CHAPTER XV: EXCELLENT RINGS

§1: REGULAR MORPHISMS

(15.1) Definition: (a) Let $\varphi: R \longrightarrow S$ be a morphism of Noetherian rings. For all $P \in \operatorname{Spec}(R)$ the fiber (ring) at P is defined to be the ring $S \otimes_R k(P)$.
(b) Let (R, m) be a local Noetherian ring and $\hat{R}$ its m-adic completion. The formal fibers of R are the fibers of the natural morphism $\varphi: R \longrightarrow \hat{R}$.

Recall that a local ring (R, m) has geometrically regular formal fibers if for all $P \in \operatorname{Spec}(R)$ the ring $\hat{R} \otimes_R k(P)$ is geometrically regular over $k(P)$, that is, if for all finite field extensions $k(P) \subseteq L$ the ring $\hat{R} \otimes_R L$ is regular.

(15.2) Definition: Let $\varphi: R \longrightarrow S$ be a morphism of Noetherian rings. φ is called a regular morphism if
(a) φ is flat
(b) For all prime ideals $P \subseteq R$ the fiber ring $S \otimes_R k(P)$ is geometrically regular over $k(P)$.

(15.3) Remark: A local Noetherian ring (R, m) has geometrically regular formal fibers if and only if the natural map $\varphi: R \longrightarrow \hat{R}$ is regular.

(15.4) Theorem: Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be morphisms of Noetherian rings. Then
(a) If φ and ψ are regular, then so is $\psi\varphi$.
(b) If $\psi\varphi$ is regular and ψ is faithfully flat, then φ is regular.

Proof: Obviously, $\psi\varphi$ is flat. Let $P \subseteq R$ be a prime and $k(P) \subseteq L$ a finite field extension. Set $S_L = S \otimes_R L$ and $T_L = T \otimes_R L$.

<u>Claim</u>: The induced morphism $\psi_L : S_L \longrightarrow T_L$ is regular.

Pf of Claim: Let $Q \subseteq S_L$ be a prime ideal and $k(Q) \subseteq F$ a finite field extension. Since $T_L = T \otimes_R L = T \otimes_S (S \otimes_R L) = T \otimes_S S_L$ it follows that

$$T_L \otimes_{S_L} F = (T \otimes_S S_L) \otimes_{S_L} F = T \otimes_S F.$$

Moreover,

$$S \otimes_R k(P) = (S/PS)_P \longrightarrow S_L = S \otimes_R L$$

is a finite extension and with $Q' = Q \cap S$ the field extension $k(Q') \subseteq k(Q)$ is finite. Hence $k(Q') \subseteq F$ is finite and $T_L \otimes_{S_L} F = T \otimes_S F$ is a regular ring, since ψ is a regular morphism. Thus $\psi_L : S_L \longrightarrow T_L$ is a regular morphism.

Note that S_L is a regular ring, since φ is a regular morphism. Let $W \subseteq T_L$ be a prime ideal and set $Q = S_L \cap W$. The induced morphism $(\psi_L)_W : (S_L)_Q \longrightarrow (T_L)_W$ is faithfully flat and the fiber $(T_L)_W \otimes_{(S_L)_Q} k(Q)$ is a localization of $T_L \otimes_{S_L} k(Q)$ and thus a regular local ring. By 911, Theorem (8.63) the ring $(T_L)_W$ is regular. This shows that T_L is a regular ring.

(b) Since $\psi\varphi$ is flat with ψ faithfully flat, φ is flat. Let $P \subseteq R$ be a prime ideal and $k(P) \subseteq L$ a finite field extension. Then $T_L = T \otimes_R L$ is a regular ring. Moreover, T_L is faithfully flat over $S_L = S \otimes_R L$, then by 911, Theorem (8.63) the ring S_L is regular.

(15.5) <u>Proposition</u>: Let $\varphi : R \longrightarrow S$ be a faithfully flat regular morphism. Then R is a regular ring if and only if S is a regular ring.

<u>Proof</u>: follows immediately from 911, Theorem (8.63) since all fibers of φ are regular rings.

(15.6) <u>Proposition</u>: Let k be a field and R a Noetherian k-algebra. If R is geometrically regular over k, for every finitely generated field extension

$k \subseteq L$ the ring $R \otimes_k L$ is regular.

Proof: Let $k \subseteq L$ be a finitely generated field extension. By (1.34) there is a finite purely inseparable field extension $k \subseteq K$ so that the extension $K \subseteq L(K)$ is separable. By assumption the ring $R \otimes_k K$ is regular and $R \otimes_k L(K) = (R \otimes_k K) \otimes_K L(K)$ is smooth over $R \otimes_k K$, since $L(K)$ is smooth over K. By (11.10) $R \otimes_k L(K)$ is a regular ring. Moreover, the extension $R \otimes_k L \longrightarrow R \otimes_k L(K)$ is faithfully flat and the ring $R \otimes_k L$ is regular.

(15.7) Lemma: Let $\{R_i, \varphi_j^i\}_{i,j \in I}$ be a direct system of regular Noetherian rings. Suppose that for all $i,j \in I$ with $i \leq j$ the morphism $\varphi_j^i : R_i \longrightarrow R_j$ is flat and that $R = \varinjlim_{i \in I} R_i$ is Noetherian. Then for all $i \in I$ the natural morphism $\varphi_i : R_i \longrightarrow R$ is flat and R is a regular ring.

Proof: Since the tensor product commutes with direct limits, for all $i \in I$ the natural morphism $\varphi_i : R_i \longrightarrow R$ is flat. Let $P \subseteq R$ be a prime ideal. Since R is Noetherian, P is finitely generated, say $P = (g_1, -, g_s)$. Then there is an $i \in I$ with $g_1, -, g_s \in R_i$. Set $P_i = \varphi_i^{-1}(P)$. The induced morphism $(\varphi_i)_P : (R_i)_{P_i} \longrightarrow R_P$ is faithfully flat with maximal fiber $k(P)$. Thus by 911, Theorem (8.63) R_P is a regular local ring.

(15.8) Proposition: Let $v: R \longrightarrow R'$ and $u: R \longrightarrow S$ be morphisms of Noetherian rings. Set $S' = S \otimes_R R'$ and $u' = u \otimes R' : R' \longrightarrow S'$. Then:

(a) If u is a regular morphism and S an R-algebra essentially of finite type, then u' is a regular morphism.

(c) Suppose that u' is a regular morphism, u is flat, R' is finite over R (via v), and $\ker v$ is nilpotent. Then u is a regular morphism.

