CHAPTER X: COHEN'S STRUCTURE THEOREMS IN THE UNEQUAL CHARACTERISTIC CASE

§1: PRELIMINARIES

(10.1) Lemma: Let R be a ring, $I \subseteq R$ an ideal, and $u: E \longrightarrow F$ an R-linear map of R-modules. Suppose:

(i) E is complete in the I-adic topology.

(ii) F is separated in the I-adic topology.

(iii) The induced map of the associated graded modules $gr(u): gr_I(E) \longrightarrow gr_I(F)$ is bijective.

Then u is bijective.

Proof: Let $x \in E$ with $u(x) = 0$. If $x \neq 0$ then there is an $n \in \mathbb{N}$ with $x \in I^n E$ and $x \notin I^{n+1}E$, since E is complete and separated in the I-adic topology. Then $x + I^{n+1}E \in I^nE/I^{n+1}E - (0)$ and $gr(u)(x + I^{n+1}E) = 0$, a contradiction. Hence u is injective.

Let $y \in F$, $y \neq 0$. Since F is I-adically separated, there is an $n \in \mathbb{N}$ with $y \in I^nF - I^{n+1}F$. The map $u_n: I^nE/I^{n+1}E \longrightarrow I^nF/I^{n+1}F$ is surjective and there is an $x_n \in I^nE$ with $u_n(x + I^{n+1}E) = y + I^{n+1}F$. Thus $u(x_n) = y + y_{n+1}$ with $y_{n+1} \in I^{n+1}F$. By a similar argument there is an $x_{n+1} \in I^{n+1}E$ with $u(x_{n+1}) = y_{n+1} + y_{n+2}$ and $y_{n+2} \in I^{n+2}F$. Thus $u(x_n - x_{n+1}) = y - y_{n+2}$. Recursively we construct a sequence $x_{n+k} \in I^{n+k}E$ so that

$$u\left(x_n - \sum_{i=1}^{k} x_{n+i}\right) - y \in I^{n+k+1}F.$$

Since E is I-adically complete, $x = x_n - \sum_{i=1}^{\infty} x_{n+i} \in E$ and $u(x) - y \in I^{n+k}F$ for all $k \in \mathbb{N}$. Since $\bigcap_{n=1}^{\infty} I^nF = 0$, $u(x) = y$ and u is surjective.

(10.2) <u>Lemma</u>: Let R be a ring, $I \subseteq R$ an ideal, and $u: E \longrightarrow F$ an R-linear map of R-modules. Suppose:

(i) E is complete in the I-adic topology.

(ii) F is separated in the I-adic topology.

(iii) F is a flat R-module.

(iv) The induced map $u_1: E/IE \longrightarrow F/IF$ is bijective.

Then u is bijective.

<u>Proof</u>: Consider the following commutative diagram:

$$\begin{tikzcd} E/I \otimes_{R/I} gr_I(R) \arrow[r, "u_1 \otimes 1"] \arrow[d, "f"'] & F/IF \otimes_{R/I} gr_I(R) \arrow[d, "g"] \\ gr_I(E) \arrow[r, "gr(u)"] & gr_I(F) \end{tikzcd}$$

where f and g are the homogeneous maps whose n-th components are the natural maps:

$$f_n: E/IE \otimes_{R/I} I^n/I^{n+1} \longrightarrow I^nE/I^{n+1}E \quad \text{and}$$

$$g_n: F/IF \otimes_{R/I} I^n/I^{n+1} \longrightarrow I^nF/I^{n+1}F.$$

We want to show that $gr(u)$ is an isomorphism. Since u_1 is an isomorphism, so is $u_1 \otimes 1$. Moreover, since F is R-flat the sequences:

$$0 \longrightarrow I^{n+1} \otimes_R F \longrightarrow I^n \otimes_R F \longrightarrow I^n/I^{n+1} \otimes_R F \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow I^n \otimes_R F \longrightarrow F \longrightarrow R/I^n \otimes_R F \longrightarrow 0$$

are exact. Hence $I^n \otimes_R F \cong I^nF$ and $I^n/I^{n+1} \otimes_R F \cong I^n/I^{n+1} \otimes_{R/I} F/IR \cong I^nF/I^{n+1}F$ and g is an isomorphism. This implies that f is injective. Since f is surjective, f is bijective and hence $gr(u)$ is an isomorphism. The assertion follows by (10.1).

