrm{Hom}_{R\text{-alg}}(E, C) \arrow[u, "\mu_*"] \arrow[r, "\Delta_C", "\cong"'] & \mathrm{idem}(S \otimes_R C) \arrow[u, "1 \otimes \mu"']
\end{tikzcd}$$

By Proposition (4.13) the map on the sets of idempotent elements is bijective.

Thus μ_* is bijective and $\bar{u}$ lifts to a unique R-algebra morphism $u: E \longrightarrow C$.

Step 4: The map $\nu': \text{idem}(S) \longrightarrow \text{idem}(\bar{S})$ is bijective.

By (4.7) ν' is injective. In order to show surjectivity consider the commutative diagram:

$$\begin{tikzcd} \text{Hom}_{R\text{-alg}}(E,k) \arrow[r, "\Delta_k", "\cong"'] & \text{idem}(\bar{S}) \\ \text{Hom}_{R\text{-alg}}(E,R) \arrow[u, "\tau_*"] \arrow[r, "\Delta_R", "\cong"'] & \text{idem}(S) \arrow[u, "\nu'"] \end{tikzcd}$$

where $\tau: R \longrightarrow k$ is the natural map and $\nu = 1 \otimes \tau$. It suffices to show that τ_* is surjective. Let $\bar{u}: E \longrightarrow k$ be an R-algebra morphism. Then $n = \ker(\bar{u})$ is a maximal ideal of E with $n \cap R = m$ and $k(n) = E/n = k$. Hence by (c) there is an R-algebra isomorphism $\varphi: E_n \longrightarrow R$. The composition $E \xrightarrow{i_n} E_n \xrightarrow{\varphi} R$, where i_n is the natural map, lifts $\bar{u}$, that is $\tau_*(\varphi i_n) = \bar{u}$.

(4.15) Corollary: Let (R, m, k) be a quasi local Henselian ring and S an R-algebra of finite type. Let $n \subseteq S$ be a prime ideal so that $n \cap R = m$ and S is quasi finite over R in a neighborhood of n. Then S_n is finite over R and S_n is a direct factor of S.

Proof: Let C' denote the integral closure of R in S. By Zariski's Main Theorem there is an element $f \in C'$, $f \notin n$, so that $C'_f = S_f$. Hence there is a finite R-subalgebra C of C' with $f \in C$ and $C_f = S_f$. In particular, $C_{\tilde{n}} = S_n$, where $\tilde{n} = n \cap C$ a maximal ideal of C. Since R is Henselian, the finite R-algebra C is decomposable, i.e. $C \cong \prod_{i=1}^{t} C_{\tilde{n}_i}$ where $m\text{-Spec}(C) = \{\tilde{n} = \tilde{n}_1, \dots, \tilde{n}_t\}$. Thus S_n is a direct factor of C and therefore finite over R.

Since C is decomposable there is an idempotent e of C with $e \notin \tilde{n} = n$, and $e \in \tilde{n}_i$ for all $2 \leq i \leq t$. Then $C \cong Ce \times C(e-1)$ and $Ce \cong C_{\tilde{n}} = S_n$. Since $C \subseteq S$, S decomposes into $S = Se \times S(e-1)$. Since $S_n \cong Ce \subseteq Se$ it follows that $Se \cong S_n$, a direct factor of S.

§3: THE HENSELIZATION

(4.16) Definition: Let (R, m, k) be a quasi local ring. The Henselization of R is a pair (R^h, i) where R^h is a quasi local Henselian ring and $i: R \to R^h$ a local ring morphism so that: for every quasi local Henselian ring S and every local morphism $u: R \to S$ there is exactly one local morphism $u^h: R^h \to S$ so that $u = u^h i$, that is, the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, "i"'] & S \\ R^h \arrow[ur, "u^h"'] & \end{tikzcd}$$

In this section we want to show that the Henselization R^h of a quasi local ring R exists and is unique up to isomorphism. We assume from now on that if S is unramified or étale over R, then S is an R-algebra of finite presentation.

(4.17) Definition: Let (R, m, k) be a quasi local ring. An R-algebra S is called locally étale over R or a local étale R-algebra if there is an étale R-algebra S' and a prime ideal $n' \subseteq S'$ with $n' \cap R = m$ and $S'_{n'} \cong S$.

(4.18) Remark: If S is locally étale over R then S is essentially of finite presentation over R. Moreover, S is smooth and formally unramified over R.

(4.19) Definition: Let (R, m, k) be a quasi local ring and (S, n, k') a local étale R-algebra. S is called an étale neighborhood of R if $S/n = k' = k = R/m$.

We want to construct the Henselization R^h of a quasi-local ring R as a direct limit:

$$R^h = \varinjlim_{i \in I} S_i$$

where S_i are local étale R-algebra. There are several issues which need to be addressed:

(i) There must be a set of étale neighborhoods of R which represent <u>all</u> étale neighborhoods of R.

(ii) We must be able to make $\{S_i\}_{i\in I}$ into a direct system, that is, we need to make sure that for elements $i,j\in I$ there are an element $k\in I$ and local R-algebra morphisms $\varphi_k^i : S_i \longrightarrow S_k$ and $\varphi_k^j : S_j \longrightarrow S_k$.

(iii) The set of R-algebra morphisms $\mathrm{Hom}_{R\text{-alg}}(S_i, S_j)$ must be fairly small. We need to pick for certain $i,j\in I$ local R-algebra morphisms $\varphi_j^i : S_i \longrightarrow S_j$. Then we have to make certain that there morphisms 'fit together' in the sense that if $\varphi_k^j : S_j \longrightarrow S_k$ is an R-algebra morphism then $\varphi_k^i = \varphi_k^j \varphi_j^i$.