Proof: (a) Let $P \in \mathrm{Spec}(R')$, $Q = v^{-1}(P) \in \mathrm{Spec}(R)$, $K = k(Q)$, $K' = k(P)$, and $K' \subseteq L$ a finite field extension. We have to show that $D' = S' \otimes_{R'} L$ is a regular local ring. By assumption the ring $D = S \otimes_R K$ is geometrically regular over K. Then $D' = S' \otimes_{R'} L = (S \otimes_R R') \otimes_{R'} L = S \otimes_R L = (S \otimes_R K) \otimes_K L = D \otimes_K L$.
Write $L = \varinjlim E_i$ where $K \subseteq E_i$ are finitely generated field extensions. By (15.6) for all $i \in I$ the ring $D \otimes_K E_i$ is regular and by (15.7) $D \otimes_K L = \varinjlim (D \otimes_K E_i)$ is a regular ring.
(b) Let $P \subseteq R$ be a prime ideal. Since R' is finite over R, there is a prime ideal $P' \subseteq R'$ with $v^{-1}(P') = P$. Since $\ker(v)$ is nilpotent, $\ker(v) \subseteq P$ and we may replace R, R', S, S' by R/P, R'/P', S/PS and $S'/P'S = (S/PS) \otimes_{R/P} (R'/P')$. Note that by (a) the induced morphism $\overline{u}' = u \otimes 1 : R'/P' \longrightarrow S'/P'S' = S' \otimes_{R'} (R'/P')$ is regular. Thus we may assume that R and R' are domains with fields of quotients $K = Q(R)$ and $L = Q(R')$. We have to show that $S \otimes_R K$ is geometrically regular over K. Let $K \subseteq E$ be a finite field extension, then $L = Q(R') \subseteq E' = L(E)$ is a finite extension and the ring $S' \otimes_{R'} E'$ is regular since $u' : R' \longrightarrow S'$ is a regular morphism. Since R' is finite over R, $R' \otimes_{R'} L = R' \otimes_R K$ and $S' \otimes_{R'} L = S \otimes_R K$ yielding $S' \otimes_{R'} E' = S \otimes_R E'$. The morphism $S \otimes_R E \longrightarrow S \otimes_R E'$ is faithfully flat with $S \otimes_R E'$ a regular ring. By 911, Theorem (8.63) the ring $S \otimes_R E$ is regular.

(15.9) Proposition: Let $u : (R, m) \longrightarrow (S, n)$ be an injective local morphism of local Noetherian domains. Suppose that S is essentially finite over R and that the formal fiber of R at (0) is geometrically regular (over $Q(R)$). Then the formal fiber of S at (0) is geometrically regular (over $Q(S)$).

Proof: Let $S = T_{\mathfrak{r}}$ where T is a finite R-algebra and $n = \mathfrak{r} T_{\mathfrak{r}}$. By assumption $u(m) \subseteq n$ and hence $\mathfrak{r} \subseteq T$ is a maximal ideal of T. Since the mT-adic completion of T is given by $\widehat{T} = T \otimes_R \widehat{R}$ it follows that $\widehat{S} = S \otimes_R \widehat{R}$.

Let $K = Q(R)$ be the quotient field of R and $L = Q(S)$ the quotient field of S with $K \subseteq L$ a finite extension. By assumption $\hat{R} \otimes_R K$ is geometrically regular over K, thus $\hat{R} \otimes_R L$ is geometrically regular over L. The assertion follows with $\hat{S} \otimes_S L = (\hat{R} \otimes_R S) \otimes_S L = \hat{R} \otimes_R L$.

(15.10) Corollary: Let (R, m) be a local Noetherian ring and (S, n) a local R-algebra essentially finite over R with $mS \subseteq n$. If the formal fibers of R are geometrically regular so are the formal fibers of S.

Proof: Let $Q \in \mathrm{Spec}(S)$ and $P = Q \cap R$. The natural morphism $u: R/P \longrightarrow S/Q$ is injective and local. By (15.9) the formal fiber of S/Q at (0) is geometrically regular over $k(Q) = (S/Q)_Q$.

§2: G-RINGS (QUASI-EXCELLENT RINGS)

(15.11) Definition: A Noetherian ring R is called a G-ring (or a quasi-excellent ring) if for all $P \in \mathrm{Spec}(R)$ the natural morphism $R_P \longrightarrow (R_P)^\wedge$ is regular.

We want to show first that R is a G-ring if and only if for all $m \in m\text{-}\mathrm{Spec}(R)$ the natural map $R_m \longrightarrow \widehat{R_m}$ is regular. In particular, a local Noetherian ring (R,m) is a G-ring if the natural map $R \longrightarrow \widehat{R}$ is regular.

(15.12) Theorem: Let $u: R \longrightarrow S$ be a regular morphism of Noetherian rings and assume that u is faithfully flat. Then:
(a) R is a regular (normal, reduced, CM, Gorenstein) ring if and only if S is regular (normal, reduced, CM, Gorenstein).
(b) If S is a G-ring, then R is a G-ring.

Proof: (a) follows from 911, since u is faithfully flat with regular fibers.
(b) Let $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(S)$ with $u^{-1}(Q) = P$. (Note that there is such a prime ideal Q since u is faithfully flat). Consider the commutative diagram:

$$\begin{array}{ccc} (R_P)^\wedge & \xrightarrow{\widehat{u_Q}} & (S_Q)^\wedge \\ \alpha \uparrow & & \uparrow \beta \\ R_P & \xrightarrow{u_Q} & S_Q \end{array}$$

where α and β are the natural maps. By assumption u_Q and β are regular morphisms and $\widehat{u_Q}$ is faithfully flat. Thus by (15.4) $\widehat{u_Q}\,\alpha = \beta u_Q$ and α are regular morphisms.

(15.13) Remark: If $u: R \longrightarrow S$ is a faithfully flat regular morphism with R a G-ring, then S may not be a G-ring. For example, there are local Noetherian rings (R,m) and (S,n) and local morphisms $R \xrightarrow{u} S \xrightarrow{v} \widehat{R} = \widehat{S}$ where vu and v are the natural maps and vu regular. Moreover, R is a G-ring and S is not a G-ring. By (15.4) u is a regular morphism.

(15.14) Proposition: Let k be a field of characteristic $p>0$ and let $\mathcal{F} = \{k_\alpha\}_{\alpha\in\Lambda}$ be a directed family of subfields with $k^p \subseteq k_\alpha \subseteq k$ for all $\alpha \in \Lambda$ and $\bigcap_{\alpha\in\Lambda} k_\alpha = k^p$. Let (R,m) be a regular local ring and a k-algebra. The following conditions are equivalent:

(a) R is geometrically regular over k.

(b) R is m-smooth over k.

(c) R is m-smooth over k relative to k_α for all $\alpha \in \Lambda$.