Recall the following proposition from Chapter $\underline{IV}$:

(4.42) <u>Proposition</u>: Let $(R_i, \varphi_j^i)_{i,j \in I}$ be a direct system of rings and morphisms which satisfies the following conditions:

(i) For all $i\in I$ R_i is a local Noetherian ring with maximal ideal m_i.
(ii) For all $i,j\in I$ with $i\le j$ the morphism $\varphi^i_j: R_i \longrightarrow R_j$ is faithfully flat.
(iii) For all $i,j\in I$ with $i\le j$: $\varphi^i_j(m_i)R_j = m_j$.
Then the direct limit $R = \varinjlim_{i\in I} R_i$ is a local Noetherian ring with maximal ideal $m = \varinjlim_{i\in I} m_i$. Moreover, the canonical maps $\varphi_i : R_i \longrightarrow R$ are faithfully flat.

(10.3) Proposition: Let (R,m,k) be a local Noetherian ring and $k\subseteq K$ a field extension. Then there is a local Noetherian R-algebra S so that:
(a) S is flat over R
(b) $S\otimes_R k \cong K$.
Moreover, one can choose S to be complete.

Proof: We first treat the case where K is generated by one element over k, say $K = k(x)$.

If x is transcendental over k, put $R' = (R[t])_{mR[t]}$ where t is a variable. R' is a flat local Noetherian R-algebra with maximal ideal $m_{R'} = mR'$. Hence $R'\otimes_R k \cong R'/mR' = k(t) \cong k(x) = K$.

If x is algebraic over k, let $f(t)\in k[t]$ be the minimal polynomial of x over k. Let $F(t)\in R[t]$ be a monic polynomial with $F(t)+mR[t] = f(t)$. Set $R' = R[t]/(F)$. R' is a finitely generated free R-module and hence flat over R. Moreover, $R'\otimes_R k \cong R'/mR' = R[t]/(F, mR[t]) \cong k[t]/(f(t)) \cong K$ and mR' is a maximal ideal of R'. Since R' is integral over R, mR' is the only maximal ideal of R' and R' is local.

If the field extension $k\subseteq K$ is finitely generated, say $K = k(x_1,\dots,x_n)$, we can first construct a flat local Noetherian R-algebra (R_1, m_1, k_1) so that $R_1\otimes_R k \cong k(x_1) \cong k_1$, in particular, $m_1 = mR_1$. Then replace R by R_1 and use the construction again to find a flat local Noetherian R_1-algebra

(R_2, m_2, k_2) so that $R_2 \otimes_{R_1} k_1 \cong k(x_1,x_2) \cong k_2$. Then $R_2 \otimes_{R_1} k_1 \cong R_2 \otimes_{R_1} (R_1 \otimes_R k) \cong R_2 \otimes_R k \cong k(x_1,x_2)$ and inductively we can construct a local Noetherian R-algebra S which satisfies conditions (a) and (b).

In the general case where $k \subseteq K$ is not finitely generated we use a transfinite induction argument. First note that there is a well-ordered set I and a family of intermediate fields $\{k_i\}_{i\in I}$ where $k \subseteq k_i \subseteq K$ for all $i \in I$ so that the following conditions are satisfied:

(α) There is an $i_0 \in I$ with $k = k_{i_0}$ and i_0 is the smallest element of I.

(β) For all $i,j \in I$ with $i \leq j$: $k_i \subseteq k_j$.

(γ) For all $i \in I$ the field extension $k_i \subseteq k_{i+1}$ is generated by one element.

(δ) If $i \in I$ is a limit number, that is, an element of I without a predecessor, then $k_i = \bigcup_{j<i} k_j$.

(ε) $K = \bigcup_{i\in I} k_i$.

By transfinite induction we want to construct for all $i \in I$ a local Noetherian R-algebra R_i so that the following conditions are satisfied:

(0) $R_{i_0} = R$

(i) For all $i \in I$ R_i is a local Noetherian R-algebra with maximal ideal m_i.

(ii) For all $i,j \in I$ with $i \leq j$ there is a faithfully flat R-algebra morphism $\varphi^i_j : R_i \longrightarrow R_j$. Moreover, if $i,j,l \in I$ with $i \leq j \leq l$ then $\varphi^j_l \varphi^i_j = \varphi^i_l$ and $\varphi^i_i = id_{R_i}$.

(iii) For all $i,j \in I$ with $i \leq j$: $\varphi^i_j(m_i)R_j = m_j$. In particular, $\varphi^{i_0}_j(m)R_j = m_j = mR_j$ for all $j \in I$.

(iv) For all $i \in I$: $R_i \otimes_R k \cong R_i/mR_i \cong k_i$.