We address issue (iii) first:

(4.20) <u>Definition</u>: Let (R,m) be a quasi local ring and (S,n), (T,w) quasi local R-algebras with $n\cap R = m = w\cap T$. Then $\mathrm{Hom}_{\mathrm{loc}R}(S,T)$ denotes the set of local R-algebra morphisms $\varphi : S \longrightarrow T$.

(4.21) <u>Proposition</u>: Let (R,m,k) be a quasi local ring, (S,n,k_S) a local étale R-algebra, and (T,w,k_T) a quasi local R-algebra with $w\cap R = m$. The natural map $\mathrm{Hom}_{\mathrm{loc}R}(S,T) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k_S, k_T)$ is injective.

<u>Proof</u>: Let $\sigma : T \longrightarrow k_T$ be the natural map and $u, v : S \longrightarrow T$ local R-algebra morphisms with $\sigma u = \sigma v$. Since S is locally étale over R, S is essentially of finite presentation over R and $S\otimes_R S$ is an R-algebra essentially of finite type. Let $\Delta : S\otimes_R S \longrightarrow S$ denote the R-algebra morphism given by $\Delta(a\otimes b) = ab$ and $I = \ker(\Delta)$ its kernel. Set $Q = \Delta^{-1}(n)$ and consider the induced local morphism $\Delta_Q : (S\otimes_R S)_Q \longrightarrow S$. By (3.31) S is essentially of finite presentation over $(S\otimes_R S)_Q$ and $I_Q = \ker(\Delta_Q)$ is a finitely generated ideal of $(S\otimes_R S)_Q$. Moreover, S is formally unramified over R and thus

$\Omega_{S/R} = I/I^2 = 0$. Hence $I_Q/I_Q^2 = 0$ and by Nakayama $I_Q = 0$. This implies that $(S\otimes_R S)_Q \cong S$. Consider the commutative diagram of R-algebra morphisms:

$$\begin{tikzcd} S\otimes_R S \arrow[r, "\Delta"] \arrow[d, "u\otimes v"'] & S \arrow[d, "\sigma u = \sigma v"] \arrow[dl, "w"'] \\ T \arrow[r, "\sigma"] & k_T \end{tikzcd}$$

For all $f \in (S\otimes_R S) - Q$ the element $\Delta(f)$ is invertible in S, hence $(u\otimes v)(f)$ is invertible in T. $u\otimes v$ factors through $S \cong (S\otimes_R S)_Q$, that is there is an R-algebra morphism $w: S \longrightarrow T$ with $w\Delta = u\otimes v$. Let $t\in S$, then $w\Delta(1\otimes t) = v(t) = w\Delta(t\otimes 1) = u(t)$ implying that $u = v$.

(4.22) <u>Corollary</u>: Let (R, m, k) be a quasi local ring and (S_1, n_1, k), (S_2, n_2, k) étale neighborhoods of R. Then $|\operatorname{Hom}_{loc\,R}(S_1, S_2)| \leq 1$, that is, there is at most one local R-algebra morphism between S_1 and S_2.

(4.23) <u>Proposition</u>: Let (R, m, k) be a quasi local ring and (S_1, n_1, k), (S_2, n_2, k) étale neighborhoods of R. Then there are an étale neighborhood (S_3, n_3, k) of R and local R-algebra morphisms $\varphi_1: S_1 \longrightarrow S_3$ and $\varphi_2: S_2 \longrightarrow S_3$.

<u>Proof</u>: By (2.5) $S_1\otimes_R S_2$ is the localization of an étale R-algebra. The kernel $n_3 = \ker(\rho)$ of the map $\rho: S_1\otimes_R S_2 \longrightarrow k$ (given by $\rho(a\otimes b) = (a+n_1)(b+n_2)$) is a maximal ideal of $S_1\otimes_R S_2$. $S_3 = (S_1\otimes S_2)_{n_3}$ is an étale neighborhood of R. Let $\lambda: S_1\otimes_R S_2 \longrightarrow S_3$ be the natural map. Then $\varphi_1: S_1 \longrightarrow S_3$ given by $\varphi_1(a) = \lambda(a\otimes 1)$ and $\varphi_2: S_2 \longrightarrow S_3$ given by $\varphi_2(b) = \lambda(1\otimes b)$ are local R-algebra morphisms.

(4.24) <u>Proposition</u>: Let (R, m, k) be a quasi local ring.

(a) There is a set Γ and a family of quasi local R-algebras $\{(S_\gamma, m_\gamma)\}_{\gamma\in\Gamma}$ such that:

(i) Every S_γ is locally étale over R.

(ii) For every quasi local R-algebra S which is locally étale over R there is exactly one $\gamma \in I$ so that S and S_γ are isomorphic as R-algebras.

(b) Let $I = \{\gamma \in \Gamma \mid S_\gamma/m_\gamma \cong k\} \subseteq \Gamma$ and define a relation on I by: $i, j \in I$, then $i \leq j \iff$ there is a local R-algebra morphism $\varphi_j^i : S_i \longrightarrow S_j$. This defines a partial order on I and $(I, \leq)$ is a directed set.

Proof: (a) Let t be a variable over R and Γ_0 the following subset of $\mathrm{Spec}(R[t]) \times R[t]$:

$$(Q, f) \in \Gamma_0 \iff \begin{cases} (\alpha) & f \in R[t] \text{ is monic} \\ (\beta) & f \in Q \text{ and } f' \notin Q \\ (\gamma) & m = Q \cap R \end{cases}$$

For an element $\gamma_0 = (Q, f) \in \Gamma_0$ set $S_{\gamma_0} = (R[t]/(f))_Q$. S_{γ_0} is locally étale over R. Furthermore, define an equivalence relation on Γ_0 by:

$\gamma_0 \sim \mu_0 \iff S_{\gamma_0}$ and S_{μ_0} are isomorphic as R-algebras.

Set $\Gamma = \Gamma_0/\sim$. By the structure theorem for étale extensions (3.25) for any local étale R-algebra S there is exactly one element $\gamma \in \Gamma$ such that S and S_γ are isomorphic as R-algebras.

