CHAPTER VI: HOMOLOGICAL ALGEBRA I

§0: CATEGORIES AND FUNCTORS

(6.1) Definition: A category $\mathcal{C}$ consists of

(1) a class of objects, denoted $\text{obj}\,\mathcal{C}$

(2) pairwise disjoint sets of morphisms, denoted $\text{Hom}_{\mathcal{C}}(A,B)$, for every ordered pair of objects (A,B)

(3) compositions $\text{Hom}_{\mathcal{C}}(A,B) \times \text{Hom}_{\mathcal{C}}(B,C) \longrightarrow \text{Hom}_{\mathcal{C}}(A,C)$, denoted $(f,g) \mapsto gf$,

so that the following conditions are satisfied:

(a) for every object A, there exists an identity morphism $1_A \in \text{Hom}_{\mathcal{C}}(A,A)$ such that $f 1_A = f$ for all $f \in \text{Hom}_{\mathcal{C}}(A,B)$ and $1_A g = g$ for all $g \in \text{Hom}_{\mathcal{C}}(C,A)$.

(b) associativity of composition holds whenever possible: if $f \in \text{Hom}_{\mathcal{C}}(A,B)$, $g \in \text{Hom}_{\mathcal{C}}(B,C)$ and $h \in \text{Hom}_{\mathcal{C}}(C,D)$, then $h(gf) = (hg)f$.

(6.2) Remark: (a) $\text{Hom}_{\mathcal{C}}(A,B)$ is required to be a set. Note that $\text{Hom}_{\mathcal{C}}(A,B)$ may be empty.

(b) For $f \in \text{Hom}_{\mathcal{C}}(A,B)$ we write $f : A \longrightarrow B$, although the elements of $\text{Hom}_{\mathcal{C}}(A,B)$ may not be maps.

(c) The identity morphism $1_A \in \text{Hom}_{\mathcal{C}}(A,A)$ is unique.

(6.3) Examples: (a) $\mathcal{C}$ = sets: The objects are sets, morphisms are functions, and the composition is the usual composition of functions.

(b) $\mathcal{C}$ = rings: Objects are rings, morphisms are homomorphisms of rings, composition is the usual composition of functions.

(c) $\mathcal{C}$ = groups: Objects are groups, morphisms are homomorphisms of groups, composition is the usual composition of functions.

(d) $\mathcal{C}$ = top: Objects are topological spaces, morphisms are continuous functions, composition is the usual composition of functions

(c) $\mathcal{C} = A$-mod: A a commutative ring with identity, objects are A-modules, morphisms are A-linear maps, composition is the composition of functions.

(6.4) Definition: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a function satisfying:
(a) If $A \in \operatorname{obj} \mathcal{C}$ then $FA \in \operatorname{obj} \mathcal{D}$
(b) If $f: A \longrightarrow B$ is a morphism in $\mathcal{C}$ then $Ff: FA \longrightarrow FB$ is a morphism in $\mathcal{D}$.
(c) If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathcal{C}$, then $F(gf) = Fg\, Ff$.
(d) For every $A \in \operatorname{obj} \mathcal{C}$: $F(1_A) = 1_{FA}$.

(6.5) Examples: (a) The identity functor $F: \mathcal{C} \longrightarrow \mathcal{C}$ defined by $FA = A$ and $Ff = f$.
(b) The localization functor: A a ring (commutative with 1), $S \subseteq A$ a multiplicative set. $F: A\text{-mod} \longrightarrow S^{-1}A\text{-mod}$ is defined by: if M is an A-module, then $FM = S^{-1}M$ and if $\varphi: M \longrightarrow N$ is an A-linear map then $F\varphi = S^{-1}\varphi: S^{-1}M \longrightarrow S^{-1}N$ is the induced $S^{-1}A$-linear map.

(6.6) Definition: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a function satisfying:
(a) If $A \in \operatorname{obj} \mathcal{C}$, then $FA \in \operatorname{obj} \mathcal{D}$
(b) If $f: A \longrightarrow B$ is a morphism in $\mathcal{C}$, then $Ff: FB \longrightarrow FA$ is a morphism in $\mathcal{D}$.
(c) If $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathcal{C}$, then $F(gf) = Ff\, Fg$.
(d) For every $A \in \operatorname{obj} \mathcal{C}$: $F(1_A) = 1_{FA}$.

Examples of contravariant functors: §1.

§1: THE FUNCTOR HOM

Througout this section: A is a commutative ring with identity, M, N, .. are A-modules and maps are A-linear.

(6.7) Definition: The set of A-linear maps from M to N is denoted by:

$$\mathrm{Hom}_A(M,N) = \{ f : M \longrightarrow N \mid f \text{ is A-linear} \}$$

(6.8) Remark: $\mathrm{Hom}_A(M,N)$ is an A-module under the operations:
For $f, g \in \mathrm{Hom}_A(M,N)$, $a \in A$, $m \in M$: $(f+g)(m) = f(m) + g(m)$ and $(af)(m) = a(f(m))$.
If A is a noncommutative ring, $\mathrm{Hom}_A(M,N)$ is an abelian group, but, in general, not an A-module.

(6.9) Definition: Let $\alpha : N \longrightarrow N'$ and $\beta : M \longrightarrow M'$ be A-linear maps of A-modules.
(a) α induces an A-linear map: $\mathrm{Hom}(M,\alpha) = \alpha_* : \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(M,N')$ defined by $\alpha_*(f) = \alpha \circ f$.
(b) β induces an A-linear map: $\mathrm{Hom}(\beta,N) = \beta^* : \mathrm{Hom}_A(M',N) \longrightarrow \mathrm{Hom}_A(M,N)$ defined by $\beta^*(g) = g \circ \beta$.

(6.10) Remark: (a) Obviously, $\mathrm{Hom}_A(M, \mathrm{id}_N) = (\mathrm{id}_N)_* = \mathrm{id}_{\mathrm{Hom}(M,N)}$. For A-linear maps $N_1 \xrightarrow{\alpha_1} N_2 \xrightarrow{\alpha_2} N_3$ we have that: $(\alpha_2 \circ \alpha_1)_* = \alpha_{2*} \circ \alpha_{1*}$. This shows that for a fixed A-module M, $\mathrm{Hom}_A(M,-)$ is a covariant functor from the category of A-modules into the category of A-modules.
(b) Similarly, for a fixed A-module N: $\mathrm{Hom}(\mathrm{id}_M, N) = (\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{Hom}(M,N)}$ and for A-linear maps $M_1 \xrightarrow{\beta_1} M_2 \xrightarrow{\beta_2} M_3$: $(\beta_2 \circ \beta_1)^* = \beta_1^* \circ \beta_2^*$. Thus, $\mathrm{Hom}_A(-,N)$ is a contravariant functor from the category of A-modules into the category of A-modules.

(6.11) <u>Theorem</u>: (Left exactness of Hom) Let $0 \longrightarrow N' \xrightarrow{\alpha_1} N \xrightarrow{\alpha_2} N''$ and $M' \xrightarrow{\beta_1} M \xrightarrow{\beta_2} M'' \longrightarrow 0$ be exact sequences of A-modules. Then:

(a) $0 \to \mathrm{Hom}_A(M, N') \xrightarrow{\alpha_{1*}} \mathrm{Hom}_A(M, N) \xrightarrow{\alpha_{2*}} \mathrm{Hom}_A(M, N'')$ is exact for all A-modules M.

(b) $0 \to \mathrm{Hom}_A(M'', N) \xrightarrow{\beta_2^*} \mathrm{Hom}_A(M, N) \xrightarrow{\beta_1^*} \mathrm{Hom}_A(M', N)$ is exact for all A-modules N.

<u>Proof</u>: (a) α_{1*} is injective: Let $f \in \mathrm{Hom}_A(M, N')$ with $\alpha_{1*}(f) = \alpha_1 \circ f = 0$. Since α_1 is injective $f = 0$.

$\mathrm{im}\,\alpha_{1*} \subseteq \ker \alpha_{2*}$: By assumption $\alpha_2 \alpha_1 = 0$. Thus $\alpha_{2*} \circ \alpha_{1*} = (\alpha_2 \circ \alpha_1)_* = 0_* = 0$.

$\ker \alpha_{2*} \subseteq \mathrm{im}\,\alpha_{1*}$: Let $f \in \ker \alpha_{2*}$. Thus $\alpha_{2*}(f) = \alpha_2 f = 0$ and $f(M) \subseteq \ker \alpha_2 = \mathrm{im}\,\alpha_1 \cong N$.
There is an A-linear map $f' : M \longrightarrow N'$ with $\alpha_1 \circ f' = f$ and $f = \alpha_{1*}(f') \in \mathrm{im}(\alpha_{1*})$.

(b) β_2^* is injective: Let $f \in \mathrm{Hom}_A(M'', N)$ with $\beta_2^*(f) = f\beta_2 = 0$. Since β_2 is surjective: $f = 0$.

$\mathrm{im}\,\beta_2^* \subseteq \ker \beta_1^*$: Since $\beta_2\beta_1 = 0$: $(\beta_2\beta_1)_* = \beta_{1*}\beta_{2*} = 0_* = 0$.

$\ker \beta_1^* \subseteq \mathrm{im}\,\beta_2^*$: Let $f \in \ker \beta_1^*$, that is, $f : M \longrightarrow N$ with $\beta_1^*(f) = f\beta_1 = 0$.
Thus $f(\mathrm{im}\,\beta_1) = 0$ and since $\mathrm{im}\,\beta_1 = \ker \beta_2$: $f(\ker \beta_2) = 0$. This implies that f factors:

$$\begin{tikzcd} & M \arrow[r, "f"] \arrow[dl, "\beta_2"'] \arrow[d] & N \\ M'' \arrow[r, phantom, "\cong"] \arrow[urr, bend right=60, "f''"'] & M/\ker(\beta_2) \arrow[ur, "\bar{f}"] & \end{tikzcd}$$

There is an A-linear map:
$f'' : M'' \longrightarrow N$ with $f'' \circ \beta_2 = f$.
Thus $f \in \mathrm{im}(\beta_2^*)$.