Proof: (a) $\Leftrightarrow$ (b): By (8.33)

(b) $\Rightarrow$ (c): Obvious

(c) $\Rightarrow$ (a): Let $F \subseteq k$ be the prime field and $K = R/m$ the residue class field of R. Since R is regular and F is perfect, R is m-smooth over F. By (8.30) we have to show that the natural map:

$$\varphi: \Omega_k \otimes_k K \longrightarrow \Omega_R \otimes_R K$$

is injective. By (8.25) for all $\alpha\in\Lambda$ the natural map

$$\varphi_\alpha: \Omega_{k/k_\alpha} \otimes_k K \longrightarrow \Omega_{R/k_\alpha} \otimes_R K$$

is injective. Thus for all $\alpha \in \Lambda$ there is a commutative diagram:

$$\begin{array}{ccc} \Omega_k \otimes_k K & \xrightarrow{\varphi} & \Omega_R \otimes_R K \\ \tau_\alpha \downarrow & & \downarrow \sigma_\alpha \\ \Omega_{k/k_\alpha} \otimes_k K & \xrightarrow{\varphi_\alpha} & \Omega_{R/k_\alpha} \otimes_R K \end{array}$$

where τ_α and σ_α are the natural maps. Passing to the inverse limit yields a commutative diagram:

$$\begin{array}{ccc} \Omega_k \otimes_k K & \xrightarrow{\varphi} & \Omega_R \otimes_R K \\ \tau_0 \downarrow & & \downarrow \sigma_0 \\ \varprojlim (\Omega_{k/k_\alpha} \otimes_k K) & \xrightarrow{\varphi_0} & \varprojlim (\Omega_{R/k_\alpha} \otimes_R K) \end{array}$$

Since the inverse limit is left exact, φ_0 is injective. Moreover, by (12.22) the natural map $\tilde{\tau}_0 : \Omega_k \longrightarrow \varprojlim \Omega_{k/k_\alpha}$ is injective. Although the inverse limit does not commute with tensor products in general, the same argument as in the proof of (12.22) yields that τ_0 is injective. Hence φ is injective and R is m-smooth over k.

(15.15) <u>Proposition</u>: Let R be a complete regular local ring, $P \subseteq R$ a prime ideal and $S = R_P$. Then the generic formal fiber of S is geometrically regular, that is, the ring $\hat{S} \otimes_S Q(S)$ is geometrically regular over $Q(S)$.

<u>Proof</u>: Set $K = Q(S)$ and let $K \subseteq L$ be a finite field extension. We have to show that the ring $\hat{S} \otimes_S L$ is regular or equivalently that for every prime ideal $Q' \subseteq \hat{S} \otimes_S L$ the local ring $(\hat{S} \otimes_S L)_{Q'}$ is regular. With $Q = Q' \cap \hat{S}$ the ring $(\hat{S} \otimes_S L)_{Q'}$ is a localization of $\hat{S}_Q \otimes_S L$ and it suffices to show that $\hat{S}_Q$ is geometrically regular over K. (Note that $Q \cap S = (0)$). By assumption $\hat{S}_Q$ is a regular local ring and the assertion follows if $\operatorname{char} K = 0$.

Suppose that $\operatorname{char} K = p > 0$, then $\operatorname{char} R = p > 0$ and R contains a coefficient field k. Moreover, R is isomorphic to a formal power series ring $R = k[[x_1, \ldots, x_n]]$. Let $\mathcal{F} = \{k_\gamma\}_{\gamma \in \Gamma}$ be a directed family of intermediate fields $k^p \subseteq k_\gamma \subseteq k$ with $[k : k_\gamma] < \infty$ for all $\gamma \in \Gamma$ and $\bigcap_{\gamma \in \Gamma} k_\gamma = k^p$. Set $y_i = x_i^p$ and consider for all $\gamma \in \Gamma$ the ring $R_\gamma = k_\gamma[[y_1, \ldots, y_n]]$. Note that R is finite over R_γ and that, since $R^p \subseteq R_\gamma$, P is the unique prime ideal of R lying over $P_\gamma = P \cap R_\gamma$. Then $S = R_P = R_{P_\gamma}$ is finite over $S_\gamma = (R_\gamma)_{P_\gamma}$ for all $\gamma \in \Gamma$. Let $K_\gamma = Q(S_\gamma) = Q(R_\gamma) = k_\gamma((y_1, \ldots, y_n))$ denote the quotient field of S_γ and R_γ. As shown in the proof of (12.25) it follows that $\bigcap_{\gamma \in \Gamma} K_\gamma = K^p$.

<u>Claim 1</u>: $\widehat{S}$ is 0-smooth over S' relative to S_γ for all $\gamma \in \Gamma$.

<u>Pf</u> of Claim 1: Let C be an S-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of morphisms of rings:

$$\begin{tikzcd}
\widehat{S_\gamma} \arrow[r] & \widehat{S} \arrow[r, "u"] \arrow[dr, "w"] & C/N \\
S_\gamma \arrow[u] \arrow[r] & S \arrow[u] \arrow[r, "\mu"] & C \arrow[u]
\end{tikzcd}$$

and suppose that $w : \widehat{S} \longrightarrow C$ is an S_γ-algebra lifting of u. Let $v' = w|\widehat{S_\gamma}$. Since S' is finite over S_γ, $\widehat{S} = S \otimes_{S_\gamma} \widehat{S_\gamma}$ and $v = \mu \otimes v' : S \otimes_{S_\gamma} \widehat{S}_\gamma \longrightarrow C$ is an S-algebra lifting of u.

<u>Claim 2</u>: $\widehat{S}_Q$ is 0-smooth over K relative to K_γ for all $\gamma \in \Gamma$.

<u>Pf</u> of Claim 2: Let C be a K-algebra and $N \subseteq C$ an ideal with $N^2 = 0$. Consider a commutative diagram of ring morphisms:

$$\begin{tikzcd}
\widehat{S} \arrow[r, "\varepsilon"] \arrow[drr, dashed, "v'"] & \widehat{S}_Q \arrow[r, "u"] \arrow[dr, "w"] & C/N \\
S \arrow[u] \arrow[r] & K \arrow[u] \arrow[r] & C \arrow[u] \\
S_\gamma \arrow[u] \arrow[r] & K_\gamma \arrow[u] &
\end{tikzcd}$$

and let w be a K_γ-algebra morphism lifting u. Since $\widehat{S}$ is 0-smooth over S relative to S_γ, there is an S'-algebra morphism $v' : \widehat{S} \longrightarrow C'$ lifting $u\varepsilon$. For every element $s \in \widehat{S} - Q$ the image $v'(s)$ is a unit in C (since $u\varepsilon(s)$ is a unit in C/N and N is nilpotent). Thus v' factors through a K-algebra morphism $v : \widehat{S}_Q \longrightarrow C'$. v is a K-algebra lifting of u.

Thus $\widehat{S}_Q$ is $Q\widehat{S}_Q$-smooth over K relative to K_γ for all $\gamma \in \Gamma$. By (15.14) $\widehat{S}_Q$ is geometrically regular over K.

