CHAPTER VIII: COHEN-MACAULAY RINGS/MODULES; REGULAR RINGS

§1: REGULAR SEQUENCES

(8.1) Definition: Let A be a ring and M an A-module.

(a) An element $a \in A$ is called M-regular if $am \neq 0$ for all $m \in M-(0)$.

(b) A sequence of elements $a_1, -, a_n \in A$ is called an M-sequence if the following conditions are satisfied.

(i) a_1 is M-regular and for all $2 \leq i \leq n$ the element a_i is $(M/\sum_{j=1}^{i-1} a_j M)$-regular.

(ii) $M \neq \sum_{i=1}^{n} a_i M$.

(8.2) Remark: A permutation of an M-sequence may not be an M-sequence (see Homework)

(8.3) Theorem: Let A be a ring, M an A-module and $a_1, -, a_n \in A$ an M-sequence. For all $\nu_i \in \mathbb{N}-(0)$, where $1 \leq i \leq n$, the sequence $a_1^{\nu_1}, -, a_n^{\nu_n}$ is an M-sequence.

Proof: It suffices to show: if $a_1, -, a_n$ is an M-sequence and $\nu \in \mathbb{N}-(0)$ then $a_1^{\nu}, a_2, -, a_n$ is an M-sequence. Condition (ii) in definition (8.1) is obvious. The proof of condition (i) is by induction on ν. We first show:

Claim: Let $b_1, -, b_n \in A$ be an M-sequence and $m_1, -, m_n \in M$ with $b_1 m_1 + - + b_n m_n = 0$. Then for all $1 \leq i \leq n$: $m_i \in \sum_{j=1}^{n} b_j M$.

Pf of Cl: by induction on n. Since b_n is $(M/\sum_{j=1}^{n-1} b_j M_j)$-regular and $b_n m_n \in M/\sum_{j=1}^{n-1} b_j M$, there are elements $l_j \in M$ so that
$$m_n = \sum_{j=1}^{n-1} b_j M$$

Thus: $\sum_{j=1}^{n} b_j m_j = \sum_{j=1}^{n-1} b_j m_j + b_n \sum_{j=1}^{n-1} b_j l_j = \sum_{j=1}^{n-1} b_j (m_j + b_n l_j) = 0$.

By induction hypothesis for all $1 \leq i \leq n-1$: $m_i + b_n l_i \in \sum_{j=1}^{n-1} b_j M$.

This proves the claim.

In order to prove the theorem we show by induction on $\nu \in \mathbb{N}-(0)$ that $a_1^\nu, a_2, \dots, a_n$ is an M-sequence. For $\nu = 1$ there is nothing to show. If $\nu > 1$, note that a_1^ν is M-regular. Suppose that for some $2 \le i \le n$ and some $w \in M$:

$$a_i w = a_1^\nu m_1 + \dots + a_{i-1} m_{i-1} \quad \text{where } m_j \in M.$$

By induction hypothesis $a_1^{\nu-1}, a_2, \dots, a_n$ is an M-sequence and there are $n_j \in M$ so that:

$$w = a_1^{\nu-1} n_1 + \dots + a_{i-1} n_{i-1}.$$

Thus:
$$\begin{aligned} a_1^\nu m_1 + \dots + a_{i-1} m_{i-1} - a_i w &= a_1^\nu m_1 + \dots + a_{i-1} m_{i-1} - a_i (a_1^{\nu-1} n_1 + \dots + a_{i-1} n_{i-1}) \\ &= a_1^{\nu-1}(a_1 m_1 - a_i n_1) + a_2(m_2 - a_i n_2) + \dots + a_{i-1}(m_{i-1} - a_i n_{i-1}) \\ &= 0 \end{aligned}$$

By the claim: $a_1 m_1 - a_i n_1 \in a_1^{\nu-1} M + \sum_{j=2}^{i-1} a_j M$ and therefore $a_i n_1 \in \sum_{j=1}^{i-1} a_j M$. Since $a_1, \dots, a_n$ is M-regular, $n_1 \in \sum_{j=1}^{i-1} a_j M$ and therefore: $w \in a_1^\nu M + \sum_{j=2}^{i-1} a_j M$.

In the following we want to show: If A is a local Noetherian ring, M a finitely generated A-module, and $a_1, \dots, a_n \in A$ an M-sequence, then for all $\sigma \in S_n$: $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ is an M-sequence.

(8.4) Remark and definition: Let A be a ring, $x_1, \dots, x_n$ variables over A, and M an A-module. We consider elements of $M \otimes_A A[x_1, \dots, x_n]$ as 'polynomials' in the x_i with coefficients in M and write
$$\sum_{|\alpha| \le t} m_{(\alpha)} \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha| \le t} m_{(\alpha)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = F(x_1, \dots, x_n)$$
where $m_{(\alpha)} \in M$. We set $M[x_1, \dots, x_n] = M \otimes_A A[x_1, \dots, x_n]$. Obviously, $M[x_1, \dots, x_n]$ is an $A[x_1, \dots, x_n]$-module and for all $b_1, \dots, b_n \in A$ there is an A-linear map $\varphi: M[x_1, \dots, x_n] \longrightarrow M$ with $\varphi(F(x_1, \dots, x_n)) = F(b_1, \dots, b_n)$.

(8.5) Definition: Let A be a ring and $a_1, \dots, a_n \in A$. Set $I = (a_1, \dots, a_n)$ and let M be an A-module with $IM \neq M$. The sequence $a_1, \dots, a_n$ is called M-quasiregular if for all $\nu \in \mathbb{N}-(0)$ the following condition $(*)$ is satisfied:

$(*)$ If $F(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree ν with

$F(a_1,-,a_n) \in I^{\nu+1}M$, then all coefficients of F are in IM.

(8.6) Remark: Condition $(*)$ is independent of the order of the sequence $a_1,-,a_n$.

(8.7) Lemma: For all $\nu \in \mathbb{N}-(0)$ condition $(*)$ is equivalent to:
$(**)$ If $F(x_1,-,x_n) \in M[x_1,-,x_n]$ is a homogeneous polynomial of degree ν with $F(a_1,-,a_n)=0$, then the coefficients of $F(x_1,-,x_n)$ are in IM.

Proof: Obviously, $(*)$ implies $(**)$. Conversely, let $F(x_1,-,x_n) \in M[x_1,-,x_n]$ be a homogeneous polynomial of degree ν with $F(a_1,-,a_n) \in I^{\nu+1}M$. Then there is a homogeneous polynomial $G(x_1,-,x_n) \in M[x_1,-,x_n]$ of degree $\nu+1$ with $F(a_1,-,a_n) = G(a_1,-,a_n)$. Write $G(x_1,-,x_n) = \sum_{i=1}^{n} x_i G_i(x_1,-,x_n)$ where $G_i(x_1,-,x_n) \in M[x_1,-,x_n]$ a homogeneous polynomial of degree ν for $1 \le i \le n$. Then $F^*(x_1,-,x_n) = F(x_1,-,x_n) - \sum_{i=1}^{n} a_i G_i(x_1,-,x_n)$ is a homogeneous polynomial of degree ν with $F^*(a_1,-,a_n)=0$. By $(**)$ the coefficients of F^* are in IM, hence the coefficients of F are in IM.

(8.8) Lemma: Let A be a ring, M an A-module and $a_1,-,a_n \in A$ an M-quasiregular sequence. Set $I=(a_1,-,a_n) \subseteq A$ and let $a \in A$ be such that $IM : a = IM$. Then for all $\nu \in \mathbb{N}-(0)$: $I^\nu M : a = I^\nu M$.

Proof by induction on ν. For the induction step assume that $I^{\nu-1}M : a = I^{\nu-1}M$ and let $m \in M$ be so that $am \in I^\nu M$. Then $m \in I^{\nu-1}M$ and we may write $m = F(a_1,-,a_n)$ where $F(x_1,-,x_n) \in M[x_1,-,x_n]$ is a homogeneous polynomial of degree $\nu-1$. Since $am = aF(a_1,-,a_n) \in I^\nu M$, the homogeneous polynomial $aF(x_1,-,x_n)$ has coefficients in IM (by $(*)$). Write $F(x_1,-,x_n) = \sum_{|i|=\nu-1} m_{(i)} x_1^{i_1} - x_n^{i_n}$ with $m_{(i)} \in M$. Then $a m_{(i)} \in IM$ and by assumption $m_{(i)} \in IM$. But then $m = F(a_1,-,a_n) \in I^\nu M$.

(8.9) Theorem: Let A be a ring, M an A-module, and $a_1,-,a_n \in A$ an M-sequence.

Then $a_1, \ldots, a_n$ is M-quasiregular.

<u>Proof</u>: by induction on n. For $n=1$ let $F(x) \in M[x]$ be a homogeneous polynomial of degree ν. Then $F(x) = mx^\nu$ for some $m \in M$. Suppose that $F(a) = ma^\nu \in a^{\nu+1}M$. Since a is M-regular, $m = an$ for some $n \in M$.

$n-1 \Rightarrow n$. Suppose that $a_1, \ldots, a_{n-1}$ is an M-quasiregular sequence. Let $F(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree ν with $F(a_1, \ldots, a_n) = 0$. We want to show by induction on ν that the coefficients of $F(x_1, \ldots, x_n)$ are in IM.

$\nu = 0$: Then $F(x_1, \ldots, x_n) = m \in M$ and $m = F(a_1, \ldots, a_n) = 0$.

$\nu - 1 \Rightarrow \nu$: Note that Lemma (8.7) shows that $(*) \Leftrightarrow (**)$ for a fixed $\nu \in \mathbb{N} - (0)$. Thus by induction hypothesis, if $H(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree $\nu - 1$ with $H(a_1, \ldots, a_n) \in I^\nu M$, then the coefficients of H are in IM.

