CHAPTER XVI: A LIFTING THEOREM

§1: LOCALIZATION OF FORMAL SMOOTHNESS

Let k be a field and (S,n) a local Noetherian k-algebra. By (8.33) S is n-smooth over k if and only if S is geometrically regular over k, that is, the natural morphism $k \longrightarrow S$ is regular. Due to M. André this result can be extended as follows: Let $\varphi: (R,m) \longrightarrow (S,n)$ be a local morphism of local Noetherian rings. If S is n-smooth over R and R is a G-ring, then φ is regular. As a corollary we obtain: Let $\varphi: (R,m) \longrightarrow (S,n)$ be as above and $Q \subseteq S$ a prime ideal with $P = Q \cap R$. Then S_Q is QS_Q-smooth over R_P, i.e. formal smoothness localizes. In this section we show André's theorem in the case that R contains a field.

(16.1) <u>Theorem</u>: Let $u: (R,m) \longrightarrow (S,n)$ be a local morphism of complete local Noetherian rings with S n-smooth over R. Then there is a commutative diagram of complete local Noetherian rings and local morphisms.

$$\begin{array}{ccc} (R',m') & \xrightarrow{u'} & (S',n') \\ \alpha \downarrow & & \downarrow \beta \\ (R,m) & \xrightarrow{u} & (S,n) \end{array}$$

where S' is n'-smooth over R', R' is a regular local ring, α and β are surjective with $S \cong S' \otimes_{R'} R$ via $\beta \otimes u$.

<u>Proof</u>: Set $k = R/m$, $C = S \otimes_R k \cong S/mS$, $\bar{u}: k \longrightarrow C$ the morphism induced by u. Note that C is n/mS-smooth over k.

<u>Case 1</u>: R is of equal characteristic

In this case R contains a coefficient field and there is an embedding

$i: k \hookrightarrow R$ so that the composition $k \overset{i}{\hookrightarrow} R \xrightarrow{\nu} k$ is the identity on k, $\nu i = \mathrm{id}_k$ where ν is the natural map. Let $\mu: S \longrightarrow C$ be the natural map. Since C is n/mS-smooth over k and S is complete, by (11.7) there is a k-algebra morphism $\gamma: C \longrightarrow S$ so that $\mu\gamma = \mathrm{id}_C$, that is, the diagram

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[dr, "\mathrm{id}_C"'] & S \arrow[d, "\mu"] \\ & C \end{tikzcd}$$

commutes.

Let $x_1, \ldots, x_n \in m$ be a system of generators of m, $t_1, \ldots, t_n$ variables over k. Set $R' = k[[t_1, \ldots, t_n]]$ and define the k-algebra morphism $\alpha: R' \longrightarrow R$ by $\alpha|_k = i$ and $\alpha(t_j) = x_j$ for all $1 \le j \le n$. Since R and R' are complete, α is surjective. Set $S' = C[[t_1, \ldots, t_n]]$ and define $u': R' \longrightarrow S'$ by $u'|_k = \bar{u}$ and $u'(t_j) = t_j$ for all $1 \le j \le n$. Since C is n/mS-smooth over k, S' is $(n/mS, t_1, \ldots, t_n)$-smooth over R'. Finally define $\beta: S' \longrightarrow S$ by $\beta|_C = \gamma$ and $\beta(t_j) = u\alpha(t_j) = u(x_j)$ for all $1 \le j \le n$. Since S and S' is complete, β is surjective with $\beta u' = u\alpha$.

Consider $\beta \otimes u: S' \otimes_{R'} R \longrightarrow S$. Then $\beta \otimes k: S' \otimes_{R'} k \longrightarrow S/mS = C$ is an isomorphism. The R-algebra $S' \otimes_{R'} R$ is m-adically complete and S is flat over R. By (10.2) $\beta \otimes u$ is an isomorphism.

<u>Case</u> 2: R is of unequal characteristic.

By (10.12) R contains a coefficient ring $R_0 \subseteq R$ and there is a complete discrete valuation ring $(D, pD, k = R/m)$ and a local surjective morphism $i: D \longrightarrow R_0 \subseteq R$. By assumption $C = S/mS$ is n/mS-smooth over k, in particular, C is a regular local ring. By (11.5) there is a complete local Noetherian D-algebra $(C', \tilde{n}, \ell)$ so that $C' \otimes_D k \cong C'/pC' \cong C$ as k-algebras. Moreover, C' is faithfully flat over D and $C \cong C'/pC'$ is geometrically regular over k. Thus C' is n'-smooth over D and by (11.7) the D-algebra morphism $\sigma: C' \longrightarrow C'/pC' \cong S/mS$ lifts to a D-algebra morphism $\gamma: C' \longrightarrow S$ so that the diagram:

$$\begin{tikzcd} C' \arrow[rr, "\sigma"] \arrow[dr, "\gamma"'] & & S/mS \\ & S \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes, that is, $\mu\gamma = \sigma$.

Let $p, x_2, -, x_n$ be a system of generators of m and $R' = D[[t_2, -, t_n]]$, $S' = C'[[t_2, -, t_n]]$ the formal power series rings over D and C'. Define $\alpha: R' \longrightarrow R$ by $\alpha|_D = i$, $\alpha(t_i) = x_i$ for all $2 \le i \le n$ and $\beta: S' \longrightarrow S$ by $\beta|_{C'} = \gamma$ and $\beta(t_i) = u(x_i)$ for all $2 \le i \le n$. Obviously, α and β are surjective. Define $u': R' \longrightarrow S'$ by $u'|_D: D \longrightarrow C'$ is the ring morphism which defines the D-algebra structure on C' and by $u'(t_i) = t_i$ for all $2 \le i \le n$. Since C' is n'-smooth over D, the ring S' is $(n', t_2, -, t_n)$-smooth over R'.