(15.16) <u>Theorem</u>: A complete local ring R is a G-ring.

<u>Proof</u>: Let $P \subseteq R$ be a prime ideal, $S' = R_P$, $Q \subseteq S$ a prime ideal and $K = Q(S/Q) = k(Q)$.

We have to show that $\hat{S} \otimes_S K$ is geometrically regular over K. Note that $\hat{S} \otimes_S K = (\hat{S}/Q\hat{S}) \otimes_{S/Q} K$, $\hat{S}/Q\hat{S} = \widehat{(R/Q')_P}$ where $Q' = Q \cap R$, $K = Q(R/Q') = k(Q')$, and that $\hat{S} \otimes_S K = \widehat{(R/Q')_P} \otimes_{(R/Q')_P} k(Q')$. Thus it suffices to show that the generic formal fiber of $S/Q = (R/Q')_P$ is geometrically regular. Hence we may assume that R is a complete local domain, $P \subseteq R$ a prime ideal and $S = R_P$. We have to show that the generic formal fiber of S is geometrically regular. By Cohen's structure theorems there is a complete regular local subring T of R so that R is finite over T. Set $P' = P \cap T$. By (15.15) the generic formal fiber of $T_{P'}$ is geometrically regular. Obviously, $S' = R_{P'}$ is finite over $T_{P'}$ and S is essentially finite over $T_{P'}$ with injective local natural map $T_{P'} \longrightarrow S = R_P$. By (15.9) the generic formal fiber of S is geometrically regular.

(15.17) <u>Corollary</u>: Let R be a Noetherian ring. The following conditions are equivalent:

(a) R is a G-ring.

(b) For all maximal ideals $m \subseteq R$ the natural map $R_m \longrightarrow (R_m)^\wedge$ is regular.

<u>Proof</u>: (b) $\Rightarrow$ (a): Let $m \subseteq R$ be a maximal ideal. By (15.16) $(R_m)^\wedge$ is a G-ring and by assumption the natural map $R_m \longrightarrow (R_m)^\wedge$ is regular. By (15.12) R_m is a G-ring. Hence R is a G-ring.

Let R be a semilocal Noetherian ring with maximal ideals $m_1, \ldots, m_n$ and Jacobson-radical $J = \bigcap_{i=1}^n m_i$. The J-adic completion $\hat{R} = (R, J)^\wedge$ is the product of complete local rings:

$$\hat{R} = (R, J)^\wedge = \prod_{i=1}^n (R_{m_i})^\wedge$$

and for all $1 \le i \le n$, $(R_{m_i})^\wedge = (\hat{R})_{m_i \hat{R}} = (R, m_i)^\wedge$. Thus R is a G-ring if and only if the natural morphism $R \longrightarrow \hat{R}$ is regular. This yields:

(15.18) Corollary: Let R be a semilocal Noetherian ring and $\widehat{R}$ the completion of R with respect to the Jacobson radical. The following conditions are equivalent:

(a) R is a G-ring.

(b) The natural morphism $R \longrightarrow \widehat{R}$ is regular.

(c) For all $P \in \mathrm{Spec}(R)$ and all finite purely inseparable field extensions $k(P) \subseteq L$ the ring $\widehat{R} \otimes_R L$ is regular.

(15.19) Proposition: Let R be a semilocal Noetherian ring and $\widehat{R}$ the completion of R with respect to the Jacobson radical. The following conditions are equivalent:

(a) R is a G-ring.

(b) For every finite R-algebra S which is a domain and for all $Q \in \mathrm{Spec}(\widehat{S})$ with $Q \cap S = (0)$ the ring $\widehat{S}_Q$ is regular.

(c) For every finite R-algebra S which is a domain and for all $Q \in \mathrm{Sing}(\widehat{S})$ the intersection $Q \cap S \neq (0)$ is not trivial.

Proof: Obviously, (b) $\Longleftrightarrow$ (c).

(a) $\Longrightarrow$ (b): Suppose that S is an R-algebra via the morphism $u: R \longrightarrow S$. Then $P = \ker u$ is a prime ideal of R. Set $\overline{R} = R/P$ and $L = Q(S)$. Since S is finite over R, the field extension $k(P) = Q(\overline{R}) \subseteq L$ is finite. Moreover, $\widehat{S} = \widehat{R} \otimes_R S = \widehat{\overline{R}} \otimes_R S$ and $\widehat{R} \otimes_R L = (\widehat{R} \otimes_R S) \otimes_S L = \widehat{S} \otimes_S L = T^{-1}(S)$ where $T = S - (0)$. The prime ideals of $\widehat{S} \otimes_S L$ correspond to the prime ideals of $\widehat{S}$ which lie over (0) in S. Since R is a G-ring, the ring $\widehat{R} \otimes_R L$ is regular. Hence $\widehat{S}_Q$ is a regular local ring for all prime ideals $Q \subseteq \widehat{S}$ with $Q \cap S = (0)$.

(b) $\Longrightarrow$ (a): Let $P \in \mathrm{Spec}(R)$ and $k(P) \subseteq L$ a finite field extension. There is a finite R-algebra S with $R/P \subseteq S \subseteq L$ and $Q(S) = L$. By assumption (b) the ring $\widehat{R} \otimes_R L = \widehat{S} \otimes_S L$ is regular.

§3: PROPERTIES OF G-RINGS

In this section we want to show:

(a) Every algebra essentially of finite type over a G-ring is a G-ring.

(b) Every semilocal G-ring is a Reg-Z ring.

(15.20) Lemma: Let R be a Noetherian ring of dimension ≥ 1, $\mathcal{U} \subseteq \mathrm{Spec}(R)$ a nonempty open subset. Then there is a prime ideal $P \in \mathcal{U}$ with $\dim R/P \leq 1$.

Proof: Since $\mathcal{U} \subseteq \mathrm{Spec}(R)$ is an open subset, there is an ideal $I \subseteq R$ with $\mathcal{U} = \mathrm{Spec}(R) - V(I)$. By assumption $\mathcal{U}$ is nonempty and there is a minimal prime ideal $P_0 \subseteq R$ with $I \not\subseteq P_0$ and thus $P_0 \in \mathcal{U}$. Since $\mathcal{U} \cap V(P_0) \neq \emptyset$, we may replace R by R/P_0 and assume that R is a domain. If $\dim R \leq 1$, the statement is trivial. Suppose that $\dim R > 1$ and let $m \subseteq R$ be a maximal ideal. Since $(0) \in \mathcal{U}$, it follows that $\mathcal{U} \cap \mathrm{Spec}(R_m) \neq \emptyset$ and we may assume that R is a local Noetherian domain. The proof is by induction on $\dim R$:

Since $\mathcal{U} \subseteq \mathrm{Spec}(R)$ is open, there is an element $f \in R-(0)$ with $D_f \subseteq \mathcal{U}$. The element f is contained in only finitely many height one prime ideals. Let $Q \subseteq R$ be a height one prime ideal with $f \notin Q$. Then $Q \in \mathcal{U}$ and $\mathcal{U} \cap \mathrm{Spec}(R/Q) \neq \emptyset$. By induction hypothesis there is a prime ideal $P \in \mathcal{U} \cap \mathrm{Spec}(R/Q)$ with $\dim(R/P) \leq 1$.