(6.12) <u>Remark</u>: In general, neither $\mathrm{Hom}_A(M, -)$ nor $\mathrm{Hom}_A(-, N)$ is right exact.
Consider the exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ where $\alpha(n) = 2n$ for all $n \in \mathbb{Z}$ and β is the canonical map.

(a) Let $M = \mathbb{Z}/2\mathbb{Z}$. The sequence $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$ is <u>not</u> exact since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

(b) Let $N = \mathbb{Z}$. The sequence $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\beta^*} \mathrm{Hom}_A(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}_A(\mathbb{Z}, \mathbb{Z}) \to 0$ is <u>not</u> exact, since α^* is not surjective. For all $f : \mathbb{Z} \to \mathbb{Z}$: $\alpha^*(f) = f\alpha$ and $f(\alpha(n)) = f(2n) = 2f(n)$ for all $n \in \mathbb{Z}$. Thus $\alpha^*(f)$ is never surjective and $\mathrm{id}_{\mathbb{Z}} \notin \mathrm{im}(\alpha^*)$.

(6.13) Definition: An A-module P is called projective if for every surjective A-linear map $\beta: M \longrightarrow N$ and every A-linear map $\alpha: P \longrightarrow N$ there is an A-linear map $\gamma: P \longrightarrow M$ so that $\alpha = \beta\gamma$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists \gamma"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

(6.14) Theorem: An A-module P is projective if and only if the functor $\mathrm{Hom}_A(P,-)$ is exact, that is, for every exact sequence $0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$ of A-modules the sequence $0 \to \mathrm{Hom}_A(P,N') \xrightarrow{\alpha_*} \mathrm{Hom}_A(P,N) \xrightarrow{\beta_*} \mathrm{Hom}_A(P,N'') \longrightarrow 0$ is exact.

Proof: "$\Rightarrow$": Suppose that P is projective and let $0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$ be an exact sequence. Since $\mathrm{Hom}_A(P,-)$ is left exact, the sequence $0 \longrightarrow \mathrm{Hom}_A(P,N') \xrightarrow{\alpha_*}$ $\mathrm{Hom}_A(P,N) \xrightarrow{\beta_*} \mathrm{Hom}_A(P,N'')$ is exact. Let $f \in \mathrm{Hom}_A(P,N'')$. Since P is projective there is a $g \in \mathrm{Hom}_A(P,N)$ with $\beta g = f$ and β_* is surjective.

"$\Leftarrow$": Let $\mathrm{Hom}_A(P,-)$ be exact and suppose that there are given A-linear maps

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

with β surjective. Consider the short exact sequence $0 \to \ker(\beta) \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$. By assumption $\mathrm{Hom}_A(P,M) \xrightarrow{\beta_*} \mathrm{Hom}_A(P,N)$ is surjective. P is projective.

(6.15) Proposition: Every free A-module is projective.

Proof: Let F be a free A-module with basis $\{e_i\}_{i\in I}$ and consider a diagram of A-linear maps:

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

with β surjective.

For every $i \in I$ choose an element $m_i \in M$ with $\beta(m_i) = f(e_i)$. Since F is free there is an A-linear map $g: F \longrightarrow M$ with $g(e_i) = m_i \ \forall\, i \in I$. Then $\beta g = f$.

(6.16) Proposition: Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of

A-linear maps and A-modules. The following are equivalent:

(a) There is an A-linear map $\gamma: M \longrightarrow M'$ with $\gamma\alpha = id_{M'}$.

(b) There is an A-linear map $\delta: M'' \longrightarrow M$ with $\beta\delta = id_{M''}$.

(c) $M = im(\alpha) \oplus N$ for a submodule $N \subseteq M$.

The submodule N of (c) is isomorphic to M''.

Proof: (a) $\Rightarrow$ (c): Let $\gamma: M \longrightarrow M'$ with $\gamma\alpha = id_{M'}$. Put $N = ker(\gamma)$. For $m \in M$ obviously, $m - \alpha\gamma(m) \in ker(\gamma)$ and $M = im(\alpha) + N$. If $n \in im(\alpha) \cap N$ then $n = \alpha(t)$ for some $t \in M'$ and $\gamma(n) = 0 = \gamma\alpha(t) = t$. Thus $M = im(\alpha) \oplus N$.

(c) $\Rightarrow$ (a): Define $\gamma: M \longrightarrow M'$ by $\gamma = \tilde{\alpha}p$ where $p: im(\alpha) \oplus N \longrightarrow im(\alpha)$ is the projection and $\tilde{\alpha}: im(\alpha) \longrightarrow M'$ is defined by $\tilde{\alpha}(\alpha(m)) = m$ (since $M' \cong im(\alpha)$). Then $\gamma \circ \alpha = id_{M'}$.

(b) $\Rightarrow$ (c): Let $\delta: M'' \longrightarrow M$ be such that $\beta\delta = id_{M'}$. Put $N = im(\delta) \cong M''$. For $m \in M$ obviously, $m - \delta\beta(m) \in ker(\beta) = im(\alpha)$ and $M = im(\alpha) + N$. If $n \in im(\alpha) \cap N$ then $n = \delta(t)$ for some $t \in M'$ and $\beta(n) = 0 = \beta\delta(t) = t$. Hence $M = im(\alpha) \oplus N$.

(c) $\Rightarrow$ (b): Define $\delta: M'' \longrightarrow M$ as follows: Since $im(\alpha) = ker(\beta)$ and $M'' \cong M/ker(\beta)$ the A-linear map $\beta|_N: N \longrightarrow M''$ is an isomorphism. Let $\delta = i \circ (\beta|_N)^{-1}$ where $i: N \longrightarrow M$ is the embedding. Then $\beta \circ \delta = id_{M''}$.

(6.17) Definition: An exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ is called split (or split exact) if there is an A-linear map $\delta: M'' \longrightarrow M$ with $\beta\delta = id_{M''}$. (or equivalently, if there is an A-linear map $\gamma: M \longrightarrow M''$ with $\gamma\alpha = id_{M'}$.)

(6.18) Proposition: Every direct summand of a projective module is projective.

Proof: Let P be a projective module, $N, Q \subseteq P$ submodules and $P = N \oplus Q$. Consider a diagram of A-linear maps:

$$\begin{tikzcd} & Q \arrow[d, "f"] & \\ M \arrow[r, "\beta"] & M'' \arrow[r] & 0 \end{tikzcd}$$

with β surjective. Extend the diagram to:

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[ddl, "g"'] & \\ & Q \arrow[u, bend right, "i"'] \arrow[d, "f"] & \\ M \arrow[r, "\beta"] & M'' \arrow[r] & 0 \end{tikzcd}$$

where $p : P \longrightarrow Q$ is the projection and $i : Q \longrightarrow P$ is the embedding. Note that $p \circ i = \mathrm{id}_Q$. Since P is projective there is an A-linear map $g : P \longrightarrow M$ with $\beta g = f p$. Then $\beta \circ (g \circ i) = f \circ (p i) = f$ and Q is projective.

(6.19) <u>Proposition</u>: Let P be an A-module. The following are equivalent:

(a) P is projective

(b) P is (isomorphic to) a direct summand of a free module.

(c) Every exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is split exact.

<u>Proof</u>: (c) $\Rightarrow$ (b): Every module is a homomorphic image of a free module. Consider an exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow P \longrightarrow 0$ with F a free module. By (6.16) P is isomorphic to a direct summand of F.

(b) $\Rightarrow$ (a): (6.15) and (6.18)

(a) $\Rightarrow$ (c): Let $0 \longrightarrow N \longrightarrow M \xrightarrow{\beta} P \longrightarrow 0$ be an exact sequence. Consider the diagram:

$$\begin{tikzcd} & P \arrow[d, "\mathrm{id}_P"] \arrow[dl, "\delta"'] & \\ M \arrow[r, "\beta"] & P \arrow[r] & 0 \end{tikzcd}$$

Since P is projective there is an A-linear map $\delta : P \longrightarrow M$ with $\beta \delta = \mathrm{id}_P$.

(6.20) <u>Examples</u>: (a) Let $A = \mathbb{Z}/6\mathbb{Z}$. By the Chinese remainder theorem: $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. $P = \mathbb{Z}/2\mathbb{Z}$ is a projective A-module, but not a free A-module.

(b) Let A be a Dedekind domain which is not factorial (for example: $\mathbb{Z}[\sqrt{-5}\,]$). We will show later that every nonzero ideal of A is projective. Note that an ideal I of a domain R is free if and only if $I \neq (0)$ and I is principal. A nonfactorial domain is not a PID. Thus every nonprincipal ideal I of A is projective but not free.

(6.21) <u>Definition</u>: An A-module E is called <u>injective</u> if for every injective A-linear map $\alpha: N \longrightarrow M$ and every A-linear map $\delta: N \longrightarrow E$ there is an A-linear map $\sigma: M \to E$ which extends δ, that is, $\delta = \sigma\alpha$.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M \arrow[dl, "\sigma"] \\ & E & \end{tikzcd}$$

(6.22) <u>Theorem</u>: An A-module E is injective if and only if $\mathrm{Hom}_A(-,E)$ is exact, that is, for every exact sequence of A-modules $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ the sequence $0 \longrightarrow \mathrm{Hom}_A(M'',E) \xrightarrow{\beta^*} \mathrm{Hom}_A(M,E) \xrightarrow{\alpha^*} \mathrm{Hom}_A(M',E) \longrightarrow 0$ is exact.

<u>Proof</u>: "$\Rightarrow$:" Assume that E is injective and that $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ is an exact sequence. We need to show that $\mathrm{Hom}_A(M,E) \xrightarrow{\alpha^*} \mathrm{Hom}_A(M',E)$ is surjective. This follows immediately from the definition of injective modules.

"$\Leftarrow$:" Conversely, suppose that $\mathrm{Hom}_A(-,E)$ is exact and consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M \\ & E & \end{tikzcd}$$

with α injective. Let $N' = \mathrm{coker}(\alpha)$. Then the sequence $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} N' \longrightarrow 0$ is exact and so is the sequence:
$$0 \to \mathrm{Hom}_A(N',E) \xrightarrow{\beta^*} \mathrm{Hom}_A(M,E) \xrightarrow{\alpha^*} \mathrm{Hom}_A(N,E) \longrightarrow 0$$
Thus there is a $\sigma \in \mathrm{Hom}_A(M,E)$ with $\alpha^*(\sigma) = \sigma\alpha = \delta$.