Homework: Show in the unequal characteristic case that $S \cong S' \otimes_R R'$ via $\beta \otimes u$.

(16.2) <u>Remark</u>: Note that the rings C and C' in the proof of (16.1) are regular. Hence (S', n') of (16.1) is also a complete regular local ring.

Recall the following definition from Chapter <u>VIII</u>:

Let R be a ring, S an R-algebra, and $I \subseteq S$ an ideal. S is called I-unramified over R if for every R-algebra C, every ideal $N \subseteq C$ with $N^2 = 0$ and every commutative diagram of R-algebra morphisms:

$$\begin{tikzcd} S \arrow[r, "\overline{u}"] \arrow[dr, dashed, "u"] & C/N \\ R \arrow[u] \arrow[r] & C \arrow[u, "v"'] \end{tikzcd}$$

where v is the natural map, C/N carries the discrete topology, S the I-adic topology and $\overline{u}$ is continuous, there is at most one R-algebra morphism $u: S \longrightarrow C$ so that $vu = \overline{u}$. Note that if (R, m) is a local Noetherian ring by (18.9) the m-adic completion $\widehat{R}$ is $\widehat{m}$-étale over R, that is, $\widehat{R}$ is $\widehat{m}$-smooth and $\widehat{m}$-unramified over R.

(16.3) <u>Proposition</u>: Let R be a ring, S an R-algebra, and $I \subseteq S$ an ideal. The following conditions are equivalent:

(a) S is I-unramified over R.

(b) $\widehat{\Omega_{S/R}} = 0$ where $\widehat{\Omega_{S/R}}$ is the I-adic completion of $\Omega_{S/R}$.

<u>Proof</u>: Note that $\widehat{\Omega_{S/R}} = 0$ if and only if $\Omega_{S/R}/I^n\Omega_{S/R} = 0$ for all $n \in \mathbb{N}$.

(b) $\Rightarrow$ (a): The proof is similar to the proof of (2.13) (b) $\Rightarrow$ (a):

Let C be an R-algebra, $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of R-algebra morphisms:

$$\begin{tikzcd} S \arrow[r, "\bar{u}"] \arrow[dr, shift left, "u"] \arrow[dr, shift right, "w"'] & C/N \\ R \arrow[u] \arrow[r] & C \arrow[u, "v"'] \end{tikzcd}$$

where $\bar{u}$ is continuous and u and w are liftings of $\bar{u}$. As in the proof of (2.13) u and v define the same S-module structure on N. Moreover, $N^2 = 0$ and $\bar{u}(I^r) = 0$ for some $r \in \mathbb{N}$, hence $u(I^r), w(I^r) \subseteq N$ and $u(I^{2r}) = w(I^{2r}) = 0$, that is, N is an S/I^{2r}-module. As in the proof of (2.13) $d = u - w : S \longrightarrow N$ defines an R-derivation from S into N. Thus $d = h\, d_{S/R}$ where $d_{S/R} : S \longrightarrow \Omega_{S/R}$ is the universal R-derivation and $h : \Omega_{S/R} \longrightarrow N$ is an S-linear map. Since $I^{2r}N = 0$, h factors through $\Omega_{S/R}/I^{2r}\Omega_{S/R} = 0$ and $h = 0$. Therefore $u = w$.

(a) $\Rightarrow$ (b): Obviously, $\widehat{\Omega_{S/R}} = 0$ if and only if for all $r \in \mathbb{N}$ and all S/I^r-modules E $\operatorname{Hom}_S(\Omega_{S/R}, E) = 0$. Let E be an S/I^r-module and $f : \Omega_{S/R} \longrightarrow E$ an S-linear map. Then f induces an R-derivation $D = f d_{S/R} : S \longrightarrow E$ where $d_{S/R} : S \longrightarrow \Omega_{S/R}$ is the universal R-derivation. Let $C = S/I^r * E$ be the trivial extension of E, $p : C \longrightarrow S/I^r$ the projection, $\nu : S \longrightarrow S/I^r$ the natural map, and $h : S \longrightarrow C$ the R-algebra morphism defined by $h(s) = (\nu(s), 0)$. Then we obtain a commutative diagram of R-algebra morphisms:

$$\begin{tikzcd} S \arrow[r, "\nu"] & S/I^r \\ R \arrow[u] \arrow[r] \arrow[ur, shift left, "h"] \arrow[ur, shift right, "h+D"'] & C \arrow[u, "p"'] \end{tikzcd}$$

where h and $h+D$ are R-algebra liftings of ν. Since S is I-unramified over R, $h = h+D$ and $D=0$. Hence $f=0$ and $\widehat{\Omega_{S/R}} = 0$.

Let $k \xrightarrow{u} (R,m) \xrightarrow{v} (S,n)$ be morphisms of rings with R and S local Noetherian rings and $v(m) \subseteq n$. Then there is an exact sequence of S-modules:

$$(*) \qquad \Omega_{R/k} \otimes_R S \xrightarrow{\alpha} \Omega_{S/k} \xrightarrow{\beta} \Omega_{S/R} \longrightarrow 0$$

(16.4) Proposition: Assumptions as above. Suppose in addition that S is n-smooth over R relative to k.