Write $F(x_1, \ldots, x_n) = G(x_1, \ldots, x_{n-1}) + x_n H(x_1, \ldots, x_n)$ where $G(x_1, \ldots, x_{n-1}) \in M[x_1, \ldots, x_{n-1}]$ is a homogeneous polynomial of degree ν and $H(x_1, \ldots, x_n) \in M[x_1, \ldots, x_n]$ is homogeneous of degree $\nu - 1$. Since $F(a_1, \ldots, a_n) = 0$, $H(a_1, \ldots, a_n) \in (a_1, \ldots, a_{n-1})^\nu M : a_n$. Since $a_1, \ldots, a_n$ is M-regular, $(a_1, \ldots, a_{n-1})M : a_n = (a_1, \ldots, a_{n-1})M$ and thus by Lemma (8.8) and the induction hypothesis (on $n-1$): $H(a_1, \ldots, a_n) \in (a_1, \ldots, a_{n-1})^\nu M$. Since $H(x_1, \ldots, x_n)$ is homogeneous of degree $\nu - 1$, by induction hypothesis, the coefficients of $H(x_1, \ldots, x_n)$ are in IM. Using $H(a_1, \ldots, a_n) \in (a_1, \ldots, a_{n-1})^\nu M$ there is a homogeneous polynomial $h(x_1, \ldots, x_{n-1}) \in M[x_1, \ldots, x_{n-1}]$ of degree ν with $h(a_1, \ldots, a_{n-1}) = H(a_1, \ldots, a_n)$.

Consider the homogeneous polynomial in $M[x_1, \ldots, x_{n-1}]$ of degree ν:

$$g(x_1, \ldots, x_{n-1}) = G(x_1, \ldots, x_{n-1}) + a_n h(x_1, \ldots, x_{n-1}).$$

Then $g(a_1, \ldots, a_{n-1}) = F(a_1, \ldots, a_n) = 0$. By induction hypothesis on n the sequence $a_1, \ldots, a_{n-1}$ is M-quasiregular. Thus the coefficients of g are in IM and so are the coefficients of G and F.

Recall: If A is a Noetherian ring, $I \subseteq \mathrm{Jrad}(A)$ an ideal in the Jacobson radical of A, and M a finitely generated A-module, then $\bigcap_{n \in \mathbb{N}} I^n M = 0$.

(8.10) Theorem: Let A be a Noetherian ring, M a finitely generated A-module and $a_1, -, a_n \in A$ with $I = (a_1, -, a_n) \subseteq \mathrm{Jrad}(A)$. The following are equivalent:

(a) $a_1, -, a_n$ is M-regular

(b) $a_1, -, a_n$ is M-quasiregular

Proof: By (8.9) (a) $\Rightarrow$ (b). For (b) $\Rightarrow$ (a) we will show:

(i) a_1 is M-regular and

(ii) The sequence $a_2, -, a_n$ is M/a_1M-regular

Then the statement follows by induction on n.

(i) Suppose that $a_1 m = 0$ for some $m \in M$. The polynomial $F = m x_1 \in M[x_1, -, x_n]$ is homogeneous of degree 1 with $F(a_1) = F(a_1, -, a_n) = a_1 m = 0$. By (**) $m \in IM$, that is, $m = \sum_{i=1}^{n} a_i w_i$ for some $w_i \in M$. The polynomial $G(x_1, -, x_n) = \sum_{i=1}^{n} w_i x_1 x_i$ is homogeneous of degree 2 with $G(a_1, -, a_n) = 0$. Again by (**) $w_i \in IM$ and thus $m \in I^2 M$. Continuing like this yields that $m \in \bigcap_{n \in \mathbb{N}} I^n M = (0)$.

(ii) Set $\overline{M} = M/a_1M$ and let $f(x_2, -, x_n) \in \overline{M}[x_2, -, x_n]$ be a homogeneous polynomial of degree ν with $f(a_2, -, a_n) = 0$. Let $F(x_2, -, x_n) \in M[x_2, -, x_n]$ be a homogeneous polynomial of degree ν with $F + a_1 M[x_2, -, x_n] = f$. Then $F(a_2, -, a_n) \in a_1 M$, say $F(a_2, -, a_n) = a_1 w$ for some $w \in M$. Let $i \in \mathbb{N}$ with $i+1 \leq \nu$ be maximal with $w \in I^i M$. Then there is a homogeneous polynomial $G(x_1, -, x_n) \in M[x_1, -, x_n]$ of degree i with $G(a_1, -, a_n) = w$ and $F(a_2, -, a_n) = a_1 G(a_1, -, a_n) \in I^\nu M$ (since F is of degree ν). The polynomial $H(x_1, -, x_n) = x_1 G(x_1, -, x_n)$ is homogeneous of degree $i+1 \leq \nu$.

Case 1: $i+1 < \nu$

Since $H(a_1, -, a_n) = a_1 G(a_1, -, a_n) \in I^\nu M$, by (*) the coefficients of H and G are in IM. This implies that $w = G(a_1, -, a_n) \in I^{i+1} M$, a contradiction.

Case 2: $i+1 = \nu$

$F^* = F(x_2, -, x_n) - H(x_1, -, x_n) \in M[x_1, -, x_n]$ is a homogeneous polynomial of degree ν with $F^*(a_1, -, a_n) = 0$. By (**) the coefficients of F^* are in IM.

Since the monomial terms of $F(x_2, \ldots, x_n)$ do not involve x_1, it follows that the coefficients of F are in IM and the coefficients of f are in $I\overline{M} = (a_2, \ldots, a_n)\overline{M}$.

(8.11) <u>Definition</u>: Let A be a ring, M an A-module and $I \subseteq A$ an ideal. A sequence of elements $a_1, \ldots, a_n \in I$ is called a <u>maximal</u> M-<u>sequence</u> in I if
(a) $a_1, \ldots, a_n$ is an M-sequence
(b) for all $b \in I$ the sequence $a_1, \ldots, a_n, b$ is not an M-sequence.

(8.12) <u>Corollary</u>: Let (A, m) be a local Noetherian ring and M a finitely generated A-module. If $a_1, \ldots, a_n \in m$ is an M-sequence so is the sequence $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$ for all $\sigma \in S_n$.

(8.13) <u>Remark</u>: Let A be a Noetherian ring and M an A-module. For every ideal $I \subsetneq A$ maximal M-sequences in I exists (possibly of length 0).

Let A be a Noetherian ring and M a finitely generated A-module. We want to show that maximal M-sequences have the same length.

(8.14) <u>Theorem</u>: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A-module with $IM \neq M$. For an integer $n \in \mathbb{N} - (0)$ the following conditions are equivalent:
(a) $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$ and all finitely generated A-modules N with $\mathrm{Supp}_A(N) \subseteq V(I)$.
(b) $\mathrm{Ext}^i_A(A/I, M) = 0$ for all $i < n$.
(c) There is a finitely generated A-module N with $\mathrm{Supp}_A(N) = V(I)$ and $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$.
(d) I contains an M-sequence of length n.

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (c): trivial

(c) $\Rightarrow$ (d): Suppose first that every element of I is a zerodivisor of M. This implies that there is a $P \in \mathrm{Ass}_A(M)$ with $I \subseteq P$. Thus there is an injective map $\varphi : A/P \longrightarrow M$ which extends to a nonzero map $\varphi_P : k(P) = (A/P)_P \longrightarrow M_P$. In particular, $\mathrm{Hom}_{A_P}(k(P), M_P) \neq 0$. N is a finitely generated A-module with $V(I) = \mathrm{Supp}_A(N)$, hence $P \in \mathrm{Supp}_A(N)$ and $N_P \neq 0$. By Nakayama $N \otimes_A k(P) \cong N_P/PN_P \neq 0$ and $\mathrm{Hom}_{k_P}(N \otimes_A k(P), k(P)) \neq 0$. For a nonzero A_P-linear map $\tau : N \otimes_A k(P) \longrightarrow k(P)$ consider the composition $\underbrace{N_P \xrightarrow{\text{surj}} N \otimes_A k(P) \xrightarrow{\tau} k(P) \xrightarrow[\text{inj}]{\varphi_P} M_P}_{\rho}$. Since $\rho \neq 0$,

we have that $\mathrm{Hom}_{A_P}(N_P, M_P) \cong \mathrm{Hom}_A(N, M)_P \neq 0$ and therefore $\mathrm{Hom}_A(N, M) \neq 0$. This shows that if $\mathrm{Hom}_A(N, M) = 0$ then there is an M-regular element $f \in I$. We proceed by induction on n. If $n = 1$, we are done. For $n > 1$ let $f \in I$ be an M-regular element and set $M_1 = M/fM$. The exact sequence $0 \rightarrow M \xrightarrow{f} M \longrightarrow M_1 \longrightarrow 0$ yields a long exact sequence:

$$0 \rightarrow \mathrm{Hom}_A(N,M) \longrightarrow \mathrm{Hom}_A(N,M) \longrightarrow \mathrm{Hom}_A(N,M_1) \longrightarrow \mathrm{Ext}^1_A(N,M) \longrightarrow \ldots$$
$$\ldots \ \mathrm{Ext}^{i-1}_A(N,M) \longrightarrow \mathrm{Ext}^{i-1}_A(N,M_1) \longrightarrow \mathrm{Ext}^i_A(N,M) \longrightarrow \ldots$$

Thus $\mathrm{Ext}^i_A(N, M_1) = 0$ for all $i < n-1$. By induction hypothesis I contains an M_1-regular sequence of length $n-1$.

(d) $\Rightarrow$ (a): Let $f_1, \ldots, f_n \in I$ be an M-sequence and let N be a finitely generated A-module with $\mathrm{Supp}_A(N) \subseteq V(I)$. We want to show by induction on n that $\mathrm{Ext}^i_A(N, M) = 0$ for all $i < n$. Consider the exact sequence $0 \longrightarrow M \xrightarrow{f_1} M \longrightarrow M_1 = M/f_1M \longrightarrow 0$. By left exactness the sequence $0 \longrightarrow \mathrm{Hom}_A(N,M) \xrightarrow{f_{1*}} \mathrm{Hom}_A(N,M)$ is exact and f_{1*} is multiplication by f_1. Since $f_1 \in I \subseteq \mathrm{rad}(\mathrm{ann}_A(N))$ and N finitely generated, $f_1^r N = 0$ for some $r \in \mathbb{N}$. Thus $f_1^r \mathrm{Hom}_A(N,M) = 0$ and, since f_{1*} is injective, $\mathrm{Hom}_A(N,M) = 0$. We know by induction hypothesis that $\mathrm{Ext}^i_A(N, M_1) = 0$ for all $i < n-1$. This implies that for all $i < n$ the sequence $0 \rightarrow \mathrm{Ext}^i_A(N,M) \xrightarrow[*]{f_1} \mathrm{Ext}^i_A(N,M)$ is exact, where $*$ is multiplication by f_1. Compute $\mathrm{Ext}^i_A(N,M)$ by using an injective resolution of M: $0 \rightarrow M \rightarrow Q_0 \rightarrow \ldots \rightarrow Q_s \rightarrow \ldots$. Since $f_1^r N = 0$ for some $r \in \mathbb{N}$, it follows that $f_1^r \operatorname{Hom}_A(N,Q_j)=0$ and therefore $f_1^r \operatorname{Ext}^i_A(N,M)=0$. Since $*$ is injective, $\operatorname{Ext}^i_A(N,M)=0$ for all $i<n$.