(b) Let $I \subseteq \Gamma$ be the subset of all étale neighborhoods in Γ. Obviously the relation $\leq$ is reflexive and transitive. To show that $\leq$ is also antisymmetric let $i, j \in I$ with $i \leq j$ and $j \leq i$ and local R-algebra morphisms $\varphi_j^i : S_i \longrightarrow S_j$ and $\varphi_i^j : S_j \longrightarrow S_i$. The compositions $\varphi_i^j \varphi_j^i : S_i \longrightarrow S_i$ and $\varphi_j^i \varphi_i^j : S_j \longrightarrow S_j$ are again local R-algebra morphisms. By Corollary (4.22) there is exactly one, namely the identity map. Thus S_i and S_j are isomorphic and $i = j$ by construction of Γ.

It remains to show that $(I, \leq)$ is a directed set. Let $i, j \in I$, S_i and S_j étale neighborhoods of R. By (4.23) there are an étale neighborhood S of R and R-algebra morphisms $\sigma_i : S_i \longrightarrow S$ and $\sigma_j : S_j \longrightarrow S$. Then for some $k \in I$ $S \cong S_k$ and the assertion follows.

(4.25) Definition: Let (R, m, k) be a quasi local ring and $\{(S_i, n_i)\}_{i\in I}$ the system of representatives of all étale neighborhoods of R as defined in (4.24). Then we define:

$$R^h = \varinjlim_{i\in I} S_i .$$

(4.26) Remark: R^h is a quasi local ring with maximal ideal $m^h = \varinjlim_{i\in I} n_i$ and residue field $R^h/m^h = k$. By (3.28) each S_i is faithfully flat over R. Since tensor products and direct limits commute R^h is faithfully flat over R.

(4.27) Theorem: With notations and assumptions as above the quasi local ring R^h is the Henselization of R.

Proof: We first show that R^h is Henselian by using criterion (4.14)(b). Let $P \in R[x]$ be a monic polynomial and suppose that $\overline{P} = P + m^h R^h[x] \in k[x]$ has a simple root in k, say $\overline{P} = (x-\overline{a})\overline{Q}$ where $\overline{Q} \in k[x]$ monic with $\overline{Q}(\overline{a}) \neq 0$. Since $R^h = \varinjlim_{i\in I} S_i$ there is an étale neighborhood S_i of R so that $P \in S_i[x]$. Consider the ring $D = S_i[x]/(P)$. D is a module-finite extension of S_i. Its maximal ideals are in one-to-one correspondence to the maximal ideals of $\overline{D} = D \otimes_{S_i} k = k[x]/(\overline{P}) = (k[x]/(x-\overline{a})) \times (k[x]/(\overline{Q}))$. Let n be the maximal ideal of D which corresponds to the local factor ring $k[x]/(x-\overline{a})$ in the decomposition of $\overline{D}$. Since D is semilocal, there is an element $g \in D - n$ which is contained in any other maximal ideal of D. Then $\overline{D}_g = \overline{D}_n = k[x]/(x-\overline{a}) \cong k$. Moreover, $\overline{P}'$ is invertible in $\overline{D}_g$. Hence by replacing g by the element gP' (if necessary) D_g is an étale extension of S_i. Since S_i is the localization of an étale R-algebra, there is an element $f \in S_i - n_i$ so that D_{gf} is the localization of an étale algebra over R. Thus D_n is an étale neighborhood of R and there is a $k \in I$, $i \leq k$, with $S_k \cong D_n$. Since P has a root in D, P has a root in S_k and R^h.

We now show that R^h has the universal property of definition (4.16). Let (S, n, k') be a quasi local Henselian ring and $\sigma: R \longrightarrow S$ a local ring morphism. We claim that it suffices to show $(*)$ for all $i \in I$ σ extends to a local morphism $\sigma_i: S_i \longrightarrow S$.

Suppose for all $i, j \in I$ with $i \leq j$ there are extensions $\sigma_i: S_i \longrightarrow S$ and $\sigma_j: S_j \longrightarrow S$. Then $\sigma_j \varphi_j^i: S_i \longrightarrow S$ and $\sigma_i: S_i \longrightarrow S$ are local R-algebra morphisms. By (4.21) $\sigma_i = \sigma_j \varphi_j^i$ and $(*)$ implies that there is a local R-algebra morphism $\sigma^h: R^h \rightarrow S$. Uniqueness follows from the uniqueness of the σ_i. Thus it suffices to show $(*)$.

Since S_i is an étale neighborhood of R, there is a standard étale R-algebra $S' = (R[x]/(f))_g$ where $f, g \in R[x]$, f monic, and f' invertible in S' with $S_i = S'_{n'}$ for some maximal ideal $n' \subseteq S'$ with $n' \cap R = m$. By (2.5) the ring $D = S \otimes_R S'$ is étale over S. Let τ denote the natural map: $\tau: D = S \otimes_R S' \longrightarrow S \otimes_R k \cong S/mS = \overline{S}$ and set $w = \tau^{-1}(n\overline{S})$ be the contraction of the maximal ideal of $\overline{S}$. Hence w is a maximal ideal of D with $n'D \subseteq w$ and $D/w = k(n) = k'$. In particular, D is an étale neighborhood of S and by (4.14) $D_w \cong S$, since S is Henselian. This establishes a local R-algebra morphism $\sigma_i: S_i \longrightarrow S$.

§4: THE STRICT HENSELIZATION

(4.28) Definition: A quasi local ring (R, m, k) is called strictly Henselian if R is Henselian and the residue field $k = R/m$ is separably closed.

(4.29) Proposition: A strictly Henselian quasi local ring is closed under local étale extension.

Proof: Let (R, m, k) be a quasi local, strictly Henselian ring and (S, n, k_s) a quasi local ring which is locally étale over R. Then k_s is a finite separable field extension of k. Hence $k = k_s$ and the rest of the proof is exactly the same as the proof of (4.14) (b) $\Rightarrow$ (c).