(a) By passing to the n-adic completions $(*)$ yields an exact sequence:

$$0 \longrightarrow (\Omega_{R/k} \otimes_R S)^\wedge \xrightarrow{\hat\alpha} \widehat{\Omega_{S/k}} \xrightarrow{\hat\beta} \widehat{\Omega_{S/R}} \longrightarrow 0.$$

(b) If, in addition, S is n-unramified over R, then $\hat\alpha$ is an isomorphism. In particular, for every local Noetherian k-algebra (S,n)

$$(\Omega_{S/k} \otimes_S \hat S)^\wedge \cong (\Omega_{\hat S/k})^\wedge .$$

Proof: (a) By (8.25) for all $t \in \mathbb{N}$ the natural map

$$\alpha_t : \Omega_{R/k} \otimes_R (S/n^t) \longrightarrow \Omega_{S/k} \otimes_S (S/n^t)$$

is left invertible. Thus for all $t \in \mathbb{N}$ the sequence

$$0 \longrightarrow \Omega_{R/k} \otimes_R (S/n^t) \xrightarrow{\alpha_t} \Omega_{S/k} \otimes_S (S/n^t) \xrightarrow{\beta_t} \Omega_{S/R} \otimes_S (S/n^t) \longrightarrow 0$$

is split exact. By passing to the inverse limit the sequence

$$0 \longrightarrow (\Omega_{R/k} \otimes_R S)^\wedge \xrightarrow{\hat\alpha} \widehat{\Omega_{S/k}} \xrightarrow{\hat\beta} \widehat{\Omega_{S/R}} \longrightarrow 0$$

is exact. (see 911, (7.86)).

(b) By (16.3) $\widehat{\Omega_{S/R}} = 0$ if S is n-unramified over R. If (S,n) is a local Noetherian k-algebra, by (8.9) $\hat S$ is $\hat n$-étale over S. Apply (a) to the ring morphisms $k \longrightarrow S \longrightarrow \hat{S}$. Then $\hat{\alpha}$ is an isomorphism and

$$(\Omega_{S/k} \otimes_S \hat{S})^{\wedge} \cong \widehat{\Omega_{\hat{S}/k}}.$$

(16.5) <u>Corollary</u>: (a) Let $(R,m) \longrightarrow (S,n)$ be a local morphism of local Noetherian rings and assume that $\Omega_{S/R}$ is a finitely generated S-module (or a finite free S-module of rank m). Then $(\Omega_{\hat{S}/R})^{\wedge}$ is a finitely generated $\hat{S}$-module (or a finite free $\hat{S}$-module of rank m).
(b) Let $K \subseteq L$ be a finitely generated field extension and $x_1, -, x_n$ variables over L. Then $(\Omega_{L[\![x_1,-,x_n]\!]/K})^{\wedge}$ is an $L[\![x_1,-,x_n]\!]$-module of finite type. In particular, $(\Omega_{K[\![x_1,-,x_n]\!]/K})^{\wedge}$ is a free $K[\![x_1,-,x_n]\!]$-module of rank n.

(16.6) <u>Corollary</u>: Let $k \xrightarrow{u} (R,m) \xrightarrow{v} (S,n)$ be morphisms of rings, (R,m) and (S,n) local Noetherian rings, and $v(m) \subseteq n$. If S is n-smooth over R relative to k, then the natural map

$$\hat{\alpha} : (\Omega_{R/k} \otimes_R S)^{\wedge} \longrightarrow \widehat{\Omega_{S/k}}$$

is left invertible.

<u>Proof</u>: follows from the proof of (16.4).

(16.7) <u>Proposition</u>: Let $\varphi : (R,m) \longrightarrow (S,n)$ be a local morphism of complete regular local rings. Suppose that R contains a field, hence $R = k[\![x_1,-,x_n]\!]$ with $k \cong R/m$. Let $\ell \subseteq k$ be a subfield with k finite over ℓ. If S is n-smooth over R every ℓ-derivation $D_R : R \longrightarrow R$ extends to an ℓ-derivation $D_S : S \longrightarrow S$.

<u>Proof</u>: Set $R_0 = k[x_1,-,x_n]$. By (16.4) and (16.5) $\widehat{\Omega_{R/\ell}} \cong (\Omega_{R_0/\ell} \otimes_{R_0} R)^{\wedge} = \Omega_{R_0/\ell} \otimes_{R_0} R$ and $\widehat{\Omega_{R/\ell}}$ is a finite free R-module. Moreover, by (16.6) the

natural map $\hat{\alpha}: (\Omega_{R/\ell} \otimes_R S)^\wedge \longrightarrow \widehat{\Omega_{S/\ell}}$ is left invertible, hence the induced map

$$\hat{\alpha}^* : \mathrm{Hom}_S(\widehat{\Omega_{S/\ell}}, S) \longrightarrow \mathrm{Hom}_S((\Omega_{R/\ell} \otimes_R S)^\wedge, S)$$

is surjective. Since $(\Omega_{R/\ell} \otimes_R S)^\wedge \cong (\Omega_{R_0/\ell} \otimes_{R_0} S)^\wedge = \Omega_{R_0/\ell} \otimes_{R_0} S$ is a finite free S-module, by 9II (7.90):