(8.15) <u>Corollary</u>: Let A be a Noetherian ring, $I\subseteq A$ an ideal, and M a finitely generated A-module with $IM\neq M$. If $a_1,\dots,a_n\in I$ is a maximal M-sequence in I, then $\operatorname{Ext}^n_A(A/I,M)\neq 0$

<u>Proof</u>: First note that by (8.14) $\operatorname{Hom}_A(A/I,M)\neq 0$ if $n=0$. For $0<i\le n$ set $M_i = M/(a_1,\dots,a_i)M$. We want to show by induction on $n-i$ that $\operatorname{Ext}^{n-i}_A(A/I,M_i)\neq 0$. Suppose that $\operatorname{Ext}^{n-i-1}_A(A/I,M_{i+1})\neq 0$ and consider the exact sequence: $0\to M_i \xrightarrow{a_{i+1}} M_i \to M_{i+1}\to 0$. This yields a long exact sequence:

$$\to \operatorname{Ext}^{n-i-1}_A(A/I,M_i)\to \operatorname{Ext}^{n-i-1}_A(A/I,M_{i+1})\to \operatorname{Ext}^{n-i}_A(A/I,M_i)\xrightarrow[*]{a_{i+1}} \operatorname{Ext}^{n-i}_A(A/I,M_i)$$

Note that by (8.14) $\operatorname{Ext}^{n-i-1}_A(A/I,M_i)=0$ and that $*$ is the zero map. Thus $\operatorname{Ext}^{n-i-1}_A(A/I,M_{i+1})\cong \operatorname{Ext}^{n-i}_A(A/I,M_i)$ and the statement follows.

(8.16) <u>Theorem</u>: Let A, I, and M be as in (8.15). The length of a maximal M-sequence in I is independent of the choice of the sequence. In particular, the length n of a maximal M-sequence is given by:

$$\operatorname{Ext}^i_A(A/I,M) = \begin{cases} 0 & \text{if } i<n \\ \neq 0 & \text{if } i=n \end{cases}$$

(8.17) <u>Definition</u>: Let A be a Noetherian ring, $I\subseteq A$ an ideal, and M a finitely generated A-module with $IM\neq M$.

(a) $\operatorname{depth}_I(M)$ = <u>I-depth of M</u> = $\operatorname{grade}(I,M)$ = <u>grade of I on M</u> = maximal length of an M-sequence in I. If $IM=M$ we set $\operatorname{depth}_I(M)=\infty$.

(b) <u>grade</u> I = $\operatorname{grade}(I,A)$

(c) <u>grade</u> M = $\operatorname{grade}\operatorname{ann}(M)$

(d) If (A,m) is local, then <u>depth</u> $M = \operatorname{depth}_m M$.

§2: DEPTH

(8.18) Proposition: Let A be a Noetherian ring, $I \subseteq A$ an ideal and M a finitely generated A-module.

(a) $\operatorname{depth}_I M = \min\{i \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$

(b) $\operatorname{grade} M = \min\{i \mid \operatorname{Ext}^i_A(M, A) \neq 0\}$

Proof: (a) If $IM \neq M$, the assertion follows by (8.16). If $IM = M$, then $\operatorname{Ext}^i_A(A/I, M) = 0$ if and only if $\operatorname{Ext}^i_A(A/I, M)_P = \operatorname{Ext}^i_{A_P}(A_P/I_P, M_P) = 0$ for all $P \in \operatorname{Spec}(A)$. If $I \subseteq P$, then $I_P M_P = M_P$ and by Nakayama $M_P = 0$; if $I \nsubseteq P$, then $I_P = A_P$. Thus $\operatorname{Ext}^i_A(A/I, M) = 0$ for all i.
(b) By (8.14) $\operatorname{grade} M = \operatorname{depth}_{\operatorname{ann}(M)} A = \min\{i \mid \operatorname{Ext}^i_A(M, A) = 0\}$.

(8.19) Corollary: Let A be a local Noetherian ring with residue field k and M a finitely generated A-module. Then $\operatorname{depth} M = \min\{i \mid \operatorname{Ext}^i_A(k, M) \neq 0\}$.

(8.20) Theorem: Let (A, m, k) be a local Noetherian ring, M and N nonzero finitely generated A-modules. Then $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < \operatorname{depth} M - \dim N$.

Proof: by induction on $r = \dim N$. If $r = 0$, then $\operatorname{Supp}_A(N) = \{m\}$, since $N \neq 0$. The assertion follows by (8.14). If $r > 0$ pick a filtration of N: $N = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_n = 0$ with $N_j/N_{j+1} \cong A/P_j$ for some $P_j \in \operatorname{Spec}(A)$.

Claim: Set $s = \operatorname{depth} M$ and suppose that (*) $\operatorname{Ext}^i_A(N_j/N_{j+1}, M) = 0$ for all $i < s - r$ and all $j = 0, \ldots, n-1$. Then $\operatorname{Ext}^i_A(N, M) = 0$ for all $i < s - r$.

Pf of Claim: We show by decreasing induction on j that (*) implies $\operatorname{Ext}^i_A(N_j, M) = 0$ for all $i < s - r$. By assumption (*) for $j = n-1$: $\operatorname{Ext}^i_A(N_{n-1}, M) = \operatorname{Ext}^i_A(N_{n-1}/N_n, M) = 0$ for all $i < s - r$. For the induction step $j \Rightarrow j-1$ consider the exact sequence:
$0 \to N_j \to N_{j-1} \to N_{j-1}/N_j \to 0$, which yields a long exact sequence:
$\to \operatorname{Ext}^i_A(N_{j-1}/N_j, M) \to \operatorname{Ext}^i_A(N_{j-1}, M) \to \operatorname{Ext}^i_A(N_j, M) \to \ldots$
By (*) $\operatorname{Ext}^i_A(N_{j-1}/N_j, M) = 0$ and by induction hypothesis $\operatorname{Ext}^i_A(N_j, M) = 0$.

Suppose now that the statement holds for all finitely generated A-modules of dimension $\leq r-1$ and let N be a finitely generated A-module of dimension r. Since $\dim(N_{j-1}/N_j) \leq \dim N = r$ by the claim it remains to show that $\operatorname{Ext}^i_A(A/P, M) = 0$ for all $i < s-r$ and all $P \in \operatorname{Spec}(A)$ with $\dim(A/P) = r$. For such a P let $x \in m - P$ and consider the exact sequence $0 \to A/P \xrightarrow{x} A/P \to A/P+xA \to 0$. Since $\dim(A/P+xA) \leq r-1$ by induction hypothesis $\operatorname{Ext}^i_A(A/P+xA, M) = 0$ for all $i < s-r+1$. Consider for $i < s-r$ the long exact sequence:

$$\to \underbrace{\operatorname{Ext}^i_A(A/P+xA, M)}_{=0} \to \operatorname{Ext}^i_A(A/P, M) \xrightarrow{x} \operatorname{Ext}^i_A(A/P, M) \to \underbrace{\operatorname{Ext}^{i+1}_A(A/P+xA, M)}_{=0} \to$$

Thus multiplication by x is an isomorphism and $\operatorname{Ext}^i_A(A/P, M) \cong x\operatorname{Ext}^i_A(A/P, M)$. Since $\operatorname{Ext}^i_A(A/P, M)$ is a finitely generated A-module, by Nakayama $\operatorname{Ext}^i_A(A/P, M) = 0$.

(8.21) Corollary: Let A be a local Noetherian ring, M a finitely generated A-module, and $P \in \operatorname{Ass}_A(M)$. Then $\dim(A/P) \geq \operatorname{depth} M$.

Proof: Suppose that $P \in \operatorname{Ass}_A(M)$ with $\dim(A/P) < \operatorname{depth} M$. By (8.20) $\operatorname{Hom}_A(A/P, M) = 0$, a contradiction.

(8.22) Proposition: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and $0 \to M' \to M \to M'' \to 0$ an exact sequence of finitely generated A-modules. Then:

(a) $\operatorname{depth}_I M \geq \min\{\operatorname{depth}_I M', \operatorname{depth}_I M''\}$

(b) $\operatorname{depth}_I M' \geq \min\{\operatorname{depth}_I M, \operatorname{depth}_I M'' + 1\}$

(c) $\operatorname{depth}_I M'' \geq \min\{\operatorname{depth}_I M, \operatorname{depth}_I M' - 1\}$

Proof: (8.18) and the long exact sequence for $\operatorname{Ext}^i_A(A/I, -)$.

(8.23) Theorem: (Auslander–Buchsbaum formula) Let A be a local Noetherian ring and $M \neq 0$ a finitely generated A-module with $\operatorname{projdim} M < \infty$. Then

$$\operatorname{projdim} M + \operatorname{depth} M = \operatorname{depth} A.$$

Proof: by induction on projdim M.

If projdim $M=0$, then M is projective, hence free and $M\cong A^n$. Thus depth $M=$ depth A by (8.18).

If projdim $M=1$, then there is an exact sequence $0\longrightarrow A^s \xrightarrow{\varphi} A^n \longrightarrow M\longrightarrow 0$ with $s\geq 1$ and all entries of φ are in m, the maximal ideal of A. Set $k=A/m$. The map

$$\begin{array}{ccc} \Phi_i=\operatorname{Ext}^i_A(k,\varphi): \operatorname{Ext}^i_A(k,A^s) & \longrightarrow & \operatorname{Ext}^i_A(k,A^n) \\ \| & & \| \\ \operatorname{Ext}^i_A(k,A)\otimes_A A^s & & \operatorname{Ext}^i_A(k,A)\otimes_A A^n \end{array}$$

is simply $\mathrm{id}_{\operatorname{Ext}^i(k,A)}\otimes\varphi$, which is zero since all entries of φ are in m and $m\operatorname{Ext}^i_A(k,A)=0$. By the long exact sequence for $\operatorname{Ext}^i_A(k,-)$ and (8.18) we obtain that depth $M=$ depth $A-1$ since $s\geq 1$.