(6.23) <u>Remark</u>: It is easy to show that the direct product of injective modules is injective. The direct sum of injective modules may not be injective (but finite direct sums are). There is an interesting theorem which states that a commutative ring A with identity is Noetherian if and only if every direct sum of injective A-modules is injective.

(6.24) <u>Proposition</u>: Every direct summand D of an injective A-module E is injective.

<u>Proof</u>: Let $E = D \oplus N$, $p: E \longrightarrow D$ the projection and $\lambda: D \longrightarrow E$ the embedding. In particular, $p\lambda = \mathrm{id}_D$. A diagram with α injective:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \\ & D & \end{tikzcd}$$ extends to the diagram $$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \arrow[ddl, "g"] \\ & D \arrow[d, "\lambda"] & \\ & E \arrow[u, bend left, "p"] & \end{tikzcd}$$

Since E is injective there is a map $g: M \to E$ with $g\circ\alpha = \lambda f$. Then $(p\circ g)\circ\alpha = (p\lambda)f = f$ and D is injective.

(6.25) Theorem: For an A-module E the following conditions are equivalent:

(a) E is injective

(b) Every exact sequence $0 \to E \to M \to N \to 0$ is split exact.

Proof: (a) $\Rightarrow$ (b): Let $0 \to E \xrightarrow{i} M \to N \to 0$ be exact. Since E is injective there is a map $f: M \to E$ such that
$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "i"] \arrow[d, "id"'] & M \arrow[dl, "f"] \\ & E & \end{tikzcd}$$
commutes. The sequence splits.

(b) $\Rightarrow$ (a): Consider a diagram
$$(*) \quad \begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \\ & E & \end{tikzcd}$$
with α injective.

Let $T = E\oplus M/W$ where $W = \{(f(m), -\alpha(m)) \mid m \in K\}$. W is a submodule of $E\oplus M$. Consider the A-linear maps $\alpha': E \to T$ defined by $\alpha'(e) = (e,0)+W$ and $f': M \to T$ with $f'(m) = (0,m)+W$. Diagram $(*)$ extends to the diagram:

$$(**) \quad \begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \arrow[d, "f'"] \\ & E \arrow[r, "\alpha'"] & T \end{tikzcd}$$

Claim 1: $(**)$ is commutative.

Proof of Cl 1: For any $m \in K$:
$$\begin{aligned} \alpha' f(m) &= (f(m),0) + W \\ &= (f(m),0) - (f(m),-\alpha(m)) + W \\ &= (0,\alpha(m)) + W \\ &= f'\alpha(m) \end{aligned}$$

Claim 2: α' is injective

Proof of Cl. 2: Let $e \in E$ with $\alpha'(e) = (e,0) + W = (0,0)$. Then there is an element $m \in K$ with $(e,0) = (f(m), -\alpha(m))$ in $E \oplus M$. Thus $\alpha(m) = 0$. Since α is injective, $m = 0$ and therefore $e = 0$. By assumption (b) there is a map $\beta : T \longrightarrow E$ with $\beta \circ \alpha' = id_E$. Let $g = \beta f'$. Then $g\alpha = \beta f' \alpha = \beta \alpha' f = f$. E is an injective A-module.

(6.26) Remark: The module T together with the maps α' and f' (in the proof of (6.25)) is called the pushout of the diagram

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & M \\ {\scriptstyle f}\downarrow & & \\ E. & & \end{array}$$

(6.27) Theorem: For an A-module E the following conditions are equivalent:

(a) E is injective

(b) Every A-linear map $f : I \longrightarrow E$, where $I \subseteq A$ is an ideal, extends to an A-linear map $g : A \longrightarrow E$.

Proof: (a) $\Rightarrow$ (b): trivial

(b) $\Rightarrow$ (a): Suppose there is given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M \\ & & {\scriptstyle f}\downarrow & & \\ & & E & & \end{array}$$

where α is injective.

We may assume that $N \subseteq M$ is a submodule. Consider the set:

$\mathcal{M} = \{(N', g') \mid N \subseteq N' \subseteq M$ a submodule, $g' : N' \longrightarrow E$ an A-linear map with $g'|_N = f\}$.

$\mathcal{M} \neq \emptyset$ since $(N, f) \in \mathcal{M}$. Define a partial order on $\mathcal{M}$ by:

$$(N', g') \leq (N'', g'') \Longleftrightarrow N' \subseteq N'' \text{ and } g''|_{N'} = g'.$$

In order to show that $\mathcal{M}$ is inductively ordered let $\mathcal{K} = \{(N_i, g_i)\}_{i \in I}$ be a chain in $\mathcal{M}$. Then $\tilde{N} = \bigcup_{i \in I} N_i$ is a submodule of M. The map $\tilde{g} : \tilde{N} \longrightarrow E$ with $\tilde{g}(n) = g_i(n)$ if $n \in N_i$ is well defined. Thus $(\tilde{N}, \tilde{g}) \in \mathcal{M}$ is an upper bound of $\mathcal{K}$. By Zorn's Lemma there is a maximal element $(N_0, g_0) \in \mathcal{M}$. If $N_0 = M$ we are done. If $N_0 \neq M$ let $m \in M - N_0$ and consider $I = \{a \in A \mid am \in N_0\}$. Obviously, I is an ideal of A. Define $h : I \longrightarrow E$ by $h(a) = g_0(am)$. h is an A-linear map. By assumption (b) h extends to an A-linear map $h': A \longrightarrow E$. Let $N_1 = N_0 + Am$ and define $g_1: N_1 \longrightarrow E$ by $g_1(n_0 + am) = g_0(n_0) + a h'(1)$ for all $n_0 \in N$ and $a \in A$.

<u>Claim</u>: g_1 is well defined

<u>Proof of Cl</u>: Suppose $n_0 + am = n_0' + a'm$ for some $n_0, n_0' \in N_0$ and $a, a' \in A$. Then $n_0 - n_0' = (a'-a)m \in N_0$ and $a' - a \in I$. Thus $g_0(n_0 - n_0') = g_0((a'-a)m) = h(a'-a) = (a'-a)h'(1)$ Hence $g_0(n_0) - g_0(n_0') = a'h'(1) - a h'(1)$ and $g_0(n_0) + a h'(1) = g_0(n_0') + a' h'(1)$.

g_1 is a well defined A-linear map which extends g_0. Therefore $(N_1, g_1) \in \mathcal{M}$ and $(N_0, g_0) \lneq (N_1, g_1)$, a contradiction. This implies that $N_0 = M$.

(6.28) <u>Example</u>: By (6.27) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. In general, it is not so easy to write down the elements of an injective A-module. In a later section we will show that every A-module M is a submodule of an injective A-module E.

§2: THE TENSOR PRODUCT

(6.29) Definition: Let A be a ring; M, N, and T A-modules. A map $\varphi : M \times N \longrightarrow T$ is called A-bilinear if

(a) For all $m \in M$ the map $\psi_m : N \longrightarrow T$ defined by $\psi_m(n) = \varphi(m,n)$ is A-linear.

(b) For all $n \in N$ the map $\eta_n : M \longrightarrow T$ defined by $\eta_n(m) = \varphi(m,n)$ is A-linear.

(6.30) Definition: Let M and N be A-modules. A tensor product of M and N over A is an A-module $M \otimes_A N$ together with an A-bilinear map $\tau : M \times N \longrightarrow M \otimes_A N$ such that for every A-bilinear map $\varphi : M \times N \longrightarrow T$ from $M \times N$ into some A-module T there is a unique A-linear map $\alpha : M \otimes_A N \longrightarrow T$ with $\alpha \circ \tau = \varphi$, that is, the diagram:

$$\begin{tikzcd} M \times N \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & M \otimes_A N \arrow[dl, "\alpha"] \\ & T & \end{tikzcd}$$

commutes.

(6.31) Proposition: Let M and N be A-modules. If the tensor product of M and N over A exists it is unique up to isomorphism.

Proof: Let $(M \otimes_A N, \tau : M \times N \longrightarrow M \otimes_A N)$ and $(M \tilde{\otimes}_A N, \tilde{\tau} : M \times N \longrightarrow M \tilde{\otimes}_A N)$ be two tensor products with bilinear maps τ and $\tilde{\tau}$. Then there is exactly one A-linear map $\alpha : M \otimes_A N \longrightarrow M \tilde{\otimes}_A N$ with $\alpha \circ \tau = \tilde{\tau}$ and exactly one A-linear map $\tilde{\alpha} : M \tilde{\otimes}_A N \longrightarrow M \otimes_A N$ with $\tilde{\alpha} \circ \tilde{\tau} = \tau$. Thus $\tilde{\alpha} \circ \alpha \circ \tau = \tau$ and $\alpha \circ \tilde{\alpha} \circ \tilde{\tau} = \tilde{\tau}$ implying that the following diagrams:

$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[d, "\tau"'] & M \otimes_A N \arrow[dl, shift left, "\tilde{\alpha}\alpha"] \arrow[dl, shift right, "id"'] \\ M \otimes_A N & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \times N \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\tau}"'] & M \tilde{\otimes}_A N \arrow[dl, shift left, "\alpha\tilde{\alpha}"] \arrow[dl, shift right, "id"'] \\ M \tilde{\otimes}_A N & \end{tikzcd}$$

commute. By uniqueness: $\tilde{\alpha} \circ \alpha = id_{M \otimes N}$ and $\alpha \tilde{\alpha} = id_{M \tilde{\otimes} N}$.

(6.32) Theorem: If M and N are A-modules, the tensor product of M and N over A exists.