$$\begin{aligned} \mathrm{Hom}_S((\Omega_{R/\ell} \otimes_R S)^\wedge, S) &\cong \mathrm{Hom}_S(\Omega_{R/\ell} \otimes_R S, S) \\ &\cong \mathrm{Hom}_R(\Omega_{R/\ell}, S) \otimes_R S \\ &\cong \mathrm{Hom}_R(\Omega_{R/\ell}, S) \\ &\cong \mathrm{Der}_\ell(R,S) \end{aligned}$$

and since S is n-adically complete

$$\begin{aligned} \mathrm{Hom}_S(\widehat{\Omega_{S/\ell}}, S) &\cong \mathrm{Hom}_S(\Omega_{S/\ell}, S) \\ &\cong \mathrm{Der}_\ell(S,S). \end{aligned}$$

Thus the natural map $\gamma: \mathrm{Der}_\ell(S,S) \longrightarrow \mathrm{Der}_\ell(R,S)$ is surjective and $D_R \in \mathrm{Der}_\ell(R) \subseteq \mathrm{Der}_\ell(R,S)$ lifts to an ℓ-derivation $D_S \in \mathrm{Der}_\ell(S,S)$.

(16.8) <u>Lemma</u>: Let $\varphi: (R,m) \longrightarrow (S,n)$ be a local morphism of local Noetherian rings. If S is n-smooth over R via φ, then $\hat{S}$ is $\hat{n}$-smooth over $\hat{R}$ via the induced morphism $\hat{\varphi}: \hat{R} \longrightarrow \hat{S}$.

<u>Proof</u>: S is n-smooth over R if and only if φ is flat and S/mS is geometrically regular over $k = R/m$. If S is n-smooth over R, $\hat{\varphi}$ is flat and $\hat{S}/m\hat{S}$ is $\hat{n}/m\hat{S}$-smooth over S/mS. Hence $\hat{S}/\hat{m}\hat{S}$ is $\hat{n}/\hat{m}\hat{S}$-smooth over $R/m = \hat{R}/\hat{m}$.

(16.9) <u>Lemma</u>: Let (R,m) be a local Noetherian ring and S a semilocal Noetherian R-algebra with $mS \subseteq \mathrm{Jac}(S)$. If S is J-smooth over R for every maximal ideal $n \subseteq S$ the local ring S_n is nS_n-smooth over R.

<u>Proof</u>: The assertion follows by (8.6) since S_n is smooth over S.

(16.10) <u>Theorem</u>: Let $\varphi : (R,m) \longrightarrow (S,n)$ be a local morphism of local Noetherian rings. Suppose that R contains a field. If S is n-smooth over R and R is a G-ring, then φ is regular.

<u>Proof</u>: (Brezuleanu / Radu) By (16.8) $\hat{S}$ is $\hat{n}$-smooth over R via the induced map $\hat{\varphi} : \hat{R} \longrightarrow \hat{S}$. Consider the commutative diagram of local morphisms

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \tau\downarrow & & \downarrow\sigma \\ \hat{R} & \xrightarrow{\hat{\varphi}} & \hat{S} \end{array}$$

where τ and σ are the natural maps. Since R is a G-ring, τ is a regular morphism. If $\hat{\varphi}$ is regular, $\hat{\varphi}\tau$ is regular and by (15.4) φ is regular since σ is faithfully flat. Thus we may assume that R and S are complete local rings.

We have to show that S/PS is geometrically regular over $k(P) = Q(R/P)$ for all $P \in \operatorname{Spec}(R)$. Since $\varphi \otimes R/P : R_P \longrightarrow S/PS$ is n/PS-smooth over R we may assume that R is a domain and have to show that $S \otimes_R Q(R)$ is geometrically regular over $Q(R)$.

Let $Q(R) \subseteq L$ be a finite field extension and $R \subseteq R' \subseteq L$ an intermediate ring so that:

(i) R' is finite over R

(ii) $Q(R') = L$.

R' is a complete semilocal domain. Since R is a complete local ring, R is Henselian and R' is a complete local domain. The ring $S' = S \otimes_R R'$ is a finite S-module and hence a complete semilocal ring. Let $J \subseteq S'$ denote the Jacobson radical of S'. By (8.7) S' is J-smooth over R'. We have to show that $S' \otimes_{R'} L \cong (S \otimes_R R') \otimes_{R'} L \cong S \otimes_R L$ is a regular ring or equivalently, that for all prime ideals $Q \subseteq S'$ with $Q \cap R' = (0)$ the ring S'_Q is regular. Every prime ideal $Q \subseteq S'$ is contained in a

maximal ideal $N \subseteq S'$ and by (16.9) the localization S'_N is NS'_N-smooth over R' under the induced morphism $\varphi'_N : R' \longrightarrow S'_N$ where $\varphi' = \varphi \otimes_R R'$. Thus it suffices to show that φ'_N is a regular morphism, more specifically, it suffices to show:
(*) Let $\varphi : (R, m) \longrightarrow (S, n)$ be a local morphism of complete local rings. Suppose that R is a domain of equal characteristic and that S is n-smooth over R. Then $S \otimes_R Q(R)$ is a regular ring.