Let (R, m, k) be a quasi local ring. First we will use a similar construction as in §3 to find a local R-algebra $\tilde{R}$ which is strictly Henselian and has as residue class field $\tilde{k}$ a separable closure of k.

In the proof of (4.24) we constructed sets Γ and $\{(S_\gamma, m_\gamma, k_\gamma)\}_{\gamma \in \Gamma}$ where S_γ is a quasi local ring which is locally étale over R. Note that in this case k_γ is a finite separable extension of k. Moreover, for every local étale R-algebra (S, n) there is exactly one $\gamma \in \Gamma$ so that S and S_γ are isomorphic as local R-algebras.

(4.30) Proposition: Let (R, m, k) be a quasi local ring and $k \subseteq k'$ a finite separable field extension of k. Then there is a local étale R-algebra (S, n) with $S/n = k'$.

Proof: Since $k \subseteq k'$ is finite separable there is an irreducible monic polynomial $\overline{P} \in k[x]$ so that $k' = k[x]/(\overline{P})$. Moreover, $\overline{P}' \neq 0$ in $k[x]/(\overline{P})$. Let $P \in R[x]$ be a monic polynomial with $P + mR[x] = \overline{P}$ in $k[x]$. Then $n = (m, P)$ is a maximal ideal of $R[x]$ with $P' \notin n$. The ring $(R[x]/(P))_{P'}$ is étale over R and $S = (R[x]/(P))_n$ is a local étale R-algebra with $S/n = k'$.

Fix a separable closure $\tilde{k}$ of k. For every $\gamma \in \Gamma$ there are $t_\gamma = [k_\gamma : k]$ different k-morphisms $\bar{\varphi}_\gamma : k_\gamma \longrightarrow \tilde{k}$. These correspond to t_γ different local R-algebra morphisms $\varphi_\gamma : S_\gamma \longrightarrow \tilde{k}$. The set of all these pairs $(S_\gamma, \varphi_\gamma)$ is denoted by J. Consider the following relation on J:

$(\gamma, \varphi_\gamma) \leq (\delta, \varphi_\delta) \Longleftrightarrow$ there is an R-algebra morphism $\varphi^\gamma_\delta : S_\gamma \longrightarrow S_\delta$ so that the diagram

$$\begin{tikzcd} S_\gamma \arrow[dr, "\varphi_\gamma"] \arrow[d, "\varphi^\gamma_\delta"'] & \\ S_\delta \arrow[r, "\varphi_\delta"'] & \tilde{k} \end{tikzcd}$$

commutes, that is, there is a local R-algebra morphism from S_γ to S_δ and the induced embeddings of the residue fields into $\tilde{k}$ commute.

(4.31) <u>Proposition</u>: J is partially ordered and a directed system under "$\leq$".

<u>Proof</u>: Obviously, the relation '$\leq$' is reflexive and transitive.

In order to show that "$\leq$" is antisymmetric let $(\gamma, \varphi_\gamma), (\delta, \varphi_\delta) \in J$ with $(\gamma, \varphi_\gamma) \leq (\delta, \varphi_\delta)$ and $(\delta, \varphi_\delta) \leq (\gamma, \varphi_\gamma)$. Hence there are R-algebra morphisms $\varphi^\gamma_\delta : S_\gamma \longrightarrow S_\delta$ and $\varphi^\delta_\gamma : S_\delta \longrightarrow S_\gamma$ so that the diagram:

$$\begin{tikzcd} S_\gamma \arrow[dr, "\varphi_\gamma"] \arrow[d, shift left, "\varphi^\gamma_\delta"] & \\ S_\delta \arrow[u, shift left, "\varphi^\delta_\gamma"] \arrow[r, "\varphi_\delta"'] & \tilde{k} \end{tikzcd}$$

commutes. Then $\varphi^\delta_\gamma \varphi^\gamma_\delta$ and $\varphi^\gamma_\delta \varphi^\delta_\gamma$ induce the identity maps on k_γ and k_δ. By Proposition (4.21) S_γ and S_δ are isomorphic as R-algebras and hence $\gamma = \delta$.

In order to show that J is a directed system let $(\gamma, \varphi_\gamma), (\delta, \varphi_\delta) \in J$. Consider the morphism $\varphi = \varphi_\gamma \otimes \varphi_\delta : S_\gamma \otimes_R S_\delta \longrightarrow \tilde{k}$ with kernel $n = \ker(\varphi)$. Then $n \cap R = m$ and the R-algebra $(S_\gamma \otimes_R S_\delta)_n$ is locally étale over R. Hence there is an $\alpha \in \Gamma$ with $(S_\gamma \otimes_R S_\delta)_n \cong S_\alpha$. The induced morphism $\varphi_\alpha \,\hat{=}\, (\varphi_\gamma \otimes \varphi_\delta)_n : S_\alpha \longrightarrow \widetilde{k}$ defines a pair $(\alpha, \varphi_\alpha) \in \mathcal{J}$ with $(\gamma, \varphi_\gamma) \leq (\alpha, \varphi_\alpha)$ and $(\delta, \varphi_\delta) \leq (\alpha, \varphi_\alpha)$.

(4.32) <u>Remark</u>: The proof of (4.31) shows that if (γ, φ_γ) and $(\gamma, \varphi_\gamma') \in \mathcal{J}$ (with $\overline{\varphi_\gamma}, \overline{\varphi_\gamma'} : k_\gamma \longrightarrow \widetilde{k}$ two different k-embeddings of k_γ into $\widetilde{k}$) then there is a larger local étale R-algebra $(S_\alpha, \varphi_\alpha)$ so that $\overline{\varphi_\alpha} : k_\alpha \longrightarrow \widetilde{k}$ 'includes' the two different k-embeddings $\overline{\varphi_\gamma}, \overline{\varphi_\gamma'} : k_\gamma \longrightarrow \widetilde{k}$.