(15.21) Corollary: Let R be a Noetherian ring, $Z \subseteq \mathrm{Spec}(R)$ a nonempty subset with $Z = \mathcal{U} \cap V(I)$ where $\mathcal{U} \subseteq \mathrm{Spec}(R)$ is open and $I \subseteq R$ an ideal. Then there is a $P \in Z$ with $\dim R/P \leq 1$.

Proof: By (15.20), since Z corresponds to the open subset $\mathcal{U} \cap V(I)$ of $\mathrm{Spec}(R/I)$.

(15.22) <u>Proposition:</u> Let (R, m) be a complete local Noetherian domain, $S = R[t]$ an R-algebra generated by one element t, and $P \subseteq S$ a maximal ideal with $P \cap R = m$. Set $T = S_P$ and suppose that D is a finite T-algebra and a domain. Let $\widehat{D}$ be the completion of D with respect to the Jacobson radical and $\varphi: \operatorname{Spec}(\widehat{D}) \to \operatorname{Spec}(D)$ the natural map given by $\varphi(\widehat{Q}) = \widehat{Q} \cap D$. Then $\varphi^{-1}(\operatorname{Reg}(D)) = \operatorname{Reg}(\widehat{D})$.

<u>Proof:</u> Since R is complete, by (13.12) and (13.13) R is Reg-2. Hence $\operatorname{Reg}(D)$ is open in $\operatorname{Spec}(D)$. Moreover, $\operatorname{Reg}(\widehat{D})$ is open in $\operatorname{Spec}(\widehat{D})$ and by flatness $\operatorname{Reg}(\widehat{D}) \subseteq \varphi^{-1}(\operatorname{Reg}(D))$. Suppose that $\varphi^{-1}(\operatorname{Reg}(D)) \nsubseteq \operatorname{Reg}(\widehat{D})$, then $Z = \varphi^{-1}(\operatorname{Reg}(D)) \cap \operatorname{Sing}(\widehat{D}) \neq \emptyset$ is a nonempty subset of $\operatorname{Spec}(\widehat{D})$ which is intersection of an open and a closed subset. By (15.21) there is a prime ideal $\widehat{Q} \in Z$ with $\dim(\widehat{D}/\widehat{Q}) \leq 1$. Set $Q = \varphi(\widehat{Q}) = \widehat{Q} \cap D$.

If $\dim(\widehat{D}/\widehat{Q}) = 0$, then $\widehat{Q}$ is a maximal ideal of $\widehat{D}$ and $\widehat{D}_{\widehat{Q}} = (D_Q)^\wedge$. Thus $\widehat{D}_{\widehat{Q}}$ is a regular local ring, a contradiction. Hence $\dim \widehat{D}/\widehat{Q} = 1$. Consider the natural morphism $\tau: D_Q \longrightarrow \widehat{D}_{\widehat{Q}}$. τ is faithfully flat and D_Q is regular while $\widehat{D}_{\widehat{Q}}$ is not. By 911, (8.63) the fiber ring $\widehat{D}_{\widehat{Q}} \otimes_D k(Q) = (\widehat{D}/Q\widehat{D})_{\widehat{Q}}$ is not regular. Replace R by $R/Q \cap R$, S by $S/Q \cap S$, T by $S_P/Q \cap S_P$, and D by D/Q and consider the sequence of injective morphisms:

$$(*) \quad R \longrightarrow S = R[t] \longrightarrow T = S_P \longrightarrow D \longrightarrow \widehat{D}/\widehat{Q} = E.$$

<u>Case 1:</u> E is finite over R.

Then D is finite over R, hence D is complete with respect to the Jacobson radical of D. Hence $D = \widehat{D}$, a contradiction.

<u>Case 2:</u> E is not finite over R.

By assumption $P \subseteq S = R[t]$ is a maximal ideal with $P \cap R = m$. If $S = R[y]/I$, where y is a variable over R and $I \subseteq R[y]$ an ideal, then $S/mS \cong (R/m)[y]/\bar{I}$ and the extension $R/m \longrightarrow S/P = T/PT = (R/m)[y]/P_0$ (where $P_0 \subseteq (R/m)[y]$ a maximal ideal) is finite.

The ring $E/\mathrm{Jrad}(E)$ is a quotient ring of $\widehat{D}/\mathrm{Jrad}(\widehat{D}) = D/\mathrm{Jrad}(D)$. Considering the natural morphisms

$$S/P = T/PT \xrightarrow{\lambda} D/\mathrm{Jrad}(D) \xrightarrow{\tau} E/\mathrm{Jrad}(E)$$

with λ finite and τ surjective, it follows that $E/\mathrm{Jrad}(E)$ is a finite T/PT-module. Thus $E/\mathrm{Jrad}(E)$ is finite over R/m.

Note that $\mathrm{rad}(mE) \subseteq \mathrm{Jrad}(E)$. If $\mathrm{Jrad}(E) = \mathrm{rad}(mE)$, then by 911, (9.29) E is a finite R-module since R is complete. This is the contradiction of case 1. Hence $\mathrm{rad}(mE) \subseteq \mathrm{Jrad}(E)$ and $\mathrm{rad}(mE) \neq \mathrm{Jrad}(E)$. Since E is a semilocal Noetherian domain of dimension one, this implies that $mE = 0$. Since $(*)$ is a sequence of injective morphisms, $m = 0$ and R is a field. Then $\dim D \leq 1$ and $\dim \widehat{D} \leq 1$. $\widehat{Q}$ is not maximal in $\widehat{D}$, hence $\widehat{Q}$ is a minimal prime ideal of $\widehat{D}$. D is essentially finitely generated over a field, thus D is a Nagata ring and a domain, $\widehat{D}$ is reduced, and $\widehat{D}_{\widehat{Q}}$ is regular, a contradiction. Hence $\varphi^{-1}(\mathrm{Reg}(D)) \subseteq \mathrm{Reg}(\widehat{D})$.

(15.23) <u>Theorem:</u> Let R be a G-ring and S an R-algebra of finite type. Then S is a G-ring.