If projdim $M\geq 2$, there is an exact sequence $0\to N\to A^n\to M\to 0$ with projdim $N=$ projdim $M-1$. By induction hypothesis, depth $N=$ depth $A-$ projdim $N<$ depth A. Thus by (8.22) depth $M=$ depth $N-1$, and we are done.

(8.24) Remark: If A is a Noetherian ring and M a finitely generated A-module then, by (8.18), grade $M\leq$ projdim M.

(8.25) Definition: Let A be a Noetherian ring, $I\subseteq A$ an ideal, and M a finitely generated A-module

(a) M is called perfect if grade $M=$ projdim M ($\Rightarrow M\neq 0$ and projdim $M<\infty$).

(b) I is called perfect if A/I is perfect as an A-module.

(8.26) Lemma: Let A be a Noetherian ring, $I\subseteq A$ an ideal, and M a finitely generated A-module. If I consists of zero divisors of M then $I\subseteq P$ for some $P\in\operatorname{Ass}(M)$.

Proof: Obviously, $M\neq 0$. Since $\bigcup_{P\in\operatorname{Ass}(M)} P$ is the set of zerodivisors of M. Thus $I\subseteq \bigcup_{P\in\operatorname{Ass}(M)} P$. Since $\operatorname{Ass}(M)$ is finite, the statement follows.

(8.27) Proposition: Let A be a Noetherian ring, $I,J\subseteq A$ ideals, and M a finitely generated A-module.

(a) $\operatorname{depth}_I M=\min\{\operatorname{depth} M_P \mid P\in V(I)\}$

(b) $\operatorname{depth}_I M = \operatorname{depth}_{\sqrt{I}} M$

(c) $\operatorname{depth}_{I\cap J} M = \min\{\operatorname{depth}_I M, \operatorname{depth}_J M\}$

(d) If $\underline{a} = a_1, \dots, a_n$ is an M-sequence in I, then $\operatorname{depth}_{I/(\underline{a})} M/(\underline{a})M = \operatorname{depth}_I M/(\underline{a})M = \operatorname{depth}_I M - n$.

Proof: (a) Since $IM = M$ if and only if $M_P = 0$ for all $P \in V(I)$ by Nakayama's Lemma, we may assume $IM \neq M$. "$\leq$" is clear by (8.14). In order to prove "$\geq$" let $a_1, \dots, a_n$ be an M-regular sequence of maximal length in I. By (8.26) $I \subseteq P$ for some $P \in \operatorname{Ass}_A(M/(a_1,\dots,a_n)M)$. Then $P_P \in \operatorname{Ass}_{A_P}(M_P/(\underline{a})M_P)$ and $a_1, \dots, a_n$ form a maximal M_P-regular sequence contained in P_P and therefore $\operatorname{depth} M_P = n = \operatorname{depth}_I M$.

(b) and (c) follow from (a) since $V(I) = V(\sqrt{I})$ and $V(I \cap J) = V(I) \cup V(J)$.

(d) follows from (8.16).

(8.28) Definition: Let A be a local Noetherian ring and M a finitely generated A-module.

(a) M is called Cohen-Macaulay (CM) if $M = 0$ or if $\operatorname{depth} M = \dim M$.

(b) M is called maximal Cohen-Macaulay (MCM) if $\operatorname{depth} M = \dim A$ ($\Leftrightarrow$ $M \neq 0$ CM and $\dim M = \dim A$).

(c) A is called a Cohen-Macaulay ring if A is CM as module over itself.

(8.29) Theorem: Let A be a local Noetherian ring and M a finitely generated A-module.

(a) If M is a CM-module, then for all $P \in \operatorname{Ass}_A(M)$: $\dim A/P = \dim M = \operatorname{depth} M$. In particular, M has no embedded prime ideals.

(b) Let $a_1, \dots, a_r \in \mathfrak{m}$ be an M-sequence and $M' = M/(a_1,\dots,a_r)M$. Then M is CM if and only if M' is CM.

(c) If M is CM and $P \in \operatorname{Spec}(A)$, then the A_P-module M_P is CM. In particular, if $M_P \neq 0$, then $\operatorname{depth}_P(M) = \operatorname{depth}_{A_P}(M_P)$.

Proof: (a) $\dim M = \sup\{\dim(A/P) \mid P \in \operatorname{Ass}(M)\} \geq \inf\{\dim(A/P) \mid P \in \operatorname{Ass}(M)\} \underset{(8.21)}{\geq} \operatorname{depth} M$.

(b) Since all maximal M-sequences have the same length $\operatorname{depth} M' = \operatorname{depth} M - r$.

By (4.44) we know if $a \in \mathfrak{m}$ and $a \notin P$ for all minimal primes $P \in \operatorname{Supp}(M)$ then $\dim(M/aM) \leq \dim M - 1$. This implies that $\dim M' \leq \dim M - r$ and therefore $\dim M' \leq \operatorname{depth} M'$. By (8.21) $\dim M' \geq \operatorname{depth} M'$ and M' is CM.

(c) Let $P \in \operatorname{Spec}(A)$ with $M_P \neq 0$. Then $\operatorname{ann}(M) \subseteq P$ and $\dim M_P \geq \operatorname{depth} M_P \geq \operatorname{depth}_P M$. We show by induction on $\operatorname{depth}_P M$ that $\dim M_P = \operatorname{depth}_P M$.

If $\operatorname{depth}_P M = 0$, then P is contained in the set of zerodivisors of M and there is a prime $Q \in \operatorname{Ass}_A(M)$ with $P \subseteq Q$ (8.26). Since $M_P \neq 0$, $P \in \operatorname{Supp}(M)$ and there is a prime $Q' \in \operatorname{Ass}(M)$ with $Q' \subseteq P$. M is CM and by (a) M has no embedded prime ideals. Thus $Q' = P = Q \in \operatorname{Ass}_A(M)$ and P is minimal in $\operatorname{Supp}(M)$. Hence $\dim M_P = 0$.

If $\operatorname{depth}_P M \neq 0$, let $a \in P$ be an M-regular element and put $M' = M/aM$. By (b) M' is a CM-module with $\operatorname{depth}_P M' = \operatorname{depth}_P M - 1$. Note that $M'_P \neq 0$. By induction hypothesis $\dim M'_P = \operatorname{depth}_P M' = \operatorname{depth}_P M - 1$ and by (4.44) $\dim M'_P = \dim M_P - 1$, since a is regular on M_P. Thus $\dim M_P = \operatorname{depth}_P M$.

(8.30) Definition: Let A be a Noetherian ring. A is called catenary if for all prime ideals $P, Q \in \operatorname{Spec}(A)$ with $P \subseteq Q$ all saturated chains of prime ideals $P \subsetneq P_1 \subsetneq - \subsetneq P_s \subsetneq Q$ have the same length.

(8.31) Remark: (a) Nagata constructed in 1956 the first non-catenary Noetherian ring.
(b) Ratliff showed 1972 that a local Noetherian domain A is catenary if and only if $\operatorname{ht} P + \dim A/P = \dim A$ for all $P \in \operatorname{Spec}(A)$.

(8.32) Definition: Let $(A, \mathfrak{m})$ be a local Noetherian ring with $\dim A = n$. Elements $a_1, -, a_n \in \mathfrak{m}$ are called a system of parameters (SOP) if $I = (a_1, -, a_n)$ is an ideal of definition of A, that is, if $\operatorname{rad}(I) = \mathfrak{m}$. Elements $b_1, -, b_r \in \mathfrak{m}$ are called part of a system of parameters of A if there are elements $b_{r+1}, -, b_n \in \mathfrak{m}$ so that $b_1, -, b_n$ is a SOP.

Note that in a local Noetherian ring systems of parameters always exist.

(8.33) <u>Theorem</u>: Let $(A, \mathfrak{m})$ be a local Noetherian CM-ring.

(a) For elements $a_1, \text{—}, a_r \in \mathfrak{m}$ the following are equivalent:

(i) $a_1, \text{—}, a_r$ is a regular sequence.

(ii) $\operatorname{ht}(a_1, \text{—}, a_i) = i$ for all $1 \leq i \leq r$.

(iii) $\operatorname{ht}(a_1, \text{—}, a_r) = r$.

(iv) $a_1, \text{—}, a_r$ is part of a system of parameters of A.

(b) For an ideal $I \subsetneq A$ the following holds:

(i) $\operatorname{ht} I = \operatorname{depth}_I A$

(ii) $\operatorname{ht} I + \dim A/I = \dim A$

(c) A is catenary.

<u>Proof</u>: (a) (i)$\Rightarrow$(ii): Since $a_1, \text{—}, a_r$ is regular, a_{i+1} is regular on $A/(a_1, \text{—}, a_i)$ and $a_{i+1} \notin P$ for all $P \in \operatorname{Ass}(A/(a_1, \text{—}, a_i))$. In particular, a_{i+1} is not contained in any minimal prime ideal of $(a_1, \text{—}, a_i)$ and $\operatorname{ht}(a_1, \text{—}, a_{i+1}) \geq \operatorname{ht}(a_1, \text{—}, a_i) + 1$. In particular, $\operatorname{ht}(a_1, \text{—}, a_i) \geq i$. By Krull's generalized principal ideal theorem $\operatorname{ht}(a_1, \text{—}, a_i) \leq i$, hence $\operatorname{ht}(a_1, \text{—}, a_i) = i$ for all $1 \leq i \leq r$.

(ii)$\Rightarrow$(iii): trivial

(iii)$\Rightarrow$(iv): If $\dim A = r$, we are done. Suppose that $\dim A = r$. Since $(a_1, \text{—}, a_r)$ is generated by r elements, by Krull's generalized principal ideal theorem the maximal ideal $\mathfrak{m}$ is not a minimal prime ideal over $(a_1, \text{—}, a_r)$. Pick an element $a_{r+1} \in \mathfrak{m}$ which is not contained in any minimal prime ideal over $(a_1, \text{—}, a_r)$. Then $\operatorname{ht}(a_1, \text{—}, a_{r+1}) = r+1$. Continue like this.