Set $\quad R^{hs} = \varinjlim_{(\gamma, \varphi_\gamma) \in \mathcal{J}} (S_\gamma, \varphi_\gamma)$

Similar to the proof of (4.27) it follows that R^{hs} is a quasi local Henselian ring with maximal ideal $m^{hs} = \varinjlim n_\gamma$ and residue field $k^{hs} = \varinjlim k_\gamma = R^{hs}/m^{hs}$. Moreover, the $\overline{\varphi_\gamma}$ induce a k-morphism $\varphi^{hs} : k^{hs} \longrightarrow \widetilde{k}$. By (4.30) φ^{hs} is surjective and the residue field k^{hs} of R^{hs} is separably closed. R^{hs} is strictly Henselian. This is summarized in the next Theorem:

(4.33) <u>Theorem</u>: The quasi local ring R^{hs} is strictly Henselian. R^{hs} is called the <u>strict Henselization</u> of R.

In order to characterize R^{hs} in terms of a universal property we first need a modification of Proposition (4.21).

(4.34) <u>Definition</u>: Let (R, m, k) be a quasi local ring and (S, n, k_S) a local R-algebra. S is called a local <u>ind-étale</u> R-algebra if S is the direct limit of local étale R-algebras (and local transition maps).

(4.35) <u>Remark</u>: R^h and R^{hs} are local ind-étale R-algebras.

(4.36) <u>Proposition</u>: Let (R, m, k) be a quasi local ring, (S, n, k_S) a local ind-étale R-algebra, and (T, w, k_T) a local Hensilian R-algebra with $n \cap R = m = w \cap R$. The natural map $\Delta : \mathrm{Hom}_{locR}(S, T) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k_S, k_T)$ is bijective.

<u>Proof</u>: By assumption $S = \varinjlim_{i \in I} S_i$ where (S_i, n_i, k_i) are local étale R-algebras. Then $n = \varinjlim n_i$ and $k_S = \varinjlim k_i$. Moreover,

$$\mathrm{Hom}_{locR}(S,T) = \mathrm{Hom}_{locR}(\varinjlim S_i, T) = \varprojlim \mathrm{Hom}_{locR}(S_i, T)$$

and $\mathrm{Hom}_{k\text{-alg}}(k_S, k_T) = \varprojlim \mathrm{Hom}_{k\text{-alg}}(k_i, k_T)$. By (4.21) we already know that the natural maps $\mathrm{Hom}_{R\text{-alg}}(S_i, T) \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k_i, k_T)$ are injective. Since the inverse limit is 'left exact' the natural map Δ is injective. It remains to show that Δ is surjective.

Set $D = S \otimes_R T$. By the universal property of the tensor product

$$\mathrm{Hom}_{locR}(S,T) \overset{(*)}{=} \mathrm{Hom}_{R\text{-alg}}(S,T) = \mathrm{Hom}_{T\text{-alg}}(D,T)$$

where $(*)$ follows since for all $i \in I$: $\mathrm{rad}(mS_i) = n_i$. Similarly,

$$\mathrm{Hom}_{k\text{-alg}}(k_S, k_T) = \mathrm{Hom}_{R\text{-alg}}(S, k_T) = \mathrm{Hom}_{T\text{-alg}}(D, k_T).$$

Hence it suffices to show that the natural map $\Delta_D : \mathrm{Hom}_{T\text{-alg}}(D,T) \longrightarrow \mathrm{Hom}_{T\text{-alg}}(D, k_T)$ is surjective. Notice that every T-algebra morphism $\varphi : D \longrightarrow k_T$ is surjective. Hence there is a bijection:

$$\mathrm{Hom}_{T\text{-alg}}(D, k_T) \overset{\cong}{\longleftrightarrow} \{ q \subseteq D \mid q \text{ a maximal ideal of } D \text{ with } k(q) = k_T,\ q \cap T = w \}$$

Set $D_i = S_i \otimes_R T$ and let q_i be the preimage of q in D_i. Then $k(q_i) = k_T$ and D_{iq_i} is a local étale T-algebra. Since T is Henselian, by (4.14) $D_{iq_i} \cong T$ for all $i \in I$ and therefore $D_q \cong T$. For every T-algebra morphism $\bar{u} : D \longrightarrow k_T$ with $\ker(\bar{u}) = q$ there is a commutative diagram:

$$\begin{tikzcd}
D \arrow[r] & D_q \arrow[r, "\bar{u}"] \arrow[d, "\cong"'] & k_T \\
 & T \arrow[ur] &
\end{tikzcd}$$

and $\bar{u}$ lifts to a T-algebra morphism $u : D \longrightarrow T$.

(4.37) Theorem: Let (R, m, k) be a quasi local ring and (R^{hs}, m^{hs}, k^{hs}) its strict Henselization as defined before. Suppose that (S, n, k_s) is a quasi local strictly Henselian ring and that there is given a commutative diagram of ring morphisms:

$$(*)\qquad \begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "u"'] & R^{hs} \arrow[r, "\alpha"] \arrow[d, dashed, "u^{hs}"] & K \arrow[d, "j"] \\ & S \arrow[r, "\beta"] & K' \end{tikzcd}$$

where K, K' are fields, i is the canonical map, and u, α, β are local morphisms. Then there is a unique local morphism $u^{hs}: R \longrightarrow S$ so that diagram $(*)$ commutes.