<u>Proof:</u> It suffices to prove the statement for $S = R[t]$, that is, S is generated by one element over R. Let $P \in \mathrm{Spec}(S)$ and $Q = P \cap R$. We have to show that the natural morphism $S_P \longrightarrow (S_P)^\wedge$ is regular. Since S_P is a localization of $R_Q[t]$ we may assume that R is a local ring with maximal ideal $m = P \cap R$. The natural map $R \longrightarrow \widehat{R}$ is regular, hence by (15.8) the morphism $\varphi: S \longrightarrow S' = S \otimes_R \widehat{R} = \widehat{R}[t]$ is regular. Moreover, φ is faithfully flat and there is a prime ideal $P' \subseteq S'$ with $P' \cap S = P$. Consider the commutative diagram:

$$\begin{array}{ccc} S_P & \xrightarrow{\varphi_{P'}} & S'_{P'} \\ u\downarrow & & \downarrow v \\ (S_P)^\wedge & \xrightarrow{\widehat{\varphi_{P'}}} & (S'_{P'})^\wedge \end{array}$$

where u and v are the natural maps. Since $\widehat{\varphi_{P'}}$ is faithfully flat and $\varphi_{P'}$ is regular, by (15.3) it suffices to show that v is regular. Thus we may assume that $R = \widehat{R}$ is complete.

With $T = S_P$ by (15.19) we have to show that for every finite T-algebra D which is a domain and all prime ideals $Q \subseteq \widehat{D}$ with $Q \cap D = (0)$ the ring $\widehat{D}_Q$ is regular. Consider the morphisms:

$$R = \widehat{R} \xrightarrow{\sigma} S_P = T \xrightarrow{\sigma} D$$

and set $\rho = \tau\sigma$. Then $\ker(\rho) \subseteq R$ and $\ker(\tau) \subseteq T$ are prime ideals and σ, τ, ρ factor through $R/\ker(\rho) = \overline{R}$ and $T/\ker(\tau) = \overline{T}$ inducing injective morphisms:

$$\overline{R} = R/\ker(\rho) \xrightarrow{\overline{\sigma}} \overline{T} = T/\ker(\tau) \xrightarrow{\overline{\tau}} D.$$

Moreover, $\overline{\sigma}$ is a local morphism since $P \cap \overline{R} = m$. By (15.22)

$$\varphi^{-1}(\mathrm{Reg}(D)) = \mathrm{Reg}(\widehat{D})$$

where $\varphi : \mathrm{Spec}(\widehat{D}) \longrightarrow \mathrm{Spec}(D)$ is the natural map. Then $Q \in \varphi^{-1}(\mathrm{Reg}(D))$, since $(0) \in \mathrm{Reg}(D)$ and $\widehat{D}_Q$ is a regular local ring.

(15.24) <u>Corollary</u>: Let R be a G-ring. Every algebra essentially of finite type over R is a G-ring.

(15.25) <u>Theorem</u>: A semilocal G-ring is Reg-2.

<u>Proof</u>: Let R be a semilocal G-ring, $\widehat{R}$ its completion with respect to the Jacobson radical, and $\varphi : \mathrm{Spec}(\widehat{R}) \longrightarrow \mathrm{Spec}(R)$ the natural map. We claim that $\varphi^{-1}(\mathrm{Reg}(R)) = \mathrm{Reg}(\widehat{R})$. By faithful flatness $\mathrm{Reg}(\widehat{R}) \subseteq \varphi^{-1}(\mathrm{Reg}(R))$. Let $P \in \mathrm{Reg}(R)$ and $Q \in \mathrm{Spec}(\widehat{R})$ with $Q \cap R = P$, i.e. $Q \in \varphi^{-1}(P)$. The natural morphism $\lambda : R_P \longrightarrow \widehat{R}_Q$ is faithfully flat and the fiber ring $(\widehat{R}/P\widehat{R})_Q$ is regular, since R is a G-ring and $(\widehat{R}/P\widehat{R})_Q$ is a localization of $(R_m)^\wedge \otimes_R k(P)$ where $m \subseteq R$ is a maximal ideal with $Q \subseteq m\widehat{R}$. By

911, (8.63) $\hat{R}_Q$ is regular. This implies that $\mathrm{Reg}(R)$ is open in $\mathrm{Spec}(R)$. In order to show that R is Reg-2 by (13.11) it suffices to show that every finite R-algebra S is Reg-1, that is, $\mathrm{Reg}(S)$ is open in $\mathrm{Spec}(S)$. Since S is a semilocal G-ring, the statement follows from the previous argument.

(15.26) <u>Corollary</u>: Let R be a semilocal G-ring. Then $\mathrm{Reg}(R)$ is open in $\mathrm{Spec}(R)$.

(15.27) <u>Theorem</u>: Let (R, m) be a local Noetherian ring. The natural morphism $\varphi : R \longrightarrow R^h$ is regular.

<u>Proof</u>: By Chapter $\underline{\mathrm{IV}}$ there is a direct system of étale neighborhoods $\{S_i, \varphi^i_j\}_{i,j \in I}$ of R so that $R^h = \varinjlim_{i \in I} S_i$. Each (S_i, n_i) is a local Noetherian R-algebra which is the localization of an étale extension of R. In particular, for all $i, j \in I$ with $i \leq j$ $\varphi^i_j : S_i \longrightarrow S_j$ is faithfully flat with $n_i S_j = n_j = m S_j$. Then $\varphi : R \longrightarrow R^h$ is faithfully flat. It remains to show that for all $P \in \mathrm{Spec}(R)$ the ring $R^h \otimes_R k(P)$ is geometrically regular over $k(P)$. Let $k(P) \subseteq L$ be a finite field extension. Then $R^h \otimes_R L = \varinjlim_{i \in I} (S_i \otimes_R L)$. For a prime ideal $Q \in \mathrm{Spec}(R^h \otimes_R L)$ let $Q_i = Q \cap (S_i \otimes_R L)$ for all $i \in I$. Since $S_i \otimes_R L$ is unramified (and smooth) over L, $(S_i \otimes_R L)_{Q_i}$ is a field. Hence $(R^h \otimes_R L)_Q = \varinjlim_{i \in I} (S_i \otimes_R L)_{Q_i}$ is a field.

(15.28) <u>Theorem</u>: Let (R, m) be a local Noetherian ring. R is a G-ring if and only if R^h is a G-ring.

<u>Proof</u>: Consider the natural maps: $R \xrightarrow{\varphi} R^h \xrightarrow{\psi} \hat{R}$.
If R^h is a G-ring, φ and ψ are regular morphisms. By (15.4) $\psi\varphi$ is regular and R is a G-ring.