(iv)$\Rightarrow$(i): It suffices to show that any SoP of A is a regular sequence. The proof is by induction on $n = \dim A$. Let $a_1, \text{—}, a_n$ be an SoP of A. Since A is CM, by (8.29) every prime ideal $P \in \operatorname{Ass}(A)$ is a minimal prime ideal of A and $\dim A/P = \dim A$ for all $P \in \operatorname{Ass}(A)$. Since $a_1 + P, \text{—}, a_n + P$ is a SOP of A/P for all $P \in \operatorname{Ass}(A)$, $a_1 \notin P$ for all $P \in \operatorname{Ass}(A)$ and a_1 is regular on A. Set $A' = A/a_1A$. By (8.29) A' is a local Noetherian CM-ring with $\dim A' = n-1$. Moreover,

$a_2+a_1A, \ldots, a_n+a_1A$ is a SoP of A'. By induction hypothesis, $a_2+a_1A, \ldots, a_n+a_1A$ is a regular sequence of A'.

(b) (i) Suppose $\mathrm{ht}\, I = r$. Then there are elements $a_1, \ldots, a_r \in I$ with $\mathrm{ht}(a_1, \ldots, a_r) = r$. By (a) the sequence $a_1, \ldots, a_r$ is regular and $\mathrm{depth}_I A \geq \mathrm{ht}\, I = r$. Let $P \subseteq A$ be a prime ideal with $I \subseteq P$ and $\mathrm{ht}\, P = r$. By (8.29) A_P is a CM-ring and $\mathrm{depth}_P A = \mathrm{depth}\, A_P = \dim A_P = r$. Since $\mathrm{depth}_I A \leq \mathrm{depth}_P A$ the statement follows.

(ii) Let $S = \{P \in \mathrm{Spec}(A) \mid I \subseteq P \text{ minimal}\}$ be the set of all prime ideals which are minimal over I. By definition:

$$\mathrm{ht}\, I = \inf\{\mathrm{ht}\, P \mid P \in S\} \quad \text{and} \quad \dim(A/I) = \sup\{\dim(A/P) \mid P \in S\}.$$

<u>Claim</u>: For all $P \in \mathrm{Spec}(A)$: $\dim A = \dim A/P + \mathrm{ht}\, P$.

<u>Pf of claim</u>: Set $\dim A = n$ and $\mathrm{ht}\, P = \dim A_P = r$. By (8.29) A_P is a CM-ring and $\mathrm{depth}\, A_P = \mathrm{depth}_P A = \dim A_P = r$. Let $a_1, \ldots, a_r \in P$ be a maximal regular sequence in P. Then $A/(a_1, \ldots, a_r)$ is a CM-ring of dimension $n-r$. By (a) $\mathrm{ht}\, P = r = \mathrm{ht}(a_1, \ldots, a_r)$ and P is a minimal prime over $(a_1, \ldots, a_r)$. Thus $P \in \mathrm{Ass}(A/(a_1, \ldots, a_r))$ and by (8.29)(a) $\dim(A/P) = \dim(A/(a_1, \ldots, a_r)) = n-r$.

In order to finish the proof of (ii) let $P \in S$ with $\mathrm{ht}\, P = \mathrm{ht}\, I$. Then $\dim(A/I) \geq \dim(A/P)$ and $\mathrm{ht}\, I + \dim(A/I) \geq \mathrm{ht}\, P + \dim(A/P) = \dim A$. Now let $Q \in S$ be such that $\dim(A/I) = \dim(A/Q)$. Then $\mathrm{ht}\, I \leq \mathrm{ht}\, Q$ and $\mathrm{ht}\, I + \dim(A/I) \leq \mathrm{ht}\, Q + \dim A/Q = \dim A$. Thus $\mathrm{ht}\, I + \dim(A/I) = \dim A$.

(c) Consider prime ideals $P \subseteq Q$ of A. Since A_Q is a CM-ring, by (b) $\dim A_Q = \mathrm{ht}\, PA_Q + \dim(A/P)_Q$. Note that $\dim A_Q = \mathrm{ht}\, Q$, $\mathrm{ht}\, PA_Q = \mathrm{ht}\, P$, and $\dim(A/P)_Q = \mathrm{ht}(Q/P)$. This shows that $\mathrm{ht}(Q/P) = \mathrm{ht}\, Q - \mathrm{ht}\, P$. Consider an intermediate prime ideal: $P \subseteq W \subseteq Q$. Then $\mathrm{ht}(Q/W) = \mathrm{ht}\, Q - \mathrm{ht}\, W$ and $\mathrm{ht}(W/P) = \mathrm{ht}\, W - \mathrm{ht}\, P$. Thus $\mathrm{ht}(Q/P) = \mathrm{ht}\, Q - \mathrm{ht}\, P = \mathrm{ht}(Q/W) + \mathrm{ht}(W/P)$ and A is catenary.

(8.34) <u>Definition</u>: Let A be a Noetherian ring.

(a) An ideal $I \subseteq A$ is called <u>unmixed</u> if for all $P \in \mathrm{Ass}(A/I)$ $\mathrm{ht}\, P = \mathrm{ht}\, I$.

(b) The <u>unmixedness theorem</u> holds for the ring A if for every ideal $I = (a_1, \ldots, a_r) \subseteq A$

with $\mathrm{ht}\, I = r$ is unmixed.

(8.35) Remark: (a) Let A be a Noetherian ring and $I = (a_1, \ldots, a_r) \subseteq A$ an ideal with $\mathrm{ht}\, I = r$. I is unmixed if and only if I has no embedded primes.
(b) By (8.29) the zero ideal (0) in a local CM-ring A is unmixed.

(8.36) Definition: Let A be a Noetherian ring and M a finitely generated A-module.
(a) A is a Cohen-Macaulay ring if A_m is CM for all $m \in m\text{-Spec}(A)$.
(b) M is a Cohen-Macaulay module if M_m is CM (over A_m) for all $m \in m\text{-Spec}(A)$.

(8.37) Theorem: Let A be a Noetherian ring. A is CM if and only if the unmixedness theorem holds for A.

Proof: "$\Rightarrow$": Let A be a CM-ring and $I = (a_1, \ldots, a_r) \subseteq A$ an ideal with $\mathrm{ht}\, I = r$. For all $P \in \mathrm{Ass}(A/I)$ the ring A_P is CM and $a_1, \ldots, a_r$ is an A_P-sequence by (8.33). By (8.29) the ideal $I_P = (a_1, \ldots, a_r) A_P$ has no embedded prime ideals. Thus P is minimal over I and $\mathrm{ht}\, P = r = \mathrm{ht}\, I$.
"$\Leftarrow$": Suppose that the unmixedness theorem holds for A and let $P \in \mathrm{Spec}(A)$. We claim that A_P is a CM-ring. Suppose $\mathrm{ht}\, P = r$ and let $a_1, \ldots, a_r \in P$ with $\mathrm{ht}(a_1, \ldots, a_i) = i$ for $1 \le i \le r$. By the unmixedness theorem all $P \in \mathrm{Ass}(A/(a_1, \ldots, a_i))$ have height i. This implies that a_{i+1} is a NZD of $A/(a_1, \ldots, a_i)$ and $a_1, \ldots, a_r$ is a regular sequence in A_P. Thus $\dim A_P = \mathrm{ht}\, P = r = \mathrm{depth}\, A_P$ and A_P is CM.

(8.38) Theorem: Let A be a CM-ring. The polynomial ring $A[x_1, \ldots, x_n]$ in finitely many variables over A is a CM-ring.

Proof: We only need to show that the polynomial ring $A[x]$ in one variable is CM. Let $P \subseteq A[x]$ be a prime ideal and $m = P \cap A$ its contraction to A. The ring

$A[x]_P$ is a localization of $A_m[x]$ and we may assume that A is a local CM-ring with maximal ideal m and that $P \subseteq A[x]$ is a prime ideal with $P \cap A = m$. We claim that $A[x]_P$ is CM. Suppose $\dim A = n$ and let $a_1, \text{—}, a_n \in m$ be a regular sequence in A. Since $A[x]_P$ is flat over A, $a_1, \text{—}, a_n$ is also a regular sequence in $A[x]_P$, in particular, $\operatorname{depth} A[x]_P \geq n$.

1. <u>case</u>: $P = mA[x]$, then $\dim A[x]_P = n$ and $A[x]_P$ is CM.

2. <u>case</u>: $P \neq mA[x]$. Set $K = A/m$ and note that $A[x]/mA[x] \cong K[x]$. Thus $\overline{P} = P/mA[x]$ is a principal ideal generated by a monic irreducible polynomial $\overline{f}$. Let $f \in A[x]$ be a monic polynomial with $f + mA[x] = \overline{f}$. Since f is monic, f is regular on $A[x]/(a_1, \text{—}, a_n) \cong (A/(a_1, \text{—}, a_n))[x]$. Hence $\operatorname{depth}(A[x]_P) \geq n+1 = \dim(A[x]_P)$. $A[x]_P$ is a CM-ring.

(8.39) <u>Corollary</u>: Let A be a CM-ring and B an A-algebra of finite type. Then B is a catenary ring.

§3: REGULAR RINGS

(8.40) Definition: Let (A,m,k) be a local Noetherian ring. A is called regular if $\dim A = \mu(m) = \dim_k(m/m^2)$.

(8.41) Examples: Let A be a local Noetherian ring.
(a) If $\dim A = 0$, then A is regular if and only if A is a field.
(b) If $\dim A = 1$, then A is regular if and only if A is a DVR.

(8.42) Theorem (Tate): Let (A,m,k) be a local Noetherian ring and $p \in m$ an element such that (p) is a prime ideal of height one. Then A is a domain.