<u>Proof</u>: Let $A^{(M\times N)}$ be the free A-module with basis $M\times N$ and let $\mathcal{U}\subseteq A^{(M\times N)}$ be the submodule which is generated by all elements of the form:

$$(m+m',n)-(m,n)-(m',n)$$
$$(m,n+n')-(m,n)-(m,n')$$
$$(am,n)-a(m,n)$$
$$(m,an)-a(m,n) \qquad \text{for } m,m'\in M,\ n,n'\in N, \text{ and } a\in A.$$

Let τ be the composition of maps: $M\times N \overset{i}{\hookrightarrow} A^{(M\times N)} \xrightarrow{\nu} A^{(M\times N)}/\mathcal{U}$ where i maps (m,n) into the basis element (m,n) of $A^{(M\times N)}$ and ν is the canonical map onto the quotient module.

<u>Claim</u>: $(A^{(M\times N)}/\mathcal{U}, \tau)$ is the tensor product of M and N over A.

<u>Proof of Cl</u>: Obviously, τ is A-bilinear.

Let T be an A-module and $\varphi: M\times N \longrightarrow T$ an A-bilinear map. Considering φ as a map from the set $M\times N$ into the A-module T we can extend φ uniquely to an A-linear map $\tilde{\alpha}: A^{(M\times N)} \longrightarrow T$. Since φ is A-bilinear, $\mathcal{U}\subseteq \ker(\tilde{\alpha})$, and there is a unique A-linear map $\alpha: A^{(M\times N)}/\mathcal{U} \longrightarrow$ such that the diagram:

$$\begin{tikzcd} M\times N \arrow[r, "\varphi"] \arrow[d, "i"'] \arrow[dr, "\tau"] & T \\ A^{(M\times N)} \arrow[r, "\nu"] \arrow[ur, bend right=60, "\tilde{\alpha}"'] & A^{(M\times N)}/\mathcal{U} \arrow[u, "\alpha"'] \end{tikzcd}$$

(6.33) <u>Remark</u>: (a) We use the following notation: $M\otimes_A N = A^{(M\times N)}/\mathcal{U}$ and $m\otimes n := \tau(m,n)$ for elements $m\in M$ and $n\in N$.

(b) For all $m,m'\in M$, $n,n'\in N$, and $a\in A$:
$$(m+m')\otimes n = m\otimes n + m'\otimes n$$
$$m\otimes(n+n') = m\otimes n + m\otimes n'$$
$$(am)\otimes n = m\otimes(an) = a(m\otimes n).$$

(c) Every element of $M\otimes N$ is of the form:
$$\textstyle\sum_{i=1}^{r} a_i(m_i\otimes n_i) = \sum_{i=1}^{r}(a_i m_i)\otimes n_i = \sum_{i=1}^{r} m_i\otimes(a n_i)$$
where $m_i\in M$, $n_i\in N$ and $a_i\in A$.

(6.34) <u>Example</u>: $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$

<u>Proof</u>: For all $q \in \mathbb{Q}/\mathbb{Z}$ there is an $n \in \mathbb{Z}-(0)$ with $nq = 0$ and for all $q \in \mathbb{Q}/\mathbb{Z}$ and all $m \in \mathbb{Z}-(0)$ there is a $q' \in \mathbb{Q}/\mathbb{Z}$ with $mq' = q$. For $a, b \in \mathbb{Q}/\mathbb{Z}$ let $n \in \mathbb{Z}-(0)$ with $na = 0$ and let $b' \in \mathbb{Q}/\mathbb{Z}$ with $nb' = b$. Then $a \otimes b = a \otimes (nb') = (na) \otimes b' = 0 \otimes b' = 0$.

Let $M_1, \dots, M_r$ be A-modules. Instead of starting with A-bilinear maps we can start with A-multilinear maps $\varphi: M_1 \times \dots \times M_r \longrightarrow T$ (these are maps which are A-linear in every 'variable'). The proofs of (6.31) and (6.32) can be adjusted accordingly to show existence and uniqueness of the 'multi-tensor product' $M_1 \otimes \dots \otimes M_r$. Note that $M_1 \otimes \dots \otimes M_r$ is generated by all products $m_1 \otimes \dots \otimes m_r$ where $m_i \in M_i$ for $1 \leq i \leq r$. This is summarized in the proposition:

(6.35) <u>Proposition</u>: Let $M_1, \dots, M_r$ be A-modules. There exists a pair (T, τ) consisting of an A-module T and an A-multilinear map $\tau: M_1 \times \dots \times M_r \longrightarrow T$ with the following property: For every A-module N and for every A-multilinear map $\varphi: M_1 \times \dots \times M_r \longrightarrow N$ there is a unique A-linear map $\alpha: T \longrightarrow N$ such that $\alpha \circ \tau = \varphi$. Moreover, if (T, τ) and (T', τ') are two pairs with this property, then there is a unique isomorphism $\psi: T \longrightarrow T'$ with $\psi \circ \tau = \tau'$.

(6.36) <u>Proposition</u>: Let M, N, P be A-modules. There are unique isomorphisms:

(a) $M \otimes_A N \cong N \otimes_A M$ with $m \otimes n \longmapsto n \otimes m$

(b) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P) \cong M \otimes_A N \otimes_A P$ with $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p) \longmapsto m \otimes n \otimes p$

(c) $A \otimes_A M \cong M$ with $a \otimes m \longmapsto am$

<u>Proof</u>: Homework

(6.37) <u>Proposition</u>: Let $M_i, i \in I$, and N be A-modules. Then:

$$\Big(\bigoplus_{i \in I} M_i \Big) \otimes_A N \cong \bigoplus_{i \in I} (M_i \otimes_A N)$$

Proof: The map $\varphi : (\bigoplus M_i) \times N \longrightarrow \bigoplus (M_i \otimes N)$ with $\varphi((m_i)_{i\in I}, n) = (m_i \otimes n)_{i\in I}$ is A-bilinear. Thus there is an A-linear map $\alpha : (\bigoplus M_i) \otimes N \longrightarrow \bigoplus (M_i \otimes N)$ with $\alpha((m_i) \otimes n) = (m_i \otimes n)$.

Conversely, for every $j \in I$ there is an A-bilinear map $\psi_j : M_j \times N \longrightarrow (\bigoplus M_i) \otimes N$ defined by $\psi_j(x,n) = (m_i) \otimes n$ where $m_i = 0$ for $i \neq j$ and $m_j = x$. Thus for every $j \in I$ there is an A-linear map $\beta_j : M_j \otimes N \longrightarrow (\bigoplus M_i) \otimes N$ with $\beta_j(x \otimes n) = (m_i) \otimes n$ where $m_i = 0$ for $i \neq j$ and $m_j = x$. By the universal property of the direct sum there is an A-linear map $\beta : \bigoplus (M_i \otimes N) \longrightarrow (\bigoplus M_i) \otimes N$ with $\beta((m_i \otimes n)_{i\in I}) = (m_i)_{i\in I} \otimes n$. The maps α and β are inverse to each other on the generators. Thus $\alpha \circ \beta = id_{\bigoplus (M_i \otimes N)}$ and $\beta \circ \alpha = id_{(\bigoplus M_i) \otimes N}$.

(6.38) <u>Definition</u>: Let A and B be rings and P a nonempty set. P is called a (A,B)-<u>bimodule</u> if P is an A-module and a B-module and the two module structures are compatible in the following sense: for all $p \in P$, $a \in A$, and $b \in B$: $a(pb) = (ap)b$.

(6.39) <u>Proposition</u>: Let A and B be rings, M an A-module, P an (A,B)-bimodule, and N a B-module. Then:

(a) $M \otimes_A P$ is naturally a B-module.

(b) $P \otimes_B N$ is naturally an A-module.

(c) $(M \otimes_A P) \otimes_B N \cong M \otimes_A (P \otimes_B N)$.

<u>Proof</u>: Homework

(6.40) <u>Remark</u>: Let $\varphi : A \longrightarrow B$ be a homomorphism of rings and N a B-module. N has an A-module structure by <u>restriction</u> <u>of</u> <u>scalars</u>: for all $a \in A$ and $n \in N$ define $an = \varphi(a)n$. N is an (A,B)-bimodule and B can be considered an A-module. If M is an A-module consider the A-module $M_B = B \otimes_A M$. M_B is a B-module via the following definition: For $b, b' \in B$ and $m \in M$ set $b(b' \otimes m) = (bb') \otimes m$. M_B is obtained from M by <u>extension</u> <u>of</u> <u>scalars</u> ore <u>base</u> <u>change</u>.

(6.41) Remark and Definition: Let $\varphi: M' \longrightarrow M$ and $\psi: N' \longrightarrow N$ be A-linear maps of A-modules. The map $\beta: M' \times N' \longrightarrow M \otimes_A N$ defined by $\beta(m', n') = \varphi(m') \otimes \psi(n')$ is A-bilinear. Thus there is a unique A-linear map $\varphi \otimes \psi: M' \otimes_A N' \longrightarrow M \otimes_A N$ with $\varphi \otimes \psi(m' \otimes n') = \varphi(m') \otimes \psi(n')$. If $\varphi_2: M'' \longrightarrow M'$ and $\varphi_1: M' \longrightarrow M$ are A-linear maps and if N is an A-module then $(\varphi_1 \circ \varphi_2) \otimes \mathrm{id}_N = (\varphi_1 \otimes \mathrm{id}_N) \circ (\varphi_2 \otimes \mathrm{id}_N)$. Similarly, if $\psi_2: N'' \longrightarrow N'$ and $\psi_1: N' \longrightarrow N$ are A-linear and M is an A-module, then $\mathrm{id}_M \otimes (\psi_1 \circ \psi_2) = (\mathrm{id}_M \otimes \psi_1)(\mathrm{id}_M \otimes \psi_2)$. This shows: for a fixed A-module M there is a covariant functor $\underline{M} \otimes_A -$ from the category of A-modules into the category of A-modules defined by: for $N \in \mathrm{obj}(A\text{-mod})$ $(M \otimes_A -)(N) = M \otimes_A N$ and for $\psi: N' \longrightarrow N$ A-linear $(M \otimes_A -)(\psi) = \mathrm{id}_M \otimes \psi$. Similarly, for a fixed N, $- \otimes_A \underline{N}$ is a covariant functor from the category of A-modules into the category of A-modules. $- \otimes_A N$ is defined by: $(- \otimes_A N)(M) = M \otimes_A N$ for $M \in \mathrm{obj}(A\text{-mod})$ and for $\varphi: M' \longrightarrow M$ A-linear: $(- \otimes_A N)(\varphi) = \varphi \otimes \mathrm{id}_N$.