Conversely, suppose that R is a G-ring and let $Q \in \mathrm{Spec}(R^h)$, $P = Q \cap R$. Since R^h is a direct limit of étale neighborhoods, Q is a minimal prime ideal over PR^h and $k(P) \subseteq k(Q)$ is an algebraic field extension. In particular, $R^h \otimes_R k(P)$ is a (finite) product of fields and $k(Q)$ is a localization of $R^h \otimes_R k(P)$. Then $\widehat{R} \otimes_R k(P) = \widehat{R} \otimes_{R^h} (R^h \otimes_R k(P))$ and $\widehat{R} \otimes_{R^h} k(Q)$ is a localization of $\widehat{R} \otimes_R k(P)$. Let $k(Q) \subseteq L$ be a finite field extension. Then there is a finite field extension $k(P) \subseteq L_0$ so that $k(Q)(L_0) = L$, hence L is a localization of $R^h \otimes_R L_0$. Then $\widehat{R} \otimes_{R^h} L$ is a localization of $\widehat{R} \otimes_R L_0$. If R is a G-ring, the rings $\widehat{R} \otimes_R L_0$ and $\widehat{R} \otimes_{R^h} L$ are regular.

(15.29) <u>Remark</u>: Let (R, m) be a local Noetherian ring. The same proof as in (15.27) shows that the natural morphism $\tilde{\varphi}: R \longrightarrow R^{hs}$ is regular.
Similar to (15.27) we also have that R is a G-ring if and only if R^{hs} is a G-ring.

§4: EXCELLENT RINGS

(15.30) Definition: A Noetherian ring R is called excellent if R satisfies the following conditions:

(a) R is a G-ring.

(b) R is Reg-2.

(c) R is universally catenary.

(15.31) Remark: (a) If R is an excellent ring and S an R-algebra essentially of finite type, then S is excellent.

(b) A semilocal Noetherian ring R is excellent if

(i) R is a G-ring

(ii) R is universally catenary.

Proof: (a) Obviously, S is universally catenary. By (15.24) S is a G-ring and by definition S is Reg-2.

(b) By (15.25) R is Reg-2.

(15.32) Examples: (a) Every complete local Noetherian ring S is excellent.

(b) Let K be a field. Then K is excellent. In particular, every algebra essentially of finite type over K is excellent.

(c) The ring of integers $\mathbb{Z}$ is excellent and every $\mathbb{Z}$-algebra essentially of finite type is excellent.

(d) There are examples of universally catenary G-rings which are not Reg-2.

(e) There are examples of local G-rings which are not universally catenary. The construction is different from the one in (14.26).

Proof: (a) Every complete local Noetherian ring S is a homomorphic image of a regular local ring and hence universally catenary. By (15.16) S is a G-ring and by (13.12) and (13.13) S is Reg-2.

(b) Obviously, every field K is a universally catenary G-ring. By Corollary (12.8) K is Reg-2.

(c) Since $\mathbb{Z}$ is regular, $\mathbb{Z}$ is universally catenary. In order to show that $\mathbb{Z}$ is a G-ring let $P \subseteq \mathbb{Z}$ be a prime ideal. If $P=(0)$, then $\mathbb{Z}_P = \mathbb{Q}$ and every field is a G-ring. Let $P=(p)$ with $p \in \mathbb{Z}$ a prime number. Since $\operatorname{char} \mathbb{Z}_p = 0$ and $\dim \mathbb{Z}_p = 1$, the formal fibers of $\mathbb{Z}_p$ are geometrically regular. Thus $\mathbb{Z}$ is a G-ring. Since $\operatorname{char} \mathbb{Z} = 0$ and $\dim \mathbb{Z} = 1$, condition (c) of Theorem (13.11) is satisfied and $\mathbb{Z}$ is Reg-2.

(15.33) <u>Theorem</u>: Let (R, m) be a local G-ring, $\widehat{R}$ the completion of R, and $Q \subseteq \widehat{R}$ a prime ideal, $P = Q \cap R$. Then:

(a) R_P satisfies Serre's condition $(R_i) \Longleftrightarrow R_Q$ satisfies (R_i)

(b) R_P satisfies Serre's condition $(S_i) \Longleftrightarrow R_Q$ satisfies (S_i)

(c) R_P is regular $\Longleftrightarrow \widehat{R}_Q$ is regular

(d) R_P is normal $\Longleftrightarrow \widehat{R}_Q$ is normal

(e) R_P is reduced $\Longleftrightarrow \widehat{R}_Q$ is reduced

(f) R_P is CM $\Longleftrightarrow \widehat{R}_Q$ is CM

(g) R_P is Gorenstein $\Longleftrightarrow \widehat{R}_Q$ is Gorenstein.

<u>Proof</u>: (a) Recall that a Noetherian ring S satisfies Serre's condition (R_i) if for all prime ideals $W \subseteq S$ with $\operatorname{ht} W \leq i$ the ring S_W is regular. Let $P_0 \subseteq P$ be a prime ideal with $\operatorname{ht} P_0 \leq i$. By flatness there is a prime ideal $Q_0 \subseteq Q$ with $Q_0 \cap R = P_0$ and $\operatorname{ht} P_0 = \operatorname{ht} Q_0 \leq i$. If $\widehat{R}_Q$ satisfies (R_i) then $\widehat{R}_Q$ is regular. Since $R_{P_0} \longrightarrow \widehat{R}_{Q_0}$ is faithfully flat, R_{P_0} is regular. Thus R_P satisfies (R_i). Conversely, suppose that R_P satisfies (R_i) and let $Q_0 \subseteq \widehat{R}$ be a prime ideal with $Q_0 \subseteq Q$ and $\operatorname{ht} Q_0 \leq i$. Set $P_0 = Q_0 \cap R$ and consider the natural map

$\tau: R_{P_0} \longrightarrow \widehat{R}_{Q_0}$. τ is faithfully flat and the fiber $(\widehat{R}/P_0\widehat{R})_{Q_0}$ is regular since R is a G-ring. By assumption R satisfies (R_i) and $\operatorname{ht} P_0 \leq \operatorname{ht} Q_0 \leq i$, thus R_{P_0} is a regular local ring. By 911, (8.63) $\widehat{R}_{Q_0}$ is regular.

(b) Recall that a Noetherian ring S satisfies Serre's condition (S_i) if for all prime ideals $W \subseteq S$ $\operatorname{depth} S_W \geq \min\{i, \operatorname{ht} W\}$. Suppose that $\widehat{R}_Q$ satisfies (S_i) and let $P_0 \subseteq P$ be a prime ideal. Since $\widehat{R}$ is faithfully flat over R, there is a prime ideal $Q_0 \subseteq Q$ with $P_0 = Q_0 \cap R$ and $\operatorname{ht} P_0 = \operatorname{ht} Q_0$, that is, Q_0 is minimal over P_0. By assumption $\operatorname{depth} \widehat{R}_{Q_0} \geq \min\{i, \operatorname{ht} Q_0\}$. Consider the induced faithfully flat morphism $\tau: R_{P_0} \longrightarrow \widehat{R}_{Q_0}$. By 911, (9.52) $\operatorname{depth} R_{P_0} = \operatorname{depth} \widehat{R}_{Q_0} - \operatorname{depth} (\widehat{R}/P_0\widehat{R})_{Q_0}$ and $\operatorname{depth} (\widehat{R}/P_0\widehat{R})_{Q_0} = 0$ since Q_0 is minimal over $P_0\widehat{R}$. Thus R_P satisfies (S_i).