Proof: We first show that for every element $a \in A-(0)$ there is an element $a' \in A-(p)$ and an integer $r \in \mathbb{N}$ so that $a = p^r a'$. Since A is local Noetherian, $\bigcap_{r\in\mathbb{N}} (p^r) = (0)$ and there is an $r \in \mathbb{N}$ with $a \in (p^r)$ and $a \notin (p^{r+1})$. Then $a = p^r a'$ for some $a' \in A-(p)$.
Next let $a,b \in A-(0)$, $a',b' \in A-(p)$ and $r,s \in \mathbb{N}$ with $a = p^r a'$ and $b = p^s b'$. Then $ab = p^{r+s} a'b'$. Since (p) is a prime ideal, $a' \cdot b' \notin (p)$ and there is a minimal prime ideal $Q \subseteq A$ with $a'b' \notin Q$. Since $ht(p) = 1$, $(p) \notin Q$ and $ab = p^{r+s} a'b' \notin Q$. Thus $ab \neq 0$.

(8.43) Corollary: Let (A,m,k) be a local Noetherian ring which is not a domain. Every principal prime ideal of A is minimal.

(8.44) Theorem: A regular local ring (A,m,k) is a domain.

Proof: by induction on $\dim A$. If $\dim A = 0$, then A is a field. Assume $\dim A = n$ and let $P_1, \dots, P_r$ be the minimal prime ideals of A.
Claim: If A is not a domain then $m \subseteq m^2 \cup P_1 \cup \dots \cup P_r$.
Pf of claim: Let $x \in m - m^2$ and $x_2, \dots, x_n \in m$ so that $x + m^2, x_2 + m^2, \dots, x_n + m^2$ is a

basis of the k-vector space m/m^2. Thus in $A/(x)$ the maximal ideal $m/(x)$ is generated by $x_2+(x), \ldots, x_n+(x)$. By (4.44)(a) $\dim A/(x) \geq n-1$. If $x \notin P_i$ for all $1 \leq i \leq r$, then $\dim A/(x) \leq n-1$ and hence $\dim A/(x) = n-1$. In this case $A/(x)$ is a regular local ring and a domain by induction hypothesis. This implies that $(x) \subseteq A$ is a principal prime ideal of height one. By (8.42), A is a domain, a contradiction. Thus $x \in P_i$ for some $1 \leq i \leq r$ and the claim follows.

Thus if A is not a domain then $m \subseteq m^2 \cup P_1 \cup \ldots \cup P_r$. If $m \neq (0)$, by Nakayama $m \not\subseteq m^2$ and $m \subseteq P_i$ for some $1 \leq i \leq r$. But then $\dim A = 0$, a contradiction.

(8.45) <u>Theorem</u>: Let (A, m, k) be a local Noetherian ring and $x_1, \ldots, x_r \in m$ a minimal system of generators of m. The following are equivalent:

(a) A is regular

(b) Let $k[z_1, \ldots, z_r]$ be the polynomial ring in r variables over k. The homomorphism of rings $\varphi : k[z_1, \ldots, z_r] \longrightarrow gr_m(A) = \bigoplus_{i=0}^{\infty} m^i/m^{i+1}$ defined by $\varphi(z_i) = x_i + m^2 \in m/m^2$ is bijective.

<u>Proof</u>: (b) $\Rightarrow$ (a): Let $S_n = \{ f \in k[z_1, \ldots, z_r] \mid f \text{ homogeneous of degree } n \}$ be the k-vector space of homogeneous polynomials of degree n. If φ is bijective, $S_n \cong m^n/m^{n+1}$ as k-vector spaces. Thus $\ell_A(m^n/m^{n+1}) = \binom{r+n-1}{r-1}$ and $\ell_A(A/m^n) = \sum_{i=0}^{n-1} \ell_A(m^i/m^{i+1}) = \binom{r+n-1}{r}$. $\ell_A(A/m^n)$ is a polynomial of degree r (in n) and therefore $\dim A = r = \dim_k(m/m^2)$.

(a) $\Rightarrow$ (b): Note that φ is always surjective and $I = \ker(\varphi)$ is a homogeneous ideal since φ is a homogeneous morphism. Thus $I = \bigoplus_{n \in \mathbb{N}} I_n$ where $I_n \subseteq S_n$. Suppose $I \neq (0)$ and let $u \in I_h$ with $u \neq 0$. Then for all $n \geq h$: $u S_{n-h} \subseteq I_n$ and $\ell_A(m^n/m^{n+1}) = \ell_A(S_n/I_n) \leq \binom{r+n-1}{r-1} - \binom{r+n-h-1}{r-1}$. $\binom{r+n-1}{r-1}$ and $\binom{r+n-h-1}{r-1}$ are polynomials in n of degree $r-1$ with the same leading coefficient. This implies that $\ell_A(m^n/m^{n+1})$ is a polynomial in n of degree $\leq r-2$. Thus $\dim A \leq r-1$, a contradiction.

(8.46) Proposition: Let (A, m, k) be a regular local ring. Every minimal system of generators of m is a regular sequence.

Proof: by induction on $n = \dim A$. If $n = 0$, then A is a field. If $n \geq 1$, let $m = (x_1, \ldots, x_n)$. Since A is a domain, x_1 is a regular element of A. Then $\dim A/(x_1) = n-1$ and $A/(x_1)$ is a regular local ring of dimension $n-1$. By induction hypothesis $x_2 + (x_1), \ldots, x_n + (x_1)$ is a regular sequence of $A/(x_1)$.

(8.47) Corollary: Every regular local ring is Cohen-Macaulay.

(8.48) Proposition: Let (A, m, k) be local Noetherian ring and $x_1, \ldots, x_n \in m$ a regular sequence of A with $m = (x_1, \ldots, x_n)$. Then:
(a) A is a regular local ring.
(b) $x_1, \ldots, x_n$ is a minimal system of generators of m.

Proof: Since $n \geq \operatorname{edim} A = \dim_k (m/m^2) \geq \dim A \geq \operatorname{depth} A = n$.

(8.49) Definition: Let (A, m, k) be a local Noetherian ring and $x_1, \ldots, x_n \in m$ a regular sequence of A. $x_1, \ldots, x_n$ is called a regular system of parameters (RSOP) if $m = (x_1, \ldots, x_n)$.

(8.50) Remark: A local Noetherian ring A has an RSOP if and only if A is regular.

(8.51) Corollary: Let (A, m, k) be a regular local ring and $x_1, \ldots, x_r \in m$. The following are equivalent:
(a) $x_1, \ldots, x_r$ are part of an RSOP.
(b) $x_1 + m^2, \ldots, x_r + m^2 \in m/m^2$ are linearly independent over k.
(c) $A/(x_1, \ldots, x_r)$ is a regular local ring with $\dim A/(x_1, \ldots, x_r) = \dim A - r$.

Proof: (a) $\Leftrightarrow$ (b): trivial.

Set $I=(x_1,\ldots,x_r)$ and $B=A/I$. B is a local Noetherian ring with maximal ideal $m_B = m/I$. Consider the exact sequence of k-vector spaces:

$$0 \longrightarrow I/I\cap m^2 \longrightarrow m/m^2 \longrightarrow m/m^2+I \cong m_B/m_B^2 \longrightarrow 0.$$

Let $s=\dim_k(I/I\cap m^2)$. Since A is regular, $n=\dim_k(m/m^2)=\dim A$ and by exactness: $\dim_k(m_B/m_B^2)=n-s$.

(b) $\Rightarrow$ (c): (b) implies that $s=r$ and therefore $\operatorname{edim} B=\dim_k(m_B/m_B^2)=n-r$. Since A is regular, A is a CM-ring and by (8.33): $\dim(A/I)=\dim A-\operatorname{ht} I$. Since $\operatorname{ht} I\le r$, $\dim B\ge n-r$ and B is regular.

(c) $\Rightarrow$ (b): $\dim B=\operatorname{edim} B=n-r$ implies that $\dim_k(I/I\cap m^2)=r$. The sequence $x_1+m^2,\ldots,x_r+m^2$ is linearly independent over k.

(8.52) Theorem: Let (A,m,k) be a regular local ring and $I\subseteq A$ an ideal. The ring A/I is regular if and only if I is generated by part of an RSOP.

Proof: "$\Leftarrow$": by (8.51)

"$\Rightarrow$": Suppose that A/I is regular. Let $x_1,\ldots,x_s\in m$ so that $x_1+I,\ldots,x_s+I$ is an RSOP of A/I. In particular, $m=(x_1,\ldots,x_s)+I$. Let $y_1,\ldots,y_r\in I$ so that $y_1+m^2,\ldots,y_r+m^2$ is a basis of $m^2+I/m^2\cong I/I\cap m^2$. From the exact sequence of k-vector spaces: $0\to m^2+I/m^2 \longrightarrow m/m^2 \longrightarrow m/m^2+I \longrightarrow 0$ we obtain that $x_1,\ldots,x_s,y_1,\ldots,y_r$ is an RSOP of A. Let $J=(y_1,\ldots,y_r)\subseteq I$. By (8.51) A/J is a regular local ring with $\dim A/J=s=\dim A/I$. Since A/I is a domain, $I=J$.

Homological description of regular local rings

(8.53) Lemma: Let (A,m,k) be a local Noetherian rings and M a finitely generated A-module. Suppose that $(F_\cdot,\partial_\cdot)$ is a minimal free resolution of M. Then

(a) $\dim_k \operatorname{Tor}_i^A(M,k)=\operatorname{rank} F_i$ for all i

(b) $\operatorname{projdim} M = \sup\{ i \mid \operatorname{Tor}_i^A(M,k) \neq 0 \} \leq \operatorname{projdim} k$.

Proof: First note that $(F_\bullet, \partial_\bullet)$ is a minimal free resolution of M if i) every F_i is a finitely generated free A-module, ii) $\partial_i(F_i) \subseteq F_{i-1}$ for all $i>0$, and iii) $F_0/mF_0 \cong M/mM_0$.
(a) Because of condition (ii), the boundary maps of the complex $(F_\bullet \otimes k, \partial_\bullet \otimes k)$ are all zero. Thus $\operatorname{Tor}_i^A(M,k) = H_i(F_\bullet \otimes k) = F_i \otimes k$. The dimension of the k-vector space $F_i \otimes k$ is the rank of F_i.
(b) follows from (a).

(8.54) Theorem: Let (A, m, k) be a local Noetherian ring, M a finitely generated A-module, and $x \in m$ an M-regular element. Then $\operatorname{projdim} M + 1 = \operatorname{projdim}(M/xM)$.