Proof: Since R^{hs} is ind-étale and S is Henselian, by (4.36):

$$\mathrm{Hom}_{R\text{-alg}}(R^{hs}, S) = \mathrm{Hom}_{k\text{-alg}}(k^{hs}, k_s).$$

Moreover, the ring morphism u is local. Let $\bar{\alpha}: k^{hs} \longrightarrow K$ and $\bar{\beta}: k_s \longrightarrow K'$ be the induced maps and consider the commutative diagram:

$$(**)\qquad \begin{tikzcd} k \arrow[r] \arrow[dr] & k^{hs} \arrow[r, "\bar{\alpha}"] \arrow[d, dashed, "\bar{u}"] & K \arrow[d, "j"] \\ & k_s \arrow[r, "\bar{\beta}"] & K' \end{tikzcd}$$

Since k^{hs} is a separable closure of k and k_s is separably closed, j induces a unique k-morphism $\bar{u}: k^{hs} \longrightarrow k_s$ so that diagram $(**)$ commutes. Hence there is a unique R-algebra morphism $u: R \longrightarrow S$ so that $(*)$ commutes.

(4.38) Definition: Let (R, m, k) be a quasi local ring. A quadruple $(\tilde{R}, \tilde{k}, i, \alpha)$ where $\tilde{R}$ is a quasi local strictly Henselian ring, $\tilde{k}$ is a separable closure of k, $i: R \longrightarrow \tilde{R}$ is a local morphism, and $\alpha: \tilde{R} \longrightarrow \tilde{k}$ is a local morphism is called a strict Henselization of R if $(\tilde{R}, \tilde{k}, i, \alpha)$ has the universal property of Theorem (4.37).

(4.39) Remark: Let (R', k', i', α') be another strict Henselization of R. Then the natural map $\mathrm{Hom}_{R\text{-alg}}(R^{hs}, R') \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k^{hs}, k')$ is bijective and R^{hs} and R'

are isomorphic (but, in general, not by a unique isomorphism). In particular, the group of R-automorphisms of R^{hs} is isomorphic to the Galois group of k-automorphisms of k^{hs}.

(4.40) <u>Remark</u>: The strict Henselization is useful in the study of divisor class groups. For example, Danilov showed that if R is a local normal Noetherian ring with perfect residue field and resolvable singularity then the natural morphism of groups $Cl(A) \longrightarrow Cl(A[\![x]\!])$ (x a variable) is bijective if and only if $Cl(A^{hs})$ is finite.

§5: PROPERTIES OF THE HENSELIZATION

Our first goal is to show that a quasi local ring R is Noetherian if and only if R^h (R^{hs}, respectively) is Noetherian. In order to do so we need the following result:

(4.41) Proposition: Let (R, m) be a quasi local ring which is complete (and separated) in the m-adic topology. R is Noetherian if and only if m is finitely generated.

Proof: Bourbaki, Chap. III, §2, Cor. 5 to Thm 2.

(4.42) Proposition: Let $(S_i, \varphi_j^i)_{i,j\in I}$ be a direct system of rings and morphisms which satisfies the following conditions:

(i) For all $i \in I$ S_i is a local Noetherian ring with maximal ideal m_i.
(ii) For all $i,j \in I$ with $i \le j$ the morphism $\varphi_j^i: S_i \longrightarrow S_j$ is faithfully flat.
(iii) For all $i,j \in I$ with $i \le j$: $\varphi_j^i(m_i) S_j = m_j$.

Then the direct limit $S = \varinjlim_{i\in I} S_i$ is a local Noetherian ring with maximal ideal $m = \varinjlim_{i\in I} m_i$. Moreover, the canonical maps $\varphi_i: S_i \longrightarrow S$ are faithfully flat for all $i \in I$.

Proof: Obviously, $S = \varinjlim S_i$ is a quasi local ring with maximal ideal $m = \varinjlim m_i$. Since the direct limit is exact and commutes with tensor products, the natural maps $\varphi_i: S_i \longrightarrow S$ are faithfully flat. Moreover, by (iii) the maximal ideal m of S is generated by $\varphi_i(m_i)$ for every $i \in I$. Hence m is finitely generated.

In order to show that S is Noetherian, let $\hat{S}$ denote the m-adic completion of S. The maximal ideal $\hat{m} = m\hat{S}$ of $\hat{S}$ is finitely generated.

Thus by (4.41) $\hat{S}$ is a local Noetherian ring.

Let $J \subseteq S$ be an ideal. Then there are $x_1, \dots, x_n \in J$ so that $J\hat{S} = (x_1, \dots, x_n)\hat{S}$. This implies that for all $k \in \mathbb{N}$: $J + m^k = (x_1, \dots, x_n) + m^k$ in S. If

$$(*) \quad \bigcap_{k \in \mathbb{N}} ((x_1, \dots, x_n) + m^k) = (x_1, \dots, x_n)$$

then $J = (x_1, \dots, x_n)$ and it suffices to show $(*)$. Let $x \in S - (x_1, \dots, x_n)$. Then there is an $h \in I$ and elements $x_1', \dots, x_n', x' \in S_h$ so that $\varphi_h(x_i') = x_i$ and $\varphi_h(x') = x$. Since $x' \notin (x_1', \dots, x_n')S_h$ and S_h local Noetherian, there is an $r \in \mathbb{N}$ so that $x' \notin (x_1', \dots, x_n')S_h + m_h^r$. Thus $x \notin (x_1, \dots, x_n)S + m^r$ since $\varphi_h : S_h \to S$ is faithfully flat and $m^r = \varphi_h(m_h^r)S$.

(4.43) <u>Proposition:</u> Let (R, m, k) be a quasi local ring and $(S, n), (S_1, n_1), (S_2, n_2)$ quasi local R-algebras.

(a) If (S, n) is locally étale over R, then S is faithfully flat over R and $mS = n$.

(b) If R is Noetherian, (S_1, n_1) and (S_2, n_2) locally étale over R, and $\varphi : S_1 \to S_2$ a local R-algebra morphism, then φ is faithfully flat.