Since R contains a field, R contains a coefficient field k. Hence R is a homomorphic image of a complete regular local ring, say $R = T/P$ where $T = k[[x_1, -, x_n]]$. By (16.1) and (16.2) there is a complete regular local ring (B, n_B) and a commutative diagram of local morphisms:

$$\begin{array}{ccc} T & \xrightarrow{\psi} & B \\ \alpha \downarrow & & \downarrow \beta \\ R & \xrightarrow{\varphi} & S \end{array}$$

where α and β are surjective, B is n_B-smooth over T via ψ and $S \cong B \otimes_T R \cong B/PB$ via $\beta \otimes \varphi$.

Since $R_{(0)} = (T/P)_P$ is regular, by (12.18) and (12.26) there is a subfield $\ell \subseteq k$ with $[k : \ell] < \infty$ and ℓ-derivations $D_1, -, D_t : T \longrightarrow T$ so that

$$\det (D_i f_j)_{1 \le i,j \le t} \notin P$$

where $f_1, -, f_t \in P$ and ht $P = t$.

By (16.7) the ℓ-derivations $D_1, —, D_t : T \longrightarrow T$ extend to ℓ-derivations $\tilde{D}_1, -, \tilde{D}_t : B \longrightarrow B$. Hence for all prime ideals $Q \subseteq B$ with $Q \cap T = P$

$$\det (\tilde{D}_i f_j)_{1 \le i,j \le t} \notin Q.$$

By the flatness of ψ ht $PB = t$. Thus $S_Q \cong (B/PB)_Q$ is regular by (12.6).

(16.11) <u>Remark</u>: Theorem (16.10) holds true without the assumption that R

contains a field. It was first proved by M. André using the André–Quillen homology. The proof of (16.10) is due to A. Brezuleanu and N. Radu. Their proof can be extended to the general case by using Jacobian criteria for complete regular local rings of unequal characteristic.

(16.12) Definition: Let $\varphi: R \longrightarrow S$ be a morphism of Noetherian rings. Then φ is called a reduced morphism if

(a) φ is flat

(b) For all prime ideals $P \subseteq R$ the ring $S \otimes_R k(P)$ is geometrically reduced over $k(P)$.

(16.13) Remark: A local Noetherian ring (R, m) is Nagata if and only if the natural map $\varphi: R \longrightarrow \widehat{R}$ is reduced.

Similar to (15.4) one can show:

(16.14) Theorem: Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be a morphism of Noetherian rings.

(a) If φ and ψ are reduced, $\psi\varphi$ is reduced.

(b) If $\psi\varphi$ is reduced and ψ is faithfully flat, then φ is reduced.

Proof: Homework

(16.15) Theorem: Let $\varphi: (R, m, k) \longrightarrow (S, n)$ be a faithfully flat morphism of local Noetherian rings. Suppose that

(a) R contains a field and is a Nagata ring.

(b) The fiber over the maximal ideal $S \otimes_R k$ is geometrically reduced over k.

Then φ is reduced.

Proof: Consider the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \alpha\downarrow & & \downarrow\beta \\ \hat{R} & \xrightarrow{\varphi^*} & S^* \end{array}$$

where $\hat{R}$ is the m-adic completion of R, $S^* = (S, mS)^\wedge$ is the mS-adic completion of S, and α, β are the natural maps. Obviously, $S^* \otimes_{\hat{R}} k \cong S \otimes_R k$ is geometrically reduced over k and by (16.14) it suffices to show that φ^* is reduced since α is reduced. Thus we may assume that R is complete and have to show that $S \otimes_R k(P)$ is geometrically reduced over $k(P)$ for every $P \in \mathrm{Spec}(R)$. Suppose that the assertion is false. Then there is a maximal prime ideal $P \subseteq R$, $P \neq m$, so that $S \otimes_R k(P)$ is not geometrically reduced over $k(P)$. We may replace R by R/P and S by S/PS and can assume that

(i) R is a complete local domain with quotient field K

(ii) For all $P \in \mathrm{Spec}(R)$ with $P \neq (0)$ the ring $S \otimes_R k(P)$ is geometrically reduced over $k(P)$.

(iii) $S \otimes_R K$ is not geometrically reduced over K.

Let $K \subseteq L$ be a finite field extension and T the integral closure of R in T. Suppose that $S \otimes_R L$ and hence $S \otimes_R T$ is not reduced. T is finite over R, hence T is a complete local domain. Since $S \otimes_R T$ is not reduced, there is a maximal ideal $Q \subseteq S \otimes_R T$ with $(S \otimes_R T)_Q$ not reduced. Moreover, the induced morphism $\varphi \otimes T_Q : T \longrightarrow (S \otimes_R T)_Q$ is local and we may replace R by T and S by $(S \otimes_R T)_Q$ and assume:

(i') R is a complete normal local domain.

(ii') For all $P \in \mathrm{Spec}(R)$ with $P \neq (0)$ the ring $S \otimes_R k(P)$ is geometrically reduced over $k(P)$

(iii') S is not reduced.