Proof: The exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ induces a long exact sequence:
$\operatorname{Tor}_{i+1}^A(k,M) \xrightarrow{x} \operatorname{Tor}_{i+1}^A(k,M) \longrightarrow \operatorname{Tor}_{i+1}^A(k, M/xM) \longrightarrow \operatorname{Tor}_i^A(k,M) \xrightarrow{x} \operatorname{Tor}_i^A(k,M) \longrightarrow \cdots$
For all $i \in \mathbb{N}$ $\operatorname{Tor}_i^A(k,M)$ is annihilated by m, thus multiplication by x is the zero map and the sequence $0 \to \operatorname{Tor}_{i+1}^A(k,M) \longrightarrow \operatorname{Tor}_{i+1}^A(k, M/xM) \longrightarrow \operatorname{Tor}_i^A(k,M) \to 0$ is exact for all $i \in \mathbb{N}$. If $\operatorname{projdim} M = r < \infty$, then $\operatorname{Tor}_{r+1}^A(k, M/xM) \neq 0$ and $\operatorname{Tor}_i^A(k, M/xM) = 0$ for all $i > r+1$. If $\operatorname{projdim} M = \infty$, then $\operatorname{Tor}_i^A(k,M) \neq 0$ for all $i \in \mathbb{N}$ and $\operatorname{Tor}_i^A(k, M/xM) \neq 0$ for all $i \in \mathbb{N}$.

(8.55) Corollary: Let (A, m, k) be a local Noetherian ring, M a finitely generated A-module, and $x_1, \ldots, x_s \in m$ an M-sequence. Then $\operatorname{projdim} M = \operatorname{projdim}(M/(x_1, \ldots, x_s)M) - s$.

(8.56) Corollary: Let (A, m, k) be a regular local ring with $\dim A = n$. Then $\operatorname{gldim} A = n$.

Proof: Let $x_1, \ldots, x_n$ be a RSOP of A. By (8.46) $x_1, \ldots, x_n$ is a regular sequence of A and by (8.55) $\operatorname{projdim} k = \operatorname{projdim}(A/(x_1, \ldots, x_n)) = \operatorname{projdim} A + n = n$. By (7.43) $\operatorname{gldim} A = \sup\{ \operatorname{projdim} A/I \mid I \text{ an } A\text{-ideal} \}$ and by (8.53) $\operatorname{projdim} M \leq \operatorname{projdim} k$

for every finitely generated A-module M. Thus $\operatorname{gldim} A = n$.

(8.57) Proposition: Let (A, m, k) be a local Noetherian ring, M a finitely generated A-module, and $x \in m$ an element which is A-regular and M-regular. Then $\operatorname{projdim}_{A/xA} M/xM \leq \operatorname{projdim}_A M$.

Proof: If $\operatorname{projdim} M = \infty$, we are done. If $\operatorname{projdim} M = n < \infty$ we proceed by induction on n. If $n = 0$, M is a projective A-module. Since A is local, M is free and M/xM is a free A/xA-module. If $n > 0$, consider an exact sequence $0 \to N \to A^r \to M \to 0$. Then $\operatorname{projdim} N = n-1$ and, since $N \subseteq A^r$, x is also an N-regular element. Consider the following commutative diagram with exact rows and colomns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "x"] & A^r \arrow[r] \arrow[d, "x"] & M \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & A^r \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & N/xN \arrow[r, "\alpha"] \arrow[d] & A^r/xA^r \arrow[r] \arrow[d] & M/xM \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the Snake Lemma α is injective and therefore $\operatorname{projdim}_{A/xA} N/xN = \operatorname{projdim}_{A/xA} M/xM - 1$. By induction hypothesis $\operatorname{projdim}_{A/x} N/xN \leq \operatorname{projdim}_A N$ and therefore:

$$\operatorname{projdim}_{A/xA} M/xM = \operatorname{projdim}_{A/xA} N/xN + 1 \leq \operatorname{projdim}_A N + 1 = \operatorname{projdim}_A M.$$

(8.58) Lemma: Let A be a Noetherian ring and M_1, M_2 finitely generated A-modules with $\operatorname{projdim}(M_1 \oplus M_2) = r < \infty$. Then $\operatorname{projdim} M_i \leq r$ for $i = 1, 2$.

Proof: For $i = 1, 2$ consider exact sequences: $0 \to K_i \to P_{i,r-1} \to \dots \to P_{i,0} \to M_i \to 0$ with $P_{i,j}$ projective A-modules. The direct sum of the two sequences:

$$0 \to K_1 \oplus K_2 \to P_{1,r-1} \oplus P_{2,r-1} \to \dots\dots \to P_{1,0} \oplus P_{2,0} \to M_1 \oplus M_2 \to 0$$

is exact. Since $\operatorname{projdim}(M_1 \oplus M_2) = r$, $K_1 \oplus K_2$ is a projective A-module. Thus K_i is a projective A-module and $\operatorname{projdim} M_i \leq r$.

(8.59) <u>Theorem</u>: Let (A,m,k) be a local Noetherian ring. A is regular if and only if $\operatorname{gldim} A<\infty$. Moreover, if A is regular then $\operatorname{gldim} A = \dim A$.

<u>Proof</u>: By (8.56) if A is regular then $\operatorname{gldim} A = \dim A$. Suppose that $\operatorname{gldim} A<\infty$ and proceed by induction on $n=\dim_k(m/m^2) = \operatorname{edim} A$. If $n=0$ then $m=m^2$ and by Nakayama $m=(0)$. A is a field. Suppose $n>0$. If every element of $m-m^2$ is a zerodivisor of A then $m\subseteq m^2 \cup \bigcup_{P\in \operatorname{Ass}(A)} P$ and $m\in \operatorname{Ass}(A)$. Thus $\operatorname{depth} A=0$. Since $\operatorname{projdim}_A k<\infty$, by Auslander – Buchsbaum (8.23) $\operatorname{projdim} k=0$ and k is a free A-module, a contradiction to $n>0$. Thus $m-m^2$ contains an A-regular element.

Let $x\in m-m^2$ be an A-regular element. The ring $B=A/xA$ is local Noetherian with maximal ideal $m_B = m/xA$ and embedding dimension $\dim_k(m_B/m_B^2) = n-1$. Moreover, $\dim B = \dim A -1$ by (4.44). If $\operatorname{gldim} B<\infty$, then B is regular by induction hypothesis and $\dim_k(m_B/m_B^2) = n-1 = \dim B$. In this case, $\dim A = n = \dim_k(m/m^2)$ and A is regular.

Thus it remains to show that $\operatorname{gldim} B<\infty$. From the proof of (8.53) we know that $\operatorname{gldim} B = \operatorname{projdim}_B k$. The exact sequence $0\longrightarrow m_B \longrightarrow B\longrightarrow k\longrightarrow 0$ yield that $\operatorname{projdim}_B k = \operatorname{projdim}_B(m_B)+1$ and it suffices to show that $\operatorname{projdim}_B(m_B)<\infty$. Note that $m_B = m/xA$. By (8.57) $\operatorname{projdim}_B(m/xm)\leq \operatorname{projdim}_A m<\infty$ and we need to compare $\operatorname{projdim}_B(m/xm)$ and $\operatorname{projdim}_B(m/xA)$.

<u>Claim</u>: $m_B = m/xA$ is a direct summand of the B-module m/xm.

<u>Pf of Cl</u>: Extend x to a minimal system of generators $x, x_2, \ldots, x_n$ of m and set $I = mx + (x_2,\ldots,x_n)$. Obviously, $Ax + I = m$. Let $a\in A$ with $ax\in Ax\cap I$. Then $ax = \ell x + a_2x_2 + \cdots + a_nx_n$ where $\ell\in m$ and $a_i\in A$. Since $x+m^2, x_2+m^2, \ldots, x_n+m^2$ is a basis of the k-vector space m/m^2, we have that $a\in m$ and $Ax\cap I = mx$. Thus $m_B = m/xA = xA+I/xA \cong I/Ax\cap I = I/mx$. Consider the B-linear map: $\varphi: I/xm \oplus xA/xm \longrightarrow m/xm$ defined by $\varphi(a,b) = a+b$. Obviously, φ is surjective. Moreover, $\varphi(a+xm, b+xm) = 0 \iff a+b\in xm = I\cap Ax \iff a,b\in xm$. φ is injective. Thus $m_B = m/xA$ is a direct summand of m/xm. By (8.58) $\operatorname{projdim}_B(m_B)<\infty$

(8.60) <u>Corollary</u>: Let (A, m, k) be a regular local ring and $P \in \mathrm{Spec}(A)$. Then A_P is a regular local ring.

<u>Proof</u>: Let $P \in \mathrm{Spec}(A)$ and $k(P) = (A/P)_P = A_P/PA_P$. We have to show that $\mathrm{gldim}\, A = \mathrm{projdim}_{A_P} k(P) < \infty$. Since $\mathrm{projdim}_A (A/P) < \infty$ there is an exact sequence of A-modules $0 \to P_m \to \dots \to P_0 \to A/P \to 0$ where P_i is a projective A-module for all i. Localizing at P yields an exact sequence of A_P-modules:
$0 \to (P_m)_P \to \cdots \to (P_0)_P \to k(P) \to 0$. Since $(P_i)_P$ are projective A_P-modules, the assertion follows.

(8.61) <u>Definition</u>: A Noetherian ring A is called <u>regular</u> if A_P is a regular local ring for all $P \in \mathrm{Spec}(A)$.

(8.62) <u>Remark</u>: Let A be a Noetherian ring.
(a) A is regular if and only if A_m is regular for all $m \in m\text{-}\mathrm{Spec}(A)$.
(b) If A is a domain of dimension one then A is regular if and only if A is a Dedekind domain.

(8.63) <u>Theorem</u>: Let $\varphi: (A, m, k) \to (B, n, k')$ be a faithfully flat morphism of local Noetherian rings. Then:
(a) If B is regular, so is A.
(b) If A and B/mB are regular, B is regular and $\dim B = \dim A + \dim B/mB$.