Let $i \in I$ and suppose for all $j<i$ R-algebras R_j have been constructed so that conditions (0)–(iv) are satisfied for all $j \in I$ with $j<i$. In order to construct R_i we need to distinguish two cases:

<u>Case</u> 1: i has a predecessor $i-1$.

Then $k_i = k_{i-1}(x)$ for some $x \in k$ and we construct R_i from R_{i-1} as in the beginning of the proof.

Case 2: i is a limit number.

Set $R_i = \varinjlim_{j<i} R_j$. By (4.42) R_i satisfies conditions (0)–(iv) for $j \in I$ with $j \leq i$.

Finally in order to obtain S we apply (4.42) again and set $S = \varinjlim_{i \in I} R_i$. Obviously, S satisfies conditions (a) and (b). The completion of S also satisfies conditions (a) and (b).

(10.4) Corollary: Assumptions as in (10.3). If $k \subseteq K$ is separable, then S is mS-smooth over R.

Proof: By (9.15).

§2: THE STRUCTURE THEOREMS

(10.5) Definition: Let (R, m, k) be a local Noetherian ring and (S, n, ℓ) a local Noetherian R-algebra. S is called a Cohen R-algebra if the following conditions are satisfied:

(a) S is a complete local ring.

(b) S is flat over R.

(c) $S \otimes_R k \cong S/mS = \ell$ and ℓ is separable over k.

(10.6) Remark: By (9.15) a Cohen R-algebra S is mS-smooth over R.

(10.7) Definition: Let $p \in \mathbb{N}$ be a prime number. A discrete valuation ring (D, m, k) (of characteristic 0) is called a p-ring if the maximal ideal m of D is generated by p.

(10.8) Theorem: Let (R, m, k) be a complete local Noetherian ring, (D, pD, ℓ) a p-ring, and $\varphi_0 : \ell \longrightarrow k$ a morphism of fields. Then there is a local morphism $\varphi : D \longrightarrow R$ which induces φ_0 on the residue fields.

Proof: Let $D_0 = \mathbb{Z}_{(p)}$ and $k_0 = D_0/pD_0 = \mathbb{F}_p$. Obviously $p \in m$ and the natural map $\mathbb{Z} \longrightarrow R$ extends to a morphism $\rho : D_0 \longrightarrow R$. Similarly, there is a natural map $\lambda : D_0 \longrightarrow D$. Since $k_0 = \mathbb{F}_p$ is perfect, ℓ and k are 0-smooth over k_0. Moreover, D is a torsion-free D_0-module, thus D is flat over D_0. By (9.15) D is pD-smooth over D_0. Consider the commutative diagram of morphisms of rings:

$$\begin{array}{ccccc} D & \xrightarrow{\nu} & \ell & \xrightarrow{\varphi_0} & k \\ \lambda \uparrow & & & & \uparrow \mu \\ D_0 & & \xrightarrow{\quad \rho \quad} & & R \end{array}$$

where ν and μ are the natural maps. Set $\tau_1 = \varphi_0 \nu$. Since D is pD-smooth over D_0, τ_1 lifts to a D_0-algebra morphism τ_2:

$$\begin{tikzcd}
D \arrow[r, "\tau_1"] \arrow[dr, "\tau_2"] \arrow[ddr, "\tau_n"] & k \\
 & R/m^2 \arrow[u] \\
 & R/m^n \arrow[u, dotted] \\
D_0 \arrow[uuu, "\lambda"] \arrow[r, "\rho"] & R \arrow[u, dotted]
\end{tikzcd}$$

Similarly, $\tau_2 : D \longrightarrow R/m^2$ lifts to a D_0-algebra morphism $\tau_3 : D \longrightarrow R/m^3$ etc, that is, for all $n \in \mathbb{N}$ we obtain a D_0-algebra morphism $\tau_n : D \longrightarrow R/m^n$ which lifts $\tau_{n-1} : D \longrightarrow R/m^{n-1}$. Since R is complete, there is a D_0-algebra morphism $\varphi : D \longrightarrow R$ which lifts τ_0. φ induces φ_0 on the residue fields.

(10.9) <u>Corollary</u>: A complete p-ring is – up to isomorphism – uniquely determined by its residue fields.

<u>Proof</u>: Let (D, pD, k) and (D', pD', k') be complete p-rings with $k \xrightarrow[\varphi_0]{\cong} k'$. By (10.8) there is a local morphism $\varphi : D \longrightarrow D'$ which induces the isomorphism $\varphi_0 : k \longrightarrow k'$ on the residue fields. Since D is complete, by 911, Theorem (9.29) φ is surjective. Since $\ker(\varphi)$ is a prime ideal of D and $\varphi(p) \neq 0$, $\ker(\varphi) \neq 0$ and φ is injective.