Let $P \subseteq S$ be a prime ideal with depth $S_P = 0$, $a \in m \subseteq R$ a nonzero element of m, and $Q \subseteq S$ a minimal prime ideal over $P+aS$. Then depth $S_Q = 1$. Let $Q_0 = Q \cap R$ and consider the induced faithfully flat morphism $\varphi_Q : R_{Q_0} \longrightarrow S_Q$. By the depth formula:

$$1 = \operatorname{depth} S_Q = \operatorname{depth} R_{Q_0} + \operatorname{depth} (S/Q_0S)_Q .$$

Since R is a domain with $Q_0 \neq 0$, depth $R_{Q_0} \geq 1$ yielding that

$$\operatorname{depth} (S/Q_0S)_Q = 0.$$

On the other hand, by (ii') the induced morphism $\overline{\varphi} : R/Q_0 \longrightarrow S/Q_0S$ is reduced and $S \otimes_R k(Q_0)$ is geometrically reduced over $k(Q_0)$. In particular, the localization $(S/Q_0S)_Q$ is geometrically reduced over $k(Q_0)$. Hence $(S/Q_0S)_Q$ is a field and geometrically regular over $k(Q_0)$. This implies that S_Q is QS_Q-smooth over R. By assumption R is complete and R_{Q_0} is a G-ring, hence $\varphi_Q : R_{Q_0} \longrightarrow S_Q$ is regular by André's theorem. Hence the induced morphism $\varphi_P : R_{P_0} \longrightarrow S_P$ is regular where $P_0 = P \cap R$. Since depth $S_P = 0$, depth $R_{P_0} = 0$ and R_{P_0} is a field. This implies that S_P is a field. By Serre's criterion for reducedness, S is a reduced ring, a contradiction.

§2: LIFTING THEOREMS

Let R be a semilocal ring and $I \subseteq \operatorname{Jrad}(R)$ an ideal in the Jacobson radical of R. In this section we want to show: If R is I-adically complete and R/I a G-ring (or a Nagata ring), then R is a G-ring (or a Nagata ring. In the following we assume André's theorem (16.10) and Theorem (16.15) in all cases, i.e. we do not assume that R contains a field.

(16.16) __Lemma__: Let R be a semilocal Noetherian ring, $I \subseteq \operatorname{Jrad}(R)$ an ideal, and $\widehat{R}$ the completion of R with respect to the Jacobson radical of R. Suppose that R is I-adically complete and that $\widehat{K} \subseteq \widehat{R}$ is an ideal so that for all $n \in \mathbb{N}$ with $n \geq n_0$ the ideal $\widehat{K} + I^n\widehat{R}$ is extended from R, that is, $(R \cap (\widehat{K} + I^n R))\widehat{R} = \widehat{K} + I^n\widehat{R}$ for all $n \geq n_0$. Then $\widehat{K}$ is extended from R.

__Proof__: Let $K_0 = \widehat{K} \cap R$. By replacing R by R/K_0 we may assume that $\widehat{K} \cap R = (0)$ and have to show that $\widehat{K} = (0)$. Suppose that $\widehat{K} \neq (0)$ and $\widehat{K} \cap R = (0)$ with $\widehat{K} + I^n\widehat{R}$ extended from R for all $n \in \mathbb{N}$ with $n \geq n_0$. Set $K_n = (\widehat{K} + I^n\widehat{R}) \cap R$ for all $n \geq n_0$. Then $K_n\widehat{R} = \widehat{K} + I^n\widehat{R}$ and therefore by faithful flatness for all $n \geq n_0$:

$$K_n = K_{n+1} + I^n.$$

Since $\widehat{K} \neq (0)$, there is an $r \in \mathbb{N}$ with $\widehat{K} \subseteq (\operatorname{Jrad}(R))^r$. Let $t \geq \max(n_0, r)$ and $f_t \in K_t - (\operatorname{Jrad}(R))^t$. Since $K_n = K_{n+1} + I^n$ for all $n \geq t$, there is a sequence $(f_n)_{n \geq t}$ in R so that:

(i) $f_n \in K_n$ for all $n \geq t$

(ii) $f_n - f_{n+1} \in I^n$ for all $n \geq t$.

By assumption R is I-adically complete, thus $(f_n)_{n \geq t}$ converges towards an element $f \in R$ with $f - f_n \in I^n$ for all $n \geq t$. Since

$f_t \notin (\mathrm{Jrad}(R))^t$ and $I^t \subseteq (\mathrm{Jrad}(R))^t$ it follows that $f \notin I^t$ and hence $f \neq 0$. Then

$$f \in \bigcap_{n\geq t} K_n = \bigcap_{n\geq t}((\hat{K}+I^n\hat{R})\cap R)$$
$$= \bigcap_{n\geq t}(\hat{K}+I^n\hat{R})\cap R = \hat{K}\cap R \neq (0),$$

a contradiction.

(16.16) Lemma: Let R be a semilocal Noetherian ring and $I \subseteq \mathrm{Jrad}(R)$ an ideal. Suppose that R/I is a G-ring (or a Nagata ring, resp.) Let $P \subseteq R$ be a prime ideal with $I \subseteq P$ and $\hat{R}$ the completion of R with respect to the Jacobson radical. If $(R_P, IR_P)^\wedge$ is a G-ring (Nagata ring), then the natural morphism

$$\varphi_P : (R_P, IR_P)^\wedge \longrightarrow (\hat{R}_P, I\hat{R}_P)^\wedge$$

is regular (reduced).

Proof: The map φ_P is faithfully flat and $(\hat{R}_P, I\hat{R}_P)^\wedge \otimes k(P) \cong \hat{R}_P \otimes_{R_P} k(P) \cong \hat{R}\otimes_R k(P)$ is geometrically regular (geometrically reduced) over $k(P)$ since $I \subseteq P$ and R/I is a G-ring (Nagata ring). The statement follows with (16.10) and (16.15).