(6.42) Theorem: (adjoint isomorphism) Let M, N, P be A-modules. Then there is an isomorphism:
$$\mathrm{Hom}_A(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

Proof: Let $\mathrm{Bilin}_A(M \times N, P) = \{\varphi: M \times N \longrightarrow P \mid \varphi \text{ A-bilinear}\}$. $\mathrm{Bilin}_A(M \times N, P)$ is an A-module which is isomorphic to $\mathrm{Hom}_A(M \otimes_A N, P)$. Thus it suffices to show that the A-modules $\mathrm{Bilin}_A(M \times N, P)$ and $\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ are isomorphic. Define:
$$\begin{aligned}
\Phi: \mathrm{Bilin}_A(M \times N, P) &\longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) \\
\varphi &\longmapsto \Phi(\varphi): M \longrightarrow \mathrm{Hom}_A(N, P) \\
&\qquad\qquad m \longmapsto \Phi(\varphi)(m) = \varphi(m, -): N \longrightarrow P \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \longmapsto \varphi(m, n).
\end{aligned}$$
Since φ is A-bilinear, $\Phi(\varphi)(m)$ is A-linear. Verify that $\Phi(\varphi)$ and Φ are A-linear. Conversely, define:
$$\begin{aligned}
\Psi: \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) &\longrightarrow \mathrm{Bilin}_A(M \times N, P) \\
\gamma &\longrightarrow \Psi(\gamma): M \times N \longrightarrow P \\
&\qquad\qquad (m, n) \longmapsto \gamma(m)(n)
\end{aligned}$$
$\Psi(\gamma)$ is A-bilinear and Ψ is A-linear. Φ and Ψ are inverse to each other, that is,

$\Psi \circ \Phi = id_{Bilin}$ and $\Phi \circ \Psi = id_{Hom}$.

(6.43) <u>Remark</u>: Theorem (6.42) is part of a larger theorem which states that the functors $- \otimes_A N$ and $Hom_A(N, -)$ are an <u>adjoint pair</u>. In particular, if $\alpha: M' \longrightarrow M$ is an A-linear map, then the diagram:

$$\begin{array}{ccc} Hom_A(M \otimes_A N, P) & \xrightarrow{\cong} & Hom_A(M, Hom_A(N,P)) \\ \downarrow (\alpha \otimes id)^* & & \downarrow \alpha^* \\ Hom_A(M' \otimes_A N, P) & \xrightarrow{\cong} & Hom_A(M', Hom_A(N,P)) \end{array}$$

commutes (Proof: Homework).

(6.44) <u>Lemma</u>: Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of A-modules. If for all A-modules N the sequence $Hom_A(M'', N) \xrightarrow{\beta^*} Hom_A(M, N) \xrightarrow{\alpha^*} Hom_A(M', N)$ is exact, then the sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is exact.

<u>Proof</u>: Suppose that $Hom_A(M'', N) \xrightarrow{\beta^*} Hom_A(M, N) \xrightarrow{\alpha^*} Hom_A(M', N)$ is exact for all N.

(a) Let $N = M''$. Then $0 = \alpha^* \circ \beta^*(id_{M''}) = id_{M''} \circ \beta \circ \alpha = \beta \circ \alpha$. Hence $im(\alpha) \subseteq ker(\beta)$.

(b) In order to show $ker(\beta) \subseteq im(\alpha)$ put $N = M/im(\alpha)$ and let $\nu: M \longrightarrow M/im(\alpha)$ be the canonical map. Since $\alpha^*(\nu) = \nu \circ \alpha = 0$ there is an A-linear map $\sigma: M'' \longrightarrow M/im(\alpha)$ with $\beta^*(\sigma) = \sigma \circ \beta = \alpha$:

$$\begin{array}{ccc} M & \xrightarrow{\beta} & M'' \\ \nu \downarrow & \swarrow \sigma & \\ M/im(\alpha) & & \end{array}.$$

Thus $im(\alpha) = ker(\nu) = ker(\sigma \circ \beta) \supseteq ker(\beta)$.

(6.45) <u>Theorem</u>: The functor $- \otimes_A N$ is right exact, that is, if $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules, then the sequence

$$M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact.

<u>Proof</u>: For every A-module P the sequence:

$$0 \to \mathrm{Hom}_A(M'', \mathrm{Hom}_A(N,P)) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P)) \longrightarrow \mathrm{Hom}_A(M', \mathrm{Hom}_A(N,P))$$

is exact. Thus by (6.42) and (6.43) the sequence:

$$0 \to \mathrm{Hom}_A(M'' \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M' \otimes_A N, P)$$

is exact for every A-module P. By (6.44) $M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$ is exact.

(6.46) Corollary: Let $I \subseteq A$ be an ideal and M an A-module. Then:

$$(A/I) \otimes_A M \cong M/IM.$$

Proof: Consider the exact sequence: $0 \to I \xrightarrow{i} A \longrightarrow A/I \longrightarrow 0$. Tensoring with M yields an exact sequence $I \otimes_A M \xrightarrow{i \otimes M} A \otimes_A M \longrightarrow (A/I) \otimes_A M \longrightarrow 0$ which can be extended to a commutative diagram with exact rows:

$$\begin{tikzcd}
 & I \otimes_A M \arrow[r, "i \otimes M"] \arrow[d, "\delta"'] \arrow[dr, phantom, "1"] & A \otimes_A M \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "2"] & (A/I) \otimes_A M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & IM \arrow[r, "\varepsilon"'] & M \arrow[r] & M/IM \arrow[r] & 0
\end{tikzcd}$$

where δ is defined by: $\delta(a \otimes m) = am$. φ is obtained from the A-bilinear map $\tau: (A/I) \times M \longrightarrow M/IM$ with $\tau(\bar{a}, m) = \overline{am}$. Thus $\varphi(\bar{a} \otimes m) = \overline{am}$ and φ is surjective. δ is also a surjective map and squares 1 and 2 are commutative. By diagram chasing φ is an isomorphism.

(6.47) Remark: If $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules and N is an A-module, in general, the sequence $0 \to M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$ is not exact. Example: Consider the sequence:

$$(*) \qquad 0 \to \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $\sigma(n) = 2n$ is the multiplication by 2. $(*)$ is exact, but

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sigma \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

is not exact since $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\sigma \otimes \mathrm{id}$ is the 0-map.

(6.48) Definition: An A-module N is called flat over A if for all exact sequences of

A-modules $0 \to M' \to M$ the sequence $0 \to M' \otimes_A N \to M \otimes_A N$ is exact.

(6.49) <u>Proposition</u>: Let $\{N_i\}_{i\in I}$ be a set of A-modules. The following are equivalent:

(a) $\bigoplus_{i\in I} N_i$ is flat over A.

(b) For all $i \in I$: N_i is flat over A.

<u>Proof</u>: If $\{M_i'\}_{i\in I}$ and $\{M_i\}_{i\in I}$ are sets of A-modules and $\{f_i : M_i' \to M_i\}$ are A-linear maps then there is a unique A-linear map $\oplus f_i : \oplus M_i' \to \oplus M_i$ with $(\oplus f_i)(m_j) = (f_i(m_i))$. Moreover, $\oplus f_i$ is injective if and only if f_i is injective for all $i \in I$.

For an A-linear map $\tau : M' \to M$ we obtain by (6.37) a commutative diagram:

$$\begin{array}{ccc} (\bigoplus_{i\in I} N_i) \otimes_A M' & \xrightarrow{id \otimes \tau} & (\bigoplus_{i\in I} N_i) \otimes_A M \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_{i\in I} (N_i \otimes_A M') & \xrightarrow{\oplus(id\otimes\tau)} & \bigoplus_{i\in I} (N_i \otimes M) \end{array}$$

where the vertical arrows are isomorphisms. The top line is injective if and only if the bottom line is. This proves the proposition.

(6.50) <u>Corollary</u>: Every projective module is flat. In particular, every free module is flat.

<u>Proof</u>: Since A is a flat A-module by (6.49) every free A-module is flat. Since projective modules are direct summands of free modules they are flat by (6.49).

(6.51) <u>Theorem</u>: Let A be a ring, $S \subseteq A$ a multiplicative set, and M an A-module. Then:

$$(S^{-1}A) \otimes_A M \cong S^{-1}M.$$

The isomorphism is natural in the following sense: if $\tau : M' \to M$ is an A-linear map then the diagram:

$$\begin{array}{ccc} (S^{-1}A) \otimes_A M' & \xrightarrow{S^{-1}A \otimes \tau} & (S^{-1}A) \otimes_A M \\ \cong \downarrow & & \downarrow \cong \\ S^{-1}M' & \xrightarrow{S^{-1}A} & S^{-1}M \end{array}$$

commutes.

Proof: The A-bilinear map $\alpha : S^{-1}A \times M \longrightarrow S^{-1}M$ with $\alpha(a/s, m) = am/s$ induces an A-linear map $\varphi : (S^{-1}A)\otimes_A M \longrightarrow S^{-1}M$ with $\varphi((a/s)\otimes m) = am/s$. Conversely, define $\psi : S^{-1}M \longrightarrow (S^{-1}A)\otimes_A M$ by $\psi(m/s) = (1/s)\otimes m$. ψ is well defined. If $m/s = m'/s'$ in $S^{-1}M$ then there is an element $t \in S$ with $tsm' = ts'm$. Thus $(1/s)\otimes m = (1/tss')\otimes ts'm = (1/tss')\otimes tsm' = (1/s')\otimes m'$. ψ is A-linear and obviously $\varphi\circ\psi = id_{S^{-1}M}$. For an element $(a/s)\otimes m \in (S^{-1}A)\otimes_A M$: $\psi\circ\varphi((a/s)\otimes m) = \psi(am/s) = (1/s)\otimes am = (a/s)\otimes m$. Since $\psi\circ\varphi$ is the identity on the generators of $(S^{-1}A)\otimes_A M$ it follows that $\psi\circ\varphi = id_{S^{-1}A\otimes M}$. It is easy to check that the isomorphism is natural.