(10.10) <u>Definition</u>: Let (R, m, k) be a complete local Noetherian ring of unequal characteristic with $\operatorname{char} k = p > 0$. A subring $R_0 \subseteq R$ is called a <u>coefficient ring</u> of R if R_0 is a complete local Noetherian ring with maximal ideal pR_0 and residue field $R_0/pR_0 = k = R/m$.

(10.11) Remark: If (R, m, k) is a complete local Noetherian ring of unequal characteristic and (R_0, pR_0, k) is a coefficient ring of R then $R = R_0 + m$.

(10.12) Theorem: Let (R, m, k) be a complete local Noetherian ring with $\operatorname{char} k = p > 0$. Then R has a coefficient ring R_0. Moreover, if $\operatorname{char} R = 0$, then R_0 is a complete discrete valuation ring.

Proof: If $\operatorname{char} R = \operatorname{char} k = p$, then by (8.17) R contains a coefficient field. In the unequal characteristic case let $D_0 = \mathbb{Z}_{(p)}$. By (10.3) there is a local Noetherian ring D so that D is flat over D_0 and $D/pD \cong k$. Hence D is a discrete valuation ring and thus a p-ring. By (10.8) there is a local morphism $\varphi: \hat{D} \longrightarrow R$ which induces an isomorphism on the residue fields. Hence $R_0 = \operatorname{im}(\varphi)$ is a coefficient ring of R. If $\operatorname{char} R = 0$, φ is injective and $R_0 = \operatorname{im}(\varphi)$ is a discrete valuation ring.

(10.13) Theorem: Every complete local Noetherian ring is a homomorphic image of a complete regular local ring.

Proof: Let (R, m, k) be a complete local Noetherian ring with $m = (x_1, \_, x_n)$. By (8.17) and (10.12) R contains a coefficient ring R_0. Moreover, R_0 is either a field or by the proof of (10.12) a homomorphic image of a complete discrete valuation ring D. Let $V = R_0$ in the equal characteristic case or $V = D$ in the unequal characteristic case. Consider the surjective morphism:

$$\varphi: V[\![t_1, \_, t_n]\!] \longrightarrow R$$

defined by $\varphi|_V$ the natural map from V onto R_0 and $\varphi(t_i) = x_i$ for all $1 \leq i \leq n$. By 911, Proposition (9.32) φ is surjective.

(10.14) Theorem: Let (R, m, k) be a complete local Noetherian ring. In the unequal charakteristic case assume in addition that R is a domain. Then there is a local Noetherian subring $(S, n, \ell) \subseteq (R, m, k)$ with $n \subseteq m$ so that:

(a) S is a complete regular local ring and $\ell = k$.

(b) R is finitely generated as an S-module.

Proof: Suppose that $\dim R = n$ and let $y_1, \ldots, y_n$ be a system of parameters of R. In the equal characteristic case let V denote a coefficient field of R and in the unequal characteristic case let V be a coefficient ring of R. In the latter case V is a complete discrete valuation ring by (10.12), since R is a domain. Moreover, in the unequal characteristic case $\operatorname{char} k = p > 0$ and we can choose $y_1 = p$. By 911, Proposition (9.32) there are morphisms of rings:

in the equal characteristic case: $\varphi: T = V[[t_1, \ldots, t_n]] \longrightarrow R$ defined by $\varphi|_V$ the embedding of the coefficient field V into R and $\varphi(t_i) = y_i$ for all $1 \leq i \leq n$ and

in the unequal characteristic case: $\varphi: T = V[[t_2, \ldots, t_n]] \longrightarrow R$ where $\varphi|_V$ is the embedding of the coefficient ring V into R and $\varphi(t_i) = y_i$ for all $2 \leq i \leq n$.

By 911, Theorem (9.29) R is a finite T-module, hence R is integral over T. Since T is a domain with $\dim T = n$, φ is injective. The image $S = \operatorname{im}(\varphi)$ satisfies conditions (a) and (b) of the theorem.

(10.15) Remark: The condition that R is Noetherian is not necessary in Theorems (8.17)(b) and (10.12). Thus every complete (quasi)-local ring contains a coefficient ring. This implies the following theorem:

' Let (R, m, k) be a complete (quasi)-local ring. Assume that

the maximal ideal $\mathfrak{m}$ of R is finitely generated. Then R is Noetherian.'
The proof is similar to the proof of (10.13).