(16.17) Definition: (a) Let R be a Noetherian ring. We say that R satisfies $\mathbb{P}$ if R is regular or reduced, respectively.

(b) Set $\mathrm{Non}\mathbb{P}(R) = \{P \in \mathrm{Spec}(R) \mid R_P$ has not property $\mathbb{P}\}$, that is, $\mathrm{Non}\mathbb{P}(R)$ is the singular locus of $\mathbb{P}$ if $\mathbb{P} \mathrel{\hat{=}}$ regular and the non-reduced locus of R if $\mathbb{P} \mathrel{\hat{=}}$ reduced.

(c) Set $$D_{\mathbb{P},R} = \bigcap_{P\in \mathrm{Non}\mathbb{P}(R)} P.$$

(16.18) Remark: If $\mathrm{Non}\mathbb{P}(R)$ is closed in $\mathrm{Spec}(R)$, for example, if R is a semilocal G-ring if $\mathbb{P} \mathrel{\hat{=}}$ regular or if R is a Nagata ring if $\mathbb{P} \mathrel{\hat{=}}$ reduced, then $V(D_{\mathbb{P},R}) = \mathrm{Non}\,\mathbb{P}(R)$.

(16.19) <u>Definition</u>: Let R and S be Noetherian rings and $\varphi: R \longrightarrow S$ a morphism. φ is called a <u>$\mathbb{P}$-morphism</u> if φ is regular and $\mathbb{P} =$ regular or if φ is reduced and $\mathbb{P} \cong$ reduced.

(16.20) <u>Lemma</u>: Let R and S be Noetherian rings and $\varphi: R \longrightarrow S$ a $\mathbb{P}$-morphism. Suppose that $\mathrm{Non}\,\mathbb{P}(R)$ is closed in $\mathrm{Spec}(R)$. Then $\mathrm{Non}\,\mathbb{P}(S)$ is closed in $\mathrm{Spec}(S)$ and

(a) $\varphi(D_{\mathbb{P},R})S = D_{\mathbb{P},S}$

(b) $\varphi^{-1}(D_{\mathbb{P},S}) \supseteq D_{\mathbb{P},R}$

<u>Proof</u>: Let $Q \in \mathrm{Spec}(S)$, $P = Q \cap R \in \mathrm{Spec}(R)$, and $\varphi_Q : R_P \longrightarrow S_Q$ the induced map. Since φ is a $\mathbb{P}$-morphism, φ_Q is a $\mathbb{P}$-morphism and the property $\mathbb{P}$ ascents and descents under the faithfully flat morphism φ_Q. This implies that $\mathrm{Non}\,\mathbb{P}(S)$ is closed in $\mathrm{Spec}(S)$ and that (a) and (b) hold.

(16.21) <u>Theorem</u>: Let R be a semilocal Noetherian ring and $I \subseteq \mathrm{Jrad}(R)$ an ideal. Suppose

(a) R is I-adically complete.

(b) R/I is a G-ring (Nagata ring).

Then R is a G-ring (Nagata ring).

<u>Proof</u>: By induction on the dimension of R. Let R be a semilocal Noetherian ring of dimension n which satisfies (a) and (b). Suppose that the assertion is true for semilocal Noetherian rings of dimension strictly less than n. After standard reductions we have to show:

(*) Let R be a semilocal Noetherian domain, $I \subseteq \mathrm{Jrad}(R)$ an ideal so that

R is I-adically complete and R/I is a G-ring (Nagata ring). Let $\widehat{R}$ be the completion of R with respect to the Jacobson radical and $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ with $\widehat{R}_{\widehat{P}}$ has not property $\mathbb{P}$. Then $\widehat{P} \cap R \neq 0$.

As before set $\widehat{D} = D_{\mathbb{P},\widehat{A}} = \bigcap_{\widehat{P} \in \operatorname{Non}\mathbb{P}(\widehat{R})} \widehat{P}$. Then it suffices to show

(**) $\quad \widehat{D} \cap R \neq 0$.

<u>Claim</u>: For all $n \in \mathbb{N}$:

(Δ) $\quad (\widehat{R} \cap (\widehat{D} + I^n \widehat{R}))\widehat{R} = \widehat{D} + I^n \widehat{R}$

that is, for all $n \in \mathbb{N}$ the ideal $\widehat{D} + I^n\widehat{R}$ is extended from R.

For all $n \in \mathbb{N}$ set $\widehat{D}_n = \widehat{D} + I^n \widehat{R}$ and let $\widehat{Q} \subseteq \widehat{R}$ be a $\widehat{P}$-primary ideal with $\widehat{D}_n \subseteq \widehat{Q}$. In order to show ($\Delta$) it suffices to show that $\widehat{D} \subseteq (\widehat{Q} \cap R)\widehat{R}$. Set $Q = \widehat{Q} \cap R$ and $P = \widehat{P} \cap R$. Obviously, Q is a P-primary ideal.

<u>Case 1</u>: $\widehat{P}$ and P are maximal ideals of $\widehat{R}$ and R, respectively. Then $\widehat{Q}$ is extended from R, i.e. $\widehat{Q} = Q\widehat{R}$ and we are done.