(6.52) Remark: The isomorphism $(S^{-1}A)\otimes_A M \cong S^{-1}M$ of (6.51) is also $S^{-1}A$-linear.

(6.53) Corollary: Let A be a ring and $S \subseteq A$ a multiplicative set. Then $S^{-1}A$ is flat as an A-module.

Proof: Let $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of A-modules. By (1.36) the sequence $0 \longrightarrow S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M'' \longrightarrow 0$ is exact. By (6.51) this sequence is naturally isomorphic to $0 \longrightarrow (S^{-1}A)\otimes_A M' \longrightarrow (S^{-1}A)\otimes_A M \longrightarrow (S^{-1}A)\otimes_A M'' \longrightarrow 0$ which has to be exact too. Thus $S^{-1}A$ is flat.

(6.54) Proposition: If N and L are flat A-modules then $N\otimes_A L$ is a flat A-module.

Proof: Let $\tau : M' \longrightarrow M$ be an injective A-linear map of A-modules. Then $\tau\otimes N : M'\otimes N \to M\otimes N$ is injective since N is flat and $(\tau\otimes N)\otimes L : (M'\otimes N)\otimes L \longrightarrow (M\otimes N)\otimes L$ is injective since L is flat. By (6.36) $\tau\otimes(N\otimes L) : M'\otimes(N\otimes L) \longrightarrow M\otimes(N\otimes L)$ is injective.

(6.55) Theorem: Let N be an A-module. The following conditions are equivalent:
(a) a sequence of A-modules $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact if and only if the sequence $N\otimes_A M' \xrightarrow{1\otimes f} N\otimes_A M \xrightarrow{1\otimes g} N\otimes_A M''$ is exact.

(b) N is flat over A and for all $m \in m\text{-Spec}(A)$: $N/mN = 0$.

(c) N is flat over A and whenever $N \otimes_A M = 0$ for an A-module M then $M = 0$.

(d) N is flat over A and whenever $1 \otimes h = 0$ for an A-linear map $h: M' \to M$ then $h = 0$.

Proof: (a) $\Rightarrow$ (b): Obviously, N is flat over A. For $m \in m\text{-Spec}(A)$ consider the exact sequence $0 \to m \to A \to A/m \to 0$. Then $0 \to N \otimes m \to N \otimes A \to N \otimes (A/m) \to 0$ is exact. If $N/mN \cong N \otimes (A/m) = 0$ then $0 \to N \otimes m \to N \otimes A \to 0$ is exact and by (a) $0 \to m \to A \to 0$ is exact, a contradiction.

(b) $\Rightarrow$ (c): Let M be a nonzero A-module and $x \in M - (0)$. Then $\operatorname{ann}(x) \neq A$ and the submodule $Ax \subseteq M$ is isomorphic to $A/\operatorname{ann}(x)$. Let $\operatorname{ann}(x) \subseteq m$ for a maximal ideal $m \subseteq A$. Consider the exact sequence $Ax \to A/m \to 0$. Then $N \otimes Ax \to N \otimes (A/m) \to 0$ is exact. By assumption $N \otimes (A/m) \cong N/mN \neq 0$ implying that $N \otimes Ax \neq 0$. By flatness of N the sequence $0 \to N \otimes Ax \to N \otimes M$ is exact. Hence $N \otimes M \neq 0$.

(c) $\Rightarrow$ (d): Let $h: M' \to M$ be an A-linear map so that $N \otimes h: N \otimes M' \to N \otimes M$ is the zero map. With $\mathcal{U} = \operatorname{im}(h)$ the map h factors into $M' \xrightarrow{\alpha} \mathcal{U} \xrightarrow{\beta} M$ where $\alpha(m) = h(m)$ for all $m \in M$ and β the embedding of $\mathcal{U}$ into M. α is surjective while β is injective and $h = \beta\alpha$. Then $N \otimes h = (N \otimes \beta)(N \otimes \alpha)$ and by the flatness of N the map $N \otimes \beta$ is injective and $N \otimes \alpha$ is surjective. Since $N \otimes h = 0$ it follows that $N \otimes \alpha = 0$ and therefore $N \otimes \mathcal{U} = 0$. Thus $\mathcal{U} = 0$ and $h = 0$.

(d) $\Rightarrow$ (a): The flatness of N implies the forward direction "$\Rightarrow$".

"$\Leftarrow$": Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence with $N \otimes M' \xrightarrow{1 \otimes f} N \otimes M \xrightarrow{1 \otimes g} N \otimes M''$ exact. Put $\mathcal{U} = \operatorname{im}(f)$ and $V = \ker(g)$. Then $1 \otimes (gf) = N \otimes (gf) = (N \otimes g)(N \otimes f) = 0$ and by assumption (d): $g \circ f = 0$ and therefore $\mathcal{U} \subseteq V$.

Consider the exact sequence $0 \to \mathcal{U} \to V \to V/\mathcal{U} \to 0$ which induces an exact sequence $0 \to N \otimes \mathcal{U} \to N \otimes V \to N \otimes (V/\mathcal{U}) \to 0$. By the flatness of N: $N \otimes \mathcal{U} = N \otimes \operatorname{im}(f) = \operatorname{im}(N \otimes f)$ and $N \otimes V = N \otimes \ker(g) = \ker(N \otimes g)$. Since $N \otimes \mathcal{U} = N \otimes V$ the map $N \otimes V \to N \otimes (V/\mathcal{U}) \to 0$ is the zero map. Thus $V \to V/\mathcal{U}$ is the zero map and $\mathcal{U} = V$.

(6.56) <u>Definition</u>: An A-module N is called <u>faithfully</u> <u>flat</u> over A if N satisfies one of the equivalent conditions of Theorem (6.55).

(6.57) <u>Examples:</u> (a) Every free A-module is faithfully flat over A.
(b) The A-module $\bigoplus_{m \in m\text{-Spec}(A)} A_m$ is a faithfully flat A-module.
(c) $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, but $\mathbb{Q}$ is <u>not</u> faithfully flat over $\mathbb{Z}$.

(6.58) <u>Remark</u>: Let $f: A \longrightarrow B$ and $g: A \longrightarrow C$ be homomorphisms of rings. The A-module $D = B \otimes_A C$ is a commutative ring under the multiplication:

$$\left({\sum_i}' b_i \otimes c_i \right) \left({\sum_j}' \tilde{b}_j \otimes \tilde{c}_j \right) = {\sum_{i,j}}' b_i \tilde{b}_j \otimes c_i \tilde{c}_j$$

The canonical maps $B \longrightarrow B \otimes C$ with $b \longmapsto b \otimes 1$ and $C \longrightarrow B \otimes C$ with $c \longmapsto 1 \otimes c$ are homomorphisms of rings and $B \otimes_A C$ is a $(B\text{-}C)$-bialgebra.

§3: HOM AND TENSOR PRODUCT

(6.59) Remark and examples: Let A be a commutative diagram with identity; $S \subseteq A$ a multiplicative subset, and M and N A-modules. There is a commutative diagram of A-modules:

$$\begin{tikzcd} \mathrm{Hom}_A(M,N) \arrow[r, "\sigma"] \arrow[d, "i"'] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \\ S^{-1}(\mathrm{Hom}_A(M,N)) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $i = i_{\mathrm{Hom},S}$; $\sigma : \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ defined by $\sigma(f) = S^{-1}f$, and $\varphi : S^{-1}(\mathrm{Hom}_A(M,N)) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ defined by $\varphi(f/s) = \frac{1}{s} S^{-1}f$.

Example: Let $A = \mathbb{Z}$ and $S = A - (0) = \mathbb{Z} - (0)$.

(a) If $M = \mathbb{Q}$ and $N = \mathbb{Z}$ then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, but $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) \cong \mathbb{Q} \neq 0$. This shows that φ may not be injective.

(b) If $M = N = \mathbb{Q}/\mathbb{Z}$ then $\mathrm{id}_{\mathbb{Q}/\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ and $n \cdot \mathrm{id}_{\mathbb{Q}/\mathbb{Z}} \neq 0$ for all $n \in \mathbb{Z} - (0)$. Thus $S^{-1}(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})) \neq 0$. Since $S^{-1}(\mathbb{Q}/\mathbb{Z}) = 0$ we have that $\mathrm{Hom}_{\mathbb{Q}}(S^{-1}(\mathbb{Q}/\mathbb{Z}), S^{-1}(\mathbb{Q}/\mathbb{Z})) = 0$ and φ may not be injective.

(6.60) Definition: Let A be a ring and M an A-module. M is called finitely presentable or of finite presentation if there is an exact sequence $A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$ for some integers $m, n \in \mathbb{N}$.

(6.61) Remark: (a) If M is an A-module of finite presentation then M is finitely generated.
(b) If A is a Noetherian ring every finitely generated A-module is of finite presentation.
(c) If P is a finitely generated projective A-module then P is of finite presentation.

Proof: There is an $n \in \mathbb{N}$ and a surjective A-linear map $\varphi : A^n \longrightarrow P$. Since P is projective $A^n \cong P \oplus \ker(\varphi)$ and $\ker(\varphi) \cong A^n/P$. Thus there is a surjective map $\psi : A^n \longrightarrow \ker(\varphi)$ and an exact sequence: $A^n \longrightarrow A^n \longrightarrow P \longrightarrow 0$.

(6.62) <u>Theorem</u>: Let M be an A-module of finite presentation, $S \subseteq A$ a multiplicative subset, and N an A-module. The map $\varphi: S^{-1}(\mathrm{Hom}_A(M,N)) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ with $\varphi(f/t) = (1/t)\, S^{-1}f$ is an isomorphism of $S^{-1}A$-modules.