<u>Proof:</u> (a) Let S' be an étale R-algebra, $n' \subseteq S'$ a prime ideal with $n' \cap R = m$, and $S = S'_{n'}$, $n = n'S$. By (3.28) S' is flat over R, hence S is flat over R. Since S' is unramified over R, $S' \otimes_R k$ is unramified over k and by (3.3) $S' \otimes_R k$ is a product of fields. Therefore the reduced, local, Artinian ring S/mS is a field and $mS = n$.

(b) By Matsumura, Thm. 22.3, S_2 is flat over S_1 if and only if $n_1 \otimes_{S_1} S_2 \cong n_1 S_2$ via the natural map. Since S_1 and S_2 are flat over R, $m \otimes_R S_1 \cong mS_1 = n_1$ and $m \otimes_R S_2 \cong mS_2 = n_2$. In particular, $n_1 S_2 = n_2$ and therefore:

$$n_1 \otimes_{S_1} S_2 = mS_1 \otimes_{S_1} S_2 \cong (m \otimes_R S_1) \otimes_{S_1} S_2 \cong m \otimes_R S_2 \cong mS_2 = n_1 S_2 = n_2.$$

Thus S_2 is flat over S_1 via φ.

Let (R, m, k) be a local Noetherian ring and $\{S_i, \varphi_j^i\}_{i \in I}$ a direct system of local R-algebras and local R-algebra morphisms where either:

(i) $\{S_i, \varphi_j^i\}_{i\in I}$ is the system of representatives of étale neighborhoods of R constructed in §3, or

(ii) $\{S_i, \varphi_j^i\}_{i\in I}$ is the system of representatives of local étale R-algebras (and embeddings into a separable closure of k) constructed in §4.

Then by (4.43)

(a) For all $i,j\in I$ with $i\leq j$ the local R-algebra morphisms $\varphi_j^i : S_i \longrightarrow S_j$ are faithfully flat

(b) For all $i\in I$: $\mathfrak{m} S_i = \mathfrak{n}_i$. In particular, if $i\leq j$ then $\varphi_j^i(\mathfrak{n}_i)S_j = \mathfrak{n}_j$.

(4.44) <u>Theorem:</u> Let $(R,\mathfrak{m},k)$ be a local Noetherian ring. Then

(a) R^h and R^{hs} are local Noetherian rings with maximal ideals $\mathfrak{m}^h = \mathfrak{m}R^h$ and $\mathfrak{m}^{hs} = \mathfrak{m}R^{hs}$.

(b) The canonical maps $\tau : R \longrightarrow R^h$ and $\lambda : R \longrightarrow R^{hs}$ are faithfully flat. Moreover, λ factors through τ, that is, there is a local R-algebra morphism $\rho : R^h \longrightarrow R^{hs}$ so that $\lambda = \rho\tau$.

<u>Proof</u>: (a) follows from (4.42).

(b) For all $i\in I$ there are canonical local morphisms:

$$R \xrightarrow{\tau_i} S_i \xrightarrow{\varphi_i} R^h \ \ (R^{hs})$$

with $\tau = \varphi_i\tau_i$ ($\lambda = \varphi_i\tau_i$ in the strict Henselian case). By (4.42) and (4.43) τ_i and φ_i are faithfully flat, hence so are τ and λ. By the universal property of the Henselization λ factors through τ.

(4.45) <u>Remark</u>: Let $(R,\mathfrak{m},k)$ be a quasi local ring. Since local étale R-algebras are faithfully flat over R, R^h and R^{hs} are faithfully flat over R. This implies that if R^h or R^{hs} are Noetherian, then R is Noetherian.

(4.46) <u>Proposition</u>: Let (R, m, k) be a local Noetherian ring and $\widehat{R}$ its m-adic completion. Then:

(a) There are natural inclusions $R \hookrightarrow R^h \hookrightarrow \widehat{R}$.

(b) R and R^h have the same m-adic completions, that is, $\widehat{R} = \widehat{R^h}$.

<u>Proof</u>: We know from 911 that complete local Noetherian rings satisfy Hensel's Lemma (Thm (9.31)). Hence by (4.11) complete local Noetherian rings are Henselian. The natural embedding $\gamma: R \longrightarrow \widehat{R}$ factors through R^h

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d, "\tau"'] & \widehat{R} \\ R^h \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Moreover, since $m R^h = m^h$ and $R^h/m^h = R/m$ we obtain by induction on t that $R^h/(m^h)^t \cong R/m^t$ for all $t \in \mathbb{N}$. Hence there is an R-isomorphism $\varphi: \widehat{R^h} \longrightarrow \widehat{R}$ yielding a diagram:

$$\begin{tikzcd} R \arrow[r, "\gamma"] \arrow[d, "\tau"'] & \widehat{R} \\ R^h \arrow[ur, "\sigma"] \arrow[r, "\delta"'] & \widehat{R^h} \arrow[u, "\varphi"'] \end{tikzcd}$$

where δ is the natural map. Since σ is unique, the diagram commutes and $\sigma = \varphi\delta$. Thus σ is surjective.

(4.47) <u>Corollary</u>: Let (R, m, k) be a local Noetherian ring. Then the following are equivalent:

(a) R is regular.

(b) R^h is regular.

(c) R^{hs} is regular.

<u>Proof</u>: Since $m^h = m R^h$ and $m^{hs} = m R^{hs}$, the statement follows immediately from 911, Theorem 8.63.

(4.48) Remark: (a) Recall that a Noetherian ring R is called normal (regular) if for all prime ideals $P \leq R$ the local ring R_P is normal (regular).
(b) If $\varphi : (R, m) \longrightarrow (S, n)$ is a faithfully flat morphism of local rings and S is regular (or normal, reduced) then R is regular (or normal, reduced). Depending on the fibers of φ, these properties may not ascend from R to S. If, however, S is a (local) étale extension of R, they do.