Conversely, suppose that R_P satisfies (S_i) and let $Q \subseteq Q_0$ be a prime ideal of $\widehat{R}$. Set $P_0 = Q_0 \cap R$. Then $\operatorname{depth} R_{P_0} \geq \min\{i, \operatorname{ht} P_0\}$. Since $R_{P_0} \longrightarrow \widehat{R}_{Q_0}$ is faithfully flat by 911, (9.43)

$$\dim \widehat{R}_{Q_0} = \dim R_{P_0} + \dim (\widehat{R}/P_0\widehat{R})_{Q_0}$$

and by 911, (9.52)

$$\operatorname{depth} \widehat{R}_{Q_0} = \operatorname{depth} R_{P_0} + \operatorname{depth} (\widehat{R}/P_0\widehat{R})_{Q_0}$$

Since R is a G-ring, $(\widehat{R}/P_0\widehat{R})_{Q_0}$ is regular and thus $\dim (\widehat{R}/P_0\widehat{R})_{Q_0} = \operatorname{depth} (\widehat{R}/P_0\widehat{R})_{Q_0}$. If $\operatorname{depth} R_{P_0} \geq i$, then $\operatorname{depth} \widehat{R}_{Q_0} \geq i$. On the other hand if $\operatorname{depth} R_{P_0} < i$, then $\operatorname{depth} R_{P_0} = \dim R_{P_0}$ and R_{P_0} is CM. In this case $\operatorname{depth} \widehat{R}_{Q_0} = \dim \widehat{R}_{Q_0}$ and $\widehat{R}_{Q_0}$ is CM.

(c) Follows by 911, (8.63) since the natural map $R_P \longrightarrow \widehat{R}_Q$ is faithfully flat and the fiber ring $(\widehat{R}/P\widehat{R})_Q$ is regular.

(d) By (a) and (b) since a Noetherian ring S is normal if and only if S satisfies (R_1) and (S_2).

(e) By (a) and (b) since a Noetherian ring S is reduced if and only if S satisfies (R_0) and (S_1).

(f) By (b) since a Noetherian ring S is CM if and only if S satisfies (S_∞).

(g) The natural map $R_p \longrightarrow \widehat{R}_Q$ is faithfully flat. Similar to 911, Theorem (8.63) the following theorem holds: $\widehat{R}_Q$ is Gorenstein $\Longleftrightarrow$ R_p and $(\widehat{R}/p\widehat{R})_Q$ are Gorenstein. (see Matsumura, Theorem 23.4). Since R is a G-ring, $(\widehat{R}/p\widehat{R})_Q$ is regular and hence Gorenstein.

(15.34) Proposition: Let R be a semilocal Noetherian ring. Then

(a) (i) R regular $\Longleftrightarrow$ $\widehat{R}$ regular

(ii) R CM $\Longleftrightarrow$ $\widehat{R}$ CM

(iii) R Gorenstein $\Longleftrightarrow$ $\widehat{R}$ Gorenstein

(b) If, in addition, R is a G-ring, then

(iv) R normal $\Longleftrightarrow$ $\widehat{R}$ normal

(v) R reduced $\Longleftrightarrow$ $\widehat{R}$ reduced

Proof: (i) is wellknown. (ii) and (iii) have been shown in 911 ((9.53) and (10.21)). (iv) and (v) follow by (15.33).

(15.35) Remark: Let R be a semilocal G-ring. Then the formal fibers of R are geometrically regular. Hence the formal fibers of R are geometrically reduced and $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$ by (7.25). Then R is a Nagata ring.

(15.36) Theorem: Let R be a semilocal G-ring and M a finitely generated R-module. The following sets are open in $\mathrm{Spec}(R)$:

(a) $\mathcal{U}_{(R_i)}(R) = \{P \in \mathrm{Spec}(R) \mid R_p \text{ satisfies } (R_i)\}$

(b) $\mathcal{U}_{(S_i)}(M) = \{P \in \mathrm{Spec}(R) \mid M_p \text{ satisfies } (S_i)\}$

(c) $\mathrm{Reg}(R) = \{P \in \mathrm{Spec}(R) \mid R_p \text{ is regular}\}$

(d) $\mathrm{Nor}(R) = \{P \in \mathrm{Spec}(R) \mid R_p \text{ is normal}\}$

(e) $\mathcal{U}_{Red}(R) = \{P \in \mathrm{Spec}(R) \mid R_p \text{ is reduced}\}$

(f) $\mathcal{U}_{CM}(M) = \{P \in \mathrm{Spec}(R) \mid M_p \text{ is CM}\}$

(g) $\mathrm{Gor}(R) = \{P \in \mathrm{Spec}(R) \mid R_P$ is Gorenstein$\}$.

<u>Proof</u>: (c) follows by (15.25) and (c) implies (a). By (13.4) and (13.5) $U_{CM}(\hat{R})$ and $\mathrm{Gor}(\hat{R})$ are open in $\mathrm{Spec}(\hat{R})$. Then (f) and (g) follow by (15.33) in case that $R = M$. Every semilocal G-ring is a Nagata ring and $\mathrm{Nor}(R)$ is open in $\mathrm{Spec}(R)$ by (7.25). (e) follows by (a), (b) and (15.33). The proofs of (b) and (f) are much more involved and omitted here (see EGA).

(15.37) <u>Theorem</u>: Let $R = R^h$ be a local Henselian G-ring. Then R is excellent.

<u>Proof</u>: We have to show that R is universally catenary. By (14.21) it suffices to show that R is formally catenary, that is, that $(R/P)^\wedge$ is equidimensional for all $P \in \mathrm{Spec}(R)$. Thus we may assume that R is a domain and have to show that $\hat{R}$ is equidimensional. We claim that $\hat{R}$ is a domain. By (15.35) R is a Nagata ring, thus the integral closure S of R in $Q(R)$ is a finite R-module. R is Henselian, hence S is a local Noetherian domain with completion $\hat{S} = S \otimes_R \hat{R}$. Moreover, S is a G-ring, thus $\hat{S}$ is normal by (15.34). Hence $\hat{S}$ is a domain. Since $\hat{R}$ is flat over R,
$$\hat{R} = R \otimes_R \hat{R} \subseteq S \otimes_R \hat{R} = \hat{S}$$
and $\hat{R}$ is a domain.