<u>Proof</u>: If $M = A^n$ is a finitely generated free A-module, then $\mathrm{Hom}_A(A^n, N) \cong N^n$. Thus $S^{-1}(\mathrm{Hom}_A(A^n,N)) \cong S^{-1}(N^n) \cong (S^{-1}N)^n \cong \mathrm{Hom}_{S^{-1}A}((S^{-1}A)^n, S^{-1}N) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.
For an arbitrary A-module M of finite presentation let $A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$ be exact.
The following diagram with exact rows is commutative:

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}(\mathrm{Hom}_A(M,N)) \arrow[r] \arrow[d, "\varphi"] & S^{-1}(\mathrm{Hom}_A(A^n,N)) \arrow[r] \arrow[d, "\cong"] & S^{-1}(\mathrm{Hom}_A(A^m,N)) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^n, N)) \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^m, N)
\end{tikzcd}$$

φ is an isomorphism.

(6.63) <u>Remark</u>: If M is a finitely generated A-module φ is injective but not necessarily surjective.

(6.64) <u>Corollary</u>: Let P be an A-module of finite presentation. P is projective if and only if P_m is a projective A_m-module for all $m \in m\text{-}\mathrm{Spec}(A)$.

<u>Proof</u>: "$\Rightarrow$": There is an A-module Q such that $P \oplus Q \cong A^n$ for some $n \in \mathbb{N}$. Since the tensor product commutes with direct sums for all $m \in m\text{-}\mathrm{Spec}(A)$: $P_m \oplus Q_m \cong (A_m)^n$.
P_m is a projective A_m-module.
"$\Leftarrow$": Let $M \xrightarrow{\varphi} M' \longrightarrow 0$ be an exact sequence of A-modules. The sequence $\mathrm{Hom}(P,M) \xrightarrow{\varphi_*} \mathrm{Hom}(P,M') \longrightarrow K \longrightarrow 0$ is exact where $K = \mathrm{coker}(\varphi_*)$. We have to show that $K = 0$. For all $m \in m\text{-}\mathrm{Spec}(A)$ P_m is a projective A_m-module and
$\mathrm{Hom}_{A_m}(P_m, M_m) \longrightarrow \mathrm{Hom}_{A_m}(P_m, M'_m) \longrightarrow 0$ is exact. By (6.62):

$$\begin{tikzcd}
\mathrm{Hom}_{A_m}(P_m, M_m) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{A_m}(P_m, M'_m) \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathrm{Hom}_A(P,M)_m \arrow[r] & \mathrm{Hom}_A(P,M')_m &
\end{tikzcd}$$

This shows that $K_m = 0$ for all $m \in m\text{-Spec}(A)$ and by the local-global principle $K = 0$.

(6.65) <u>Proposition</u>: Let A be a ring and $S \subseteq A$ a multiplicative subset. For A-modules M and N there are natural isomorphisms:

$$S^{-1}(M \otimes_A N) \cong (S^{-1}M) \otimes_A N \cong M \otimes_A (S^{-1}N) \cong (S^{-1}M) \otimes_A (S^{-1}N) \cong (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N).$$

<u>Proof</u>:
$$\begin{aligned} S^{-1}(M \otimes_A N) &\cong S^{-1}A \otimes_A (M \otimes_A N) \cong (S^{-1}A \otimes_A M) \otimes_A N \cong (S^{-1}M) \otimes_A N \\ &\cong (M \otimes_A N) \otimes_A S^{-1}A \cong M \otimes_A (N \otimes_A S^{-1}A) \cong M \otimes_A (S^{-1}N) \\ &\cong S^{-1}A \otimes_A (S^{-1}A \otimes_A (M \otimes_A N)) \cong S^{-1}A \otimes_A (M \otimes_A (S^{-1}N)) \cong (S^{-1}M) \otimes_A (S^{-1}N) \\ &\cong ((S^{-1}M) \otimes_{S^{-1}A} S^{-1}A) \otimes_A N \cong (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N). \end{aligned}$$

(6.66) <u>Proposition</u>: Let $\varphi : A \longrightarrow B$ be a homomorphism of rings and M a flat A-module. $M \otimes_A B$ is a flat B-module.

<u>Proof</u>: If $0 \longrightarrow N' \longrightarrow N$ is an exact sequence of B-modules, then
$0 \longrightarrow (M \otimes_A B) \otimes_B N' = M \otimes_A N' \longrightarrow (M \otimes_A B) \otimes_B N = M \otimes_A N$ is exact.

(6.67) <u>Corollary</u>: Let M be an A-module. M is flat over A if and only if M_m is flat over A_m for all $m \in m\text{-Spec}(A)$.

<u>Proof</u>: "$\Rightarrow$": By (6.66) $M_m = M \otimes_A A_m$ is flat over A_m.
"$\Leftarrow$": Let $0 \longrightarrow N' \xrightarrow{\varphi} N$ be an exact sequence of A-modules. With $K = \ker(\varphi \otimes M)$ the sequence $0 \longrightarrow K \longrightarrow N' \otimes M \xrightarrow{\varphi \otimes M} N \otimes M$ is exact. Thus for all $m \in m\text{-Spec}(A)$

$$\begin{tikzcd} 0 \arrow[r] & K_m \arrow[r] & (N' \otimes M)_m \arrow[r, "(\varphi \otimes M)_m"] \arrow[d, "\cong"'] & (N \otimes M)_m \arrow[d, "\cong"] \\ & & N'_m \otimes_{A_m} M_m \arrow[r, "\varphi_m \otimes M_m"] & N_m \otimes_{A_m} M_m \end{tikzcd}$$

is exact.

Since M_m is A_m-flat the map $\varphi_m \otimes M_m$ is injective. Since the diagram is commutative $(\varphi \otimes M)_m$ is injective and $K_m = 0$ for all $m \in m\text{-Spec}(A)$.

(6.68) Proposition: Let (A,m,K) be a local ring and M a finitely generated A-module. Then there is an $n \in \mathbb{N}$ and a surjective A-linear map $\varphi: A^n \longrightarrow M$ with $\ker(\varphi) \subseteq mA^n$.

Proof: Let $\dim_K(M/mM)=n$ and let $x_1, \ldots, x_n \in M$ so that $x_1+mM, \ldots, x_n+mM$ is a basis of the K-vector space M/mM. By Nakayama $M = Ax_1 + \ldots + Ax_n$. Define $\varphi: A^n \longrightarrow M$ by $\varphi(e_i)=x_i$ where $e_1, \ldots, e_n$ is the standard basis of A^n. The induced map $\varphi\otimes K: A^n \otimes_A K \longrightarrow M \otimes_A K$ is surjective with $\dim_K(A^n\otimes_A K) = \dim_K(M\otimes_A K) = n$. Thus $\varphi \otimes K$ is an isomorphism and $\ker(\varphi) \subseteq mA^n$.

(6.69) Corollary: Let (A,m) be a local ring. Every finitely generated projective A-module P is free.

Proof: Consider an exact sequence $(*)$ $0 \longrightarrow N \longrightarrow A^n \xrightarrow{\varphi} P \longrightarrow 0$ with $N = \ker(\varphi) \subseteq mA^n$. Since P is projective $A^n \cong N \oplus P$. Since $(*)$ is split exact for every A-module M the sequence $0 \to N\otimes_A M \longrightarrow A^n \otimes_A M \longrightarrow P \otimes_A M \longrightarrow 0$ is exact. In particular, for $K = A/m$ the sequence $0 \longrightarrow N/mN \longrightarrow A^n/mA^n \xrightarrow{\overline{\varphi}} P/mP \longrightarrow 0$ is exact. Since $N \subseteq mA^n$ $\overline{\varphi}$ is an isomorphism and $N/mN = 0$. N is a finitely generated A-module and by Nakayama $N=0$.

(6.70) Remark: Kaplansky showed that every projective module over a local ring is free.

Consider the commutative diagram of A-modules and A-maps with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & K' \arrow[r, "\alpha'"] \arrow[d, "f'"] & K \arrow[r, "\beta'"] \arrow[d, "f"] & K'' \arrow[d, "f''"] \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "g'"] & M \arrow[r, "\beta"] \arrow[d, "g"] & M'' \\
 & N' \arrow[r, "\alpha''"] \arrow[d] & N & \\
 & 0 & &
\end{tikzcd}$$

(6.71) Lemma: The A-linear map α'' in the above diagram is injective.

Proof: by diagram chasing: Let $x \in N'$ with $\alpha''(x) = 0$. Then there is a $y \in M$ with $g'(y) = x$. Thus $g\alpha(y) = \alpha'' g'(y) = \alpha''(x) = 0$ and $\alpha(y) \in \ker(g) = \operatorname{im}(f)$. There is an element $z \in K$ with $f(z) = \alpha(y)$ and $f'' \beta'(z) = \beta f(z) = \beta\alpha(y) = 0$. Since f'' is injective $\beta'(z) = 0$ and there is an element $w \in K'$ with $\alpha'(w) = z$. Then $\alpha f'(w) = f\alpha'(w) = f(z) = \alpha(y)$. Since α is injective $f'(w) = y$ and hence $x = g'(y) = g'f'(w) = 0$.

(6.72) Theorem: Let F be a flat A-module and $0 \to M' \xrightarrow{\alpha} M \to F \to 0$ an exact sequence of A-modules. For all A-modules N the sequence $0 \to M' \otimes_A N \xrightarrow{\alpha \otimes N} M \otimes_A N \to F \otimes_A N \to 0$ is exact.

Proof: We need to show that $\alpha \otimes N : M' \otimes_A N \to M \otimes_A N$ is injective. Consider an exact sequence $0 \to K \to P \to N \to 0$ with P a projective A-module. In particular, P is A-flat. This yields a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & M' \otimes K \arrow[r] \arrow[d] & M \otimes K \arrow[r] \arrow[d] & F \otimes K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \otimes P \arrow[r] \arrow[d] & M \otimes P \arrow[r] \arrow[d] & F \otimes P \arrow[r] \arrow[d] & 0 \\
 & M' \otimes N \arrow[r, "\alpha \otimes N"] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & F \otimes N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By (6.71) $\alpha \otimes N$ is injective.

(6.73) Proposition: Let (A, m, K) be a local ring, M an A-module of finite presentation, and N a finitely generated A-module. If $f : N \to M$ is an A-linear map with

$f \otimes K : N/\mathfrak{m}N \longrightarrow M/\mathfrak{m}M$ an isomorphism, then $\ker(f)$ is a finitely generated A-module.