Recall that a Noetherian ring R satisfies Serre's condition
(R_i) if for all $P \in \mathrm{Spec}(R)$ with $\mathrm{ht}\, P \leq i$ the localization R_P is regular
(S_i) if for all $P \in \mathrm{Spec}(R)$: $\mathrm{depth}\, R_P \geq \min\{i, \mathrm{ht}\, p\}$.

(4.49) Theorem: Let R be a Noetherian ring and S an étale R-algebra. Then R satisfies Serre's condition (R_i) (or (S_i)) if and only if S satisfies Serre's condition (R_i) (or (S_i)).

Proof: Notice that the Serre conditions are local properties and that for $p \in \mathrm{Spec}(R)$, $q \in \mathrm{Spec}(S)$ with $q \cap R = p$ the induced map $R_p \longrightarrow S_q$ is faithfully flat. The backward direction follows immediately from 911, Theorem 9.60. In order to show the forward direction let $n \leq S$ be a maximal ideal and $m = n \cap R$. By 911, Theorem 9.60 S_n satisfies (R_i) (or (S_i)) if for all $P \in \mathrm{Spec}(R)$ with $P \leq m$ the fiber ring $S_n \otimes_R k(P)$ satisfies (R_i) (or (S_i)). Since S is étale over R, $S_n \otimes_R k(P)$ is unramified over $k(P)$ and thus by (3.3) a finite product of fields. All fiber rings are regular and the forward direction follows.

(4.50) Corollary: Let R be a Noetherian ring and S an étale R-algebra. Then:
(a) R is reduced $\Longleftrightarrow$ S is reduced
(b) R is normal $\Longleftrightarrow$ S is normal
(c) R is regular $\Longleftrightarrow$ S is regular

(4.51) Corollary: Let (R, m) be a local Noetherian ring. Then:

(a) R is reduced $\iff R^h$ is reduced $\iff R^{hs}$ is reduced

(b) R is normal $\iff R^h$ is normal $\iff R^{hs}$ is normal.

(4.52) Theorem: Let (R, m, k) be a local Noetherian domain, $K = Q(R)$ its field of quotients, and S its integral closure (in K). Suppose that S is a finite R-module. Then there is a one-to-one correspondence between the sets:

$$m\text{-}\mathrm{Spec}(S) \cong \mathrm{Min}(R^h)$$

where $\mathrm{Min}(R^h)$ denotes the set of minimal prime ideals of R^h.

Proof: Set $\tilde{S} = S \otimes_R R^h$.

(a) We claim that there is a one-to-one correspondence between $m\text{-}\mathrm{Spec}(S)$ and $\mathrm{Min}(\tilde{S})$

Since S is a finite R-module, $\tilde{S}$ is a finite R^h-module and $m\text{-}\mathrm{Spec}(\tilde{S})$ is the set of all prime ideals of $\tilde{S}$ which lie over m^h. In particular,

$$m\text{-}\mathrm{Spec}(\tilde{S}) \cong \mathrm{Spec}(\tilde{S} \otimes_{R^h} k) = \mathrm{Spec}(S \otimes_R k) \cong m\text{-}\mathrm{Spec}(S).$$

Hence it suffices to show that $m\text{-}\mathrm{Spec}(\tilde{S}) \cong \mathrm{Min}(\tilde{S})$. Write $R^h = \varinjlim S_i$ where S_i are étale neighborhoods of R. Then $\tilde{S} = \varinjlim (S \otimes_R S_i)$ and each $S \otimes_R S_i$ is the localization of an étale extension of S. By (4.50) $S \otimes_R S_i$ is a normal ring implying that the direct limit $\tilde{S}$ is a normal ring. Let $Q \in m\text{-}\mathrm{Spec}(\tilde{S})$. Then $\tilde{S}_Q$ is a normal domain, in particular, Q contains exactly one minimal prime ideal of $\tilde{S}$. This yields a surjective map:

$$\phi: m\text{-}\mathrm{Spec}(\tilde{S}) \longrightarrow \mathrm{Min}(\tilde{S})$$

Since R^h is Henselian and $\tilde{S}$ a finite R^h-module, $\tilde{S}$ decomposes into:

$$\tilde{S} = \prod_{Q \in m\text{-}\mathrm{Spec}(\tilde{S})} \tilde{S}_Q.$$

Every minimal prime ideal of $\tilde{S}$ is contained in exactly one maximal ideal of $\tilde{S}$ and $m\text{-}\mathrm{Spec}(\tilde{S}) \cong \mathrm{Min}(\tilde{S})$.

(b) It remains to show that $\mathrm{Min}(\tilde{S}) \cong \mathrm{Min}(R^h)$. Since the natural map $\tau: R \to R^h$ is flat, by going down every minimal prime ideal of R^h lies over the zero ideal of R and $\mathrm{Min}(R^h) \cong \mathrm{Min}(R^h \otimes_R K)$. By a similar argument $S \longrightarrow \widetilde{S}$ is flat and $\widetilde{S}$ is a domain, hence $\mathrm{Min}(\widetilde{S}) \cong \mathrm{Min}(\widetilde{S} \otimes_R K)$. Since $S \subseteq K$ and R^h is flat over R, there are inclusions:

$$\widetilde{S} = R^h \otimes_R S \subseteq R^h \otimes_R K = R^h \otimes_R (S \otimes_R K) = \widetilde{S} \otimes_R K$$

(since $S \otimes_R K = K$). Therefore:

$$\mathrm{Min}(\widetilde{S}) \cong \mathrm{Min}(\widetilde{S} \otimes_R K) = \mathrm{Min}(R^h \otimes_R K) \cong \mathrm{Min}(R^h)$$

and the assertion follows.

(4.53) <u>Corollary</u>: Let (R, m, k) be a local Noetherian domain and S its integral closure. Suppose that S is a finite R-module. Then R^h is a domain if and only if S is a local ring.

(4.54) <u>Exercise</u>: Let R be a quasi local ring and $I \subseteq R$ an ideal. Show

$$(R/I)^h \cong R^h / I R^h.$$