<u>Proof</u>: (a) Suppose that $\dim B = n$. Let M be a finitely generated A-module and $0 \to K \to F_{n-1} \to \dots \to F_0 \to M \to 0$ an exact sequence with finitely generated free A-modules F_i. Since B is flat over A, the sequence:
$0 \to K \otimes_A B \to F_{n-1} \otimes_A B \to \cdots \to F_0 \otimes_A B \to M \otimes_A B \to 0$ is exact and $F_i \otimes_A B$ is a finitely generated free B-module for all $0 \le i \le n-1$. Since B is regular of

dimension n, $K\otimes_A B$ is a free B-module. Thus $K\otimes_A B$ is a flat A-module. Let $0\to N'\to N$ be an exact sequence of A-modules. Then $0\to N'\otimes_A(K\otimes_A B)\to N\otimes_A(K\otimes B)$ is exact. Since B is faithfully flat over A, the sequence $0\to N'\otimes_A K\to N\otimes_A K$ is exact and K is a flat A-module. Since K is finitely generated and A is local, K is a free A-module and $\operatorname{projdim} M\leq n$. A is a regular local ring.

(b) Let $x_1,\dots,x_t\in m$ be a RSOP of A and $y_1,\dots,y_s\in n$ with $y_1+mB,\dots,y_s+mB$ a RSOP of B/mB. Obviously, $n=(x_1,\dots,x_t,y_1,\dots,y_s)$. We claim that $x_1,\dots,x_t,y_1,\dots,y_s$ is a B-regular sequence. For $1\leq i\leq t$: $0\to A/(x_1,\dots,x_i)\xrightarrow{x_{i+1}} A/(x_1,\dots,x_i)$ is injective and since B is flat over A $0\to B/(x_1,\dots,x_i)\xrightarrow{x_{i+1}} B/(x_1,\dots,x_i)$ is exact. Hence $x_1,\dots,x_t$ is a B-regular sequence. For $1\leq j\leq s$:

$$B/(x_1,\dots,x_s,y_1,\dots,y_{j-1})B \cong (B/mB)/(y_1+mB,\dots,y_{j-1}+mB)(B/mB)$$

and $x_1,\dots,x_t,y_1,\dots,y_s$ is a B-regular sequence. Thus

$$\dim A+\dim B/mB = t+s \geq \dim_{k'}(n/n^2)\geq \dim B\geq \operatorname{depth} B\geq t+s.$$

B is a regular local ring.

(8.64) Theorem: Let A be a regular ring. The polynomial ring $A[x_1,\dots,x_n]$ in n variables over A is a regular ring.

Proof: We have to show that $A[x]$ is regular. Let $Q\in\operatorname{Spec}(A[x])$ and set $P=Q\cap A\in\operatorname{Spec}(A)$. Since $A\to A[x]$ is flat, the induced morphism $A_P\to A[x]_Q$ is faithfully flat. Moreover, $(A[x]/PA[x])_Q\cong k(P)[x]_Q$ where $k(P)=(A/P)_P$. Since $k(P)[x]$ is regular, $A[x]_Q$ is regular by (8.63).

Regular local rings are factorial

(8.65) Lemma: Let A be a domain with field of quotients $K=Q(A)$, $n\in\mathbb{N}$ an integer, and $I\subseteq A$ an ideal with $I\oplus A^n\cong A^{n+1}$. Then I is principal.

Proof: Let $e_0, \ldots, e_n$ denote the canonical basis of A^{n+1} and let $\varphi: A\oplus A^n \xrightarrow{\simeq} I\oplus A^n \subseteq A\oplus A^n$ be an isomorphism. Considering φ as an A-linear map from A^{n+1} to A^{n+1} we write $\varphi(e_i) = \sum_{j=0}^{n} a_{ij} e_j$ with $a_{ij}\in A$ for all $0\leq i\leq n$. Let $M=(a_{ij})$ be the matrix of φ (considered as a map from A^{n+1} to A^{n+1}) and $d = \det M = \det(a_{ij})$ its determinant. Then $M\tilde{M} = \tilde{M}M = d\mathcal{J}$ where $\tilde{M}$ is the adjoint matrix of M and $\mathcal{J}$ is the $(n+1)\times(n+1)$ identity matrix. Let M_{io} be the $n\times n$-matrix obtained from M by eliminating the i^{th} row and j^{th} column. The first row of $\tilde{M}$ is $(d_0, -d_1, \ldots, (-1)^n d_n)$ where $d_i = \det M_{io}$. From $\tilde{M}M = d\mathcal{J}$ we obtain that $\sum_{i=0}^{n} a_{io}(-1)^i d_i = d$ and $\sum_{i=0}^{n} a_{ij}(-1)^i d_i = 0$ for $1\leq j\leq n$. Since φ is injective, φ extends to an isomorphism on K^{n+1} and $d\neq 0$. With $f_0 = \sum_{i=0}^{n} (-1)^i d_i e_i$:

$$\varphi(f_0) = \varphi\Big(\sum_{i=0}^{n} (-1)^i d_i e_i\Big) = \sum_{i=0}^{n} (-1)^i d_i \varphi(e_i) =$$
$$= \sum_{i=0}^{n} (-1)^i d_i \Big(\sum_{j=0}^{n} a_{ij} e_j\Big) = \sum_{j=0}^{n} \Big(\sum_{i=0}^{n} a_{ij} (-1)^i d_i\Big) e_j = d e_0$$

and $d\in I$. Since φ is surjective onto $I\oplus A^n$, for all $1\leq j\leq n$ there is an $f_j\in A^{n+1}$ with $\varphi(f_j) = e_j$. Write $f_j = \sum_{k=0}^{n} c_{jk} e_k$ with $c_{jk}\in A$ for $0\leq j\leq n$ and $c_{0k} = (-1)^k d_k$. The $(n+1)\times(n+1)$ matrix $C = (c_{jk})$ defines an A-linear map $\psi: A^{n+1}\longrightarrow A^{n+1}$ with $\varphi\psi(e_0) = \varphi(f_0) = de_0$ and $\varphi\psi(e_j) = \varphi(f_j) = e_j$ for $1\leq j\leq n$. Thus

$$MC = \begin{bmatrix} d & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

and $\det MC = d = \det M \det C$. Thus $\det C = 1$ and ψ is an isomorphism on A^{n+1}. In particular, $f_0, \ldots, f_n$ is a basis of A^{n+1}. Since $\varphi(f_j) = e_j$ for $1\leq j\leq n$, we obtain that $I = (d)$.

(8.66) <u>Definition</u>: An A-module M is called <u>stably free</u> if there are finitely generated free A-modules F and F' with $M\oplus F\cong F'$.

(8.67) <u>Remark</u>: Let M be a stably free A-module. Then

(a) M is finitely generated.

(b) M is projective.

(c) There is an exact sequence $0 \to F \to F' \to M \to 0$ with F and F' finitely generated A-modules.

(8.68) <u>Lemma</u>: Let M be a finitely generated projective A-module and $0 \to F_n \to \ldots \to F_0 \to M \to 0$ an exact sequence with finitely generated free A-modules F_i. Then M is stably free.

<u>Proof</u>: We proceed by induction on n. If $n=1$, $0 \to F_1 \to F_0 \to M \to 0$ is exact and $F_0 \cong F_1 \oplus M$ since M is projective. If $n>1$ consider the exact sequences:
$0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to K \to 0$ and $0 \to K \to F_0 \to M \to 0$.
Since M is projective, $F_0 \cong K \oplus M$ and K is projective. By induction hypothesis K is stably free. Thus $F^* \oplus K \cong F'$ for finitely generated free A-modules F^* and F'. Therefore $F^* \oplus F_0 \cong F^* \oplus K \oplus M \cong F' \oplus M$. M is stably free.

Note that a Noetherian domain A is factorial if and only if every height one prime ideal of A is principal.

(8.69) <u>Theorem</u>: Let A be a Noetherian domain, $\Gamma \subseteq A$ a set of prime elements of A, and $S \subseteq A$ the multiplicative set generated by Γ, i.e. $S = \{1\} \cup \{p_1^{e_1} \cdots p_n^{e_n} \mid n \in \mathbb{N}, e_i \in \mathbb{N}, p_i \in \Gamma\}$. If $S^{-1}A$ is factorial, A is factorial.

<u>Proof</u>: Let $P \subseteq A$ be a height one prime ideal. If $P \cap S \neq \emptyset$ then there is an element $p \in \Gamma$ with $p \in P$ and $P = pA$. If $P \cap S = \emptyset$ consider the set of ideals $\Delta = \{pA \mid p \in P$ and $PS^{-1}A = pS^{-1}A\}$. Since $S^{-1}A$ is factorial, $\Delta \neq \emptyset$, and Δ contains a maximal element pA. If $q \in \Gamma$ with $q \mid p$ then $p = qt$ for some $t \in A$. Thus $t \in P$ with $pS^{-1}A = tS^{-1}A = PS^{-1}A$. Therefore $tA \in \Delta$ and $pA = tA$ be the maximality of pA. This implies that q is a unit in A, a contradiction. Thus for all $q \in \Gamma$: $q \nmid p$.
Let $x \in P$ and $s \in S$ with $sx = py$ for some $y \in A$. Write $s = p_1 \cdots p_n$ with $p_i \in \Gamma$. Then $p_i \mid y$ for all $1 \leq i \leq n$ and $x \in pA$. Thus $P = pA$.

(8.70) Theorem: A regular local ring is factorial.

Proof: Let (A, m, k) be a regular local ring. The proof is by induction on $\dim A = n$. If $n=0$, A is a field and if $n=1$, A is a discrete valuation ring. Let $n>1$ and $x \in m - m^2$. Since xA is a prime ideal (8.51), x is a prime element of A. By (8.69) we have to show that A_x is factorial. Let $P \subseteq A_x$ be a prime ideal of height one and set $Q = A \cap P$. Then $P = QA_x$. Since A is a regular local ring, there is an exact sequence $0 \to F_n \to \cdots \to F_0 \to Q \to 0$ with F_i finitely generated free A-modules for all $0 \leq i \leq n$. Localization at x yields that $0 \to F_{nx} \to \cdots \to F_{0x} \to P \to 0$ is an exact sequence with F_{ix} finitely generated free A_x-modules. We claim that P is a projective A_x-module. Every prime ideal $W \in \operatorname{Spec}(A_x)$ corresponds to a prime ideal $W_0 = W \cap A$ with $W_0 \neq m$. Moreover, $A_{W_0} \cong (A_x)_W$ and $(A_x)_W$ is factorial by induction hypothesis. The height one prime ideal $P(A_x)_W$ is principal, thus projective. This implies that P is a projective A_x-module. By (8.68) P is stably free and by (8.65) P is principal.