<u>Case 2</u>: $\widehat{P}$ and P are not maximal. Consider the following diagram of natural morphisms:

$$\begin{array}{ccccc}
R & \xrightarrow{\varepsilon_P} & R_P & \xrightarrow{\gamma_P} & (R_P, IR_P)^\wedge = R^* \\
\tau \downarrow & & \downarrow \sigma_P & & \downarrow \varphi_P \\
\widehat{R} & \xrightarrow{\mu_P} & \widehat{R}_P & \xrightarrow{\psi_P} & (\widehat{R}_P, I\widehat{R}_P)^\wedge = S^*
\end{array}$$

where $\widehat{R}_P$ denotes the localization of $\widehat{R}$ with respect to the multiplicative set $R - P$. This yields:

(i) By induction hypothesis R^* is a G-ring (Nagata ring).

(ii) By (16.16) φ_P is regular (reduced).

(iii) $\widehat{R}_P$ is a G-ring, hence by André's theorem ψ_P is a regular and reduced morphism.

This implies:

(a) Since $D_{\mathbb{P},\widehat{R}_P} = \widehat{D}\widehat{R}_P$ and since ψ_P is a $\mathbb{P}$-morphism, by (16.20)
$$\widehat{D} S^* = D_{\mathbb{P}, S^*}.$$

In particular, $\widehat{D} \subseteq \mu_P^{-1}(\psi_P^{-1}(D_{\mathbb{P},S^*}))$.

(b) Since φ_P is a $\mathbb{P}$-morphism, by (16.20)

$$\varphi_P^{-1}(D_{\mathbb{P},S^*}) \supseteq D_{\mathbb{P},R^*} \quad \text{and} \quad D_{\mathbb{P},R^*}S^* = D_{\mathbb{P},S^*}.$$

Hence $D_{\mathbb{P},R^*} \subseteq \varphi_P^{-1}(D_{\mathbb{P},S^*}) = \varphi_P^{-1}(\widehat{D}S^*) \subseteq \varphi_P^{-1}(\widehat{Q}S^*)$.

(c) By the commutativity of the diagram:

$$\varepsilon_P^{-1}\gamma_P^{-1}\varphi_P^{-1}(\widehat{Q}S^*) = \tau^{-1}\mu_P^{-1}\psi_P^{-1}(\widehat{Q}S^*) \quad \text{and}$$

$\psi_P^{-1}(\widehat{Q}S^*) = \widehat{Q}\widehat{R}_P$ since $I^n\widehat{R}_P \subseteq \widehat{Q}\widehat{R}_P$

$\mu_P^{-1}(\widehat{Q}\widehat{R}_P) = \widehat{Q}$ since $\widehat{Q}$ is $\widehat{P}$-primary and $R-P \subseteq \widehat{R}-\widehat{P}$.

Therefore

$$\varepsilon_P^{-1}\gamma_P^{-1}\varphi_P^{-1}(\widehat{Q}S^*) = Q$$

and since Q and $\gamma_P^{-1}\varphi_P^{-1}(\widehat{Q}S^*)$ are P-primary

$$\gamma_P^{-1}\varphi_P^{-1}(\widehat{Q}S^*) = QR_P.$$

(d) Similarly, since Q is P-primary with $I^n \subseteq Q$:

$$\psi_P^{-1}(QS^*) = Q\widehat{R}_P \quad \text{and} \quad \mu_P^{-1}(Q\widehat{R}_P) = Q\widehat{R}.$$

(e) By (c) $\gamma_P^{-1}\varphi_P^{-1}(\widehat{Q}S^*) = QR_P$ and therefore, since $I^n \subseteq Q$:

$$QR^* = \varphi_P^{-1}(\widehat{Q}S^*).$$

This implies

$$QS^* = \varphi_P^{-1}(\widehat{Q}S^*)S^* \supseteq \widehat{D}S^* \qquad \text{by (a) and (b)}$$

and

$$\mu_P^{-1}\psi_P^{-1}(QS^*) = \mu_P^{-1}(Q\widehat{R}_P) = Q\widehat{R} \supseteq \mu_P^{-1}\psi_P^{-1}(\widehat{D}S^*) \supseteq \widehat{D}.$$

Thus $Q\widehat{R} \supseteq \widehat{D}$ and the claim is proven. By (16.16) $\widehat{D}$ is extended from R and the theorem follows.

For Nagata rings the general statement holds:

(16.22) <u>Theorem</u>: (Marot) Let R be a Noetherian ring and $I \subseteq \operatorname{Jrad}(R)$ an ideal. Suppose

(a) R is I-adically complete

(b) R/I is a Nagata ring.
Then R is a Nagata ring.

After Marot proved his theorem, Nishimura found an example of a G-ring R and an ideal $I \subseteq R$ so that the formal fibers of $(R,I)^\wedge$ are not geometrically reduced. Nishimura's example shows that the closedness of the non-reduced locus plays a prominant role in the lifting property for Nagata rings. In order to prove a similar lifting theorem for excellent rings one has to make use of Hironaka's theorem on the resolution of singularities. Then one can show:

(16.23) <u>Theorem:</u> Let R be a Noetherian universally catenary ring. Suppose that R contains $\mathbb{Q}$ and that R is a G-ring. Let $I \subseteq \mathrm{Jrad}(R)$ be an ideal with R/I a Reg-2 ring. Then R is Reg-2.

(16.24) <u>Theorem:</u> Let R be a Noetherian universally catenary ring of finite Krull dimension. Suppose that R contains $\mathbb{Q}$ and let $I \subseteq \mathrm{Jrad}(R)$ be an ideal. If R is I-adically complete and R/I is excellent, then R is excellent.