<u>Proof</u>: $f \otimes K$ an isomorphism implies that $M = \mathrm{im}(f) + \mathfrak{m}M$ and by Nakayama $M = \mathrm{im}(f)$; f is surjective. By assumption there are integers $m, n \in \mathbb{N}$ and an exact sequence $A^m \xrightarrow{\alpha} A^n \xrightarrow{\beta} M \longrightarrow 0$. Consider the diagram:

$$\begin{tikzcd} & A^n \arrow[d, "\beta"] \arrow[dl, dotted, "g"'] & \\ N \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

Since A^n is projective there is an A-linear map $g : A^n \longrightarrow N$ with $\beta = fg$.
<u>Claim</u>: $\ker(f) = \mathrm{im}(g \circ \alpha)$ (which implies that $\ker(f)$ is finitely generated.
<u>Pf of Cl</u>: Tensoring with K yields maps:

$$\begin{tikzcd} A^n \otimes K \arrow[r, "g \otimes K"] \arrow[rr, bend right, "\beta \otimes K"'] & N \otimes K \arrow[r, "f \otimes K", "\cong"'] & M \otimes K \end{tikzcd}$$

$\beta \otimes K$ is surjective and $f \otimes K$ is an isomorphism. Thus $g \otimes K$ is surjective. By Nakayama g is surjective. Since $0 = \beta\alpha = fg\alpha$ we obtain that $\mathrm{im}(g \circ \alpha) \subseteq \ker(f)$. Let $x \in N$ with $f(x) = 0$ and let $y \in A^n$ with $g(y) = x$. Then $\beta(y) = fg(y) = f(x) = 0$ and there is a $z \in A^m$ with $\alpha(z) = y$. Then $x = g(y) = (g \circ \alpha)(z) \in \mathrm{im}(g \circ \alpha)$.

(6.74) <u>Corollary</u>: Let $(A, \mathfrak{m}, K)$ be a local ring and M a flat A-module of finite presentation. M is free.

<u>Proof</u>: By (6.68) there is an exact sequence $0 \longrightarrow N \xrightarrow{g} A^n \xrightarrow{f} M \longrightarrow 0$ with $N = \ker(f) \subseteq \mathfrak{m}A^n$. This implies that the K-linear map $f \otimes K : A^n \otimes K \longrightarrow M \otimes K$ is an isomorphism of K-vector spaces. By (6.72) the sequence $0 \longrightarrow N \otimes K \xrightarrow{g \otimes K} A^n \otimes K \xrightarrow{f \otimes K} M \otimes K \longrightarrow 0$ is exact. Thus $N \otimes_A K = N/\mathfrak{m}N = 0$. By (6.73) $N = \ker(f)$ is finitely generated and by Nakayama $N = 0$. M is free.

(6.75) <u>Corollary</u>: Let $(A, \mathfrak{m}, K)$ be a Noetherian local ring and M a finitely generated A-module. The following are equivalent:

(a) M is free.

(b) M is projective.

(c) M is flat.

(6.76) <u>Theorem</u>: Let A be a ring and P an A-module. The following are equivalent:

(a) P is finitely generated and projective.

(b) P is of finite presentation and flat.

(c) P is of finite presentation and for all $\mathfrak{m} \in \text{m-Spec}(A)$ the $A_{\mathfrak{m}}$-module $P_{\mathfrak{m}}$ is free.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (6.50) and (6.61)

(b) $\Rightarrow$ (c): By base change $P_{\mathfrak{m}} = P \otimes_A A_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module. Since $P_{\mathfrak{m}}$ is an $A_{\mathfrak{m}}$-module of finite presentation by (6.74) $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module.

(c) $\Rightarrow$ (a): By (6.64).

§4. More on Injective Modules

Recall Theorem (6.27): An A-module E is injective if and only if every A-linear map $f: I \to E$, where $I \subseteq A$ an ideal, extends to an A-linear map $g: A \to E$. In this section we want to show that every A-module M is isomorphic to a submodule of an injective A-module. In order to show this we use (6.27) to characterize injective modules over PID's.

(6.77) Definition: Let M be an A-module and $m \in M$, $a \in A$. m is divisible by a if $m = am'$ for some $m' \in M$. The A-module M is called divisible if every $m \in M$ is divisible by every nonzero divisor $a \in A$ (of A).

(6.78) Example: $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module.

(6.79) Proposition: Every injective A-module E is divisible.

Proof: Let $m \in E$ and $a \in A$ a nonzero divisor of A. Define $f: (a) \longrightarrow E$ by $f(ax) = xm$. Since a is a NZD of A, f is a well defined A-linear map with $f(a) = m$. By (6.27) f extends to an A-linear map $g: A \longrightarrow E$. If $g(1) = n$ then $g(a) = an = f(a) = m$.

(6.80) Proposition: Let A be a PID and M an A-module. The following are equivalent:

(a) M is divisible.

(b) M is injective.

Proof: (a)$\Rightarrow$(b): Let $I \subseteq A$ be an ideal and $f: I \longrightarrow M$ an A-linear map. We have to show that f extends to an A-linear map $g: A \longrightarrow M$. If $I = (0)$ there is nothing to show. If $I = (a) \neq (0)$ let $f(a) = m$. Since M is divisible and $a \in A$ is a NZD of A there is an $n \in M$ with $m = an$. The A-linear map $g: A \longrightarrow M$ with $g(1) = n$ extends f.

(b)$\Rightarrow$(a): By (6.79).

(6.81) <u>Lemma</u>: (a) Every quotient of a divisible module is divisible.
(b) Every direct summand of a divisible module is divisible.
(c) Every direct product of divisible modules is divisible.
(d) Every direct sum of divisible modules is divisible.

<u>Proof</u>: Homework

(6.82) <u>Proposition</u>: Every injective $\mathbb{Z}$-module M can be embedded into an injective $\mathbb{Z}$-module.

<u>Proof</u>: Write $M = F/S$ where $F = \bigoplus_{i\in I} \mathbb{Z}$ is a free $\mathbb{Z}$-module. Then $F = \bigoplus_{i\in I} \mathbb{Z} \subseteq \bigoplus_{i\in I} \mathbb{Q} = E$ and E is a divisible $\mathbb{Z}$-module. S is a $\mathbb{Z}$-submodule of E with $M = F/S \subseteq E/S$. E/S is divisible and thus injective by (6.80).

(6.83) <u>Lemma</u>: Let D be a divisible $\mathbb{Z}$-module and A a ring. Then $\mathrm{Hom}_{\mathbb{Z}}(A,D)$ is an injective A-module.

<u>Proof</u>: $\mathrm{Hom}_{\mathbb{Z}}(A,D)$ is an A-module under the operation: $(af)(x) = f(ax)$ for all $a, x \in A$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(A,D)$.
We have to show: For every ideal $I \subseteq A$ and every A-linear map $\varphi : I \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,D)$ φ extends to an A-linear map $\varrho : A \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,D)$. φ induces a map $g : I \longrightarrow D$ defined by $g(a) = [\varphi(a)](1_A)$. g is a homomorphism of $\mathbb{Z}$-modules and g extends to a $\mathbb{Z}$-linear map $\bar{g} : A \longrightarrow D$. Define $\varrho : A \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A,D)$ by $\varrho(a) : A \longrightarrow D$ is the map defined by $[\varrho(a)](x) = \bar{g}(ax)$ for all $a, x \in A$. For every $a \in A$ the map $\varrho(a)$ is a homomorphism of groups and ϱ is well defined. It is easy to see that ϱ is $\mathbb{Z}$-linear.
<u>Claim 1</u>: ϱ is A-linear.
<u>Pf of Cl.1</u>: Let $a, b, x \in A$. Then $\varrho(ab)(x) = \bar{g}((ab)x) = \bar{g}(b(ax)) = \varrho(b)(ax)$ and $\varrho(b)(ax) = [a\varrho(b)](x)$ by definition of the A-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A,D)$. Thus

$\rho(ab) = a\rho(b)$.

Claim 2: ρ extends φ.

Pf of Cl 2.: Let $a \in I$ and $x \in A$. Then $\rho(a)(x) = \bar{g}(ax) = g(ax) = [\varphi(ax)](1_A)$ since $ax \in I$. Then $[\varphi(ax)](1_A) = [x\varphi(a)](1_A) = \varphi(a)(x)$ by definition of the A-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, D)$. Thus $\rho(a) = \varphi(a)$ for all $a \in I$.

(6.84) Theorem: Let A be a ring and M an A-module. M can be embedded into an injective A-module.

Proof: Consider M as a $\mathbb{Z}$-module. By (6.82) there is a divisible $\mathbb{Z}$-module D and an injective $\mathbb{Z}$-linear map $\varepsilon: M \longrightarrow D$. Since $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ is left exact there is an injective $\mathbb{Z}$-linear map $\varepsilon_*: \mathrm{Hom}_{\mathbb{Z}}(A, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, D)$ with $\varepsilon_*(f) = \varepsilon f$. By definition of the A-module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ for all $a, x \in A$: $\varepsilon_*(af)(x) = (\varepsilon \circ (af))(x) = \varepsilon f(ax) = (\varepsilon f)(ax) = \varepsilon_*(f)(ax) = a(\varepsilon_*(f))(x)$. Thus $\varepsilon_*(af) = a(\varepsilon_*(f))$ and ε_* is A-linear. Every A-linear map $g: A \longrightarrow M$ is also $\mathbb{Z}$-linear. Thus $\mathrm{Hom}_A(A, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(A, M)$. By (6.82) $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is an injective A-module. Since $M \cong \mathrm{Hom}_A(A, M)$ we obtain the embedding: $M \cong \mathrm{Hom}_A(A, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, M) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{\mathbb{Z}}(A, D)$.

In the following we want to show that for every A-module M there is a smallest injective A-module which contains M.

(6.85) Definition: An essential extension of an A-module M is an A-module E which contains M such that for every nonzero submodule $N \subseteq E$ the intersection $N \cap M \neq 0$. If, in addition, $M \neq E$ then E is called a proper essential extension of M.

(6.86) Example: $\mathbb{Q}$ is a proper essential extension of $\mathbb{Z}$.

(6.87) Theorem: An A-module M is injective if and only if M has no proper essential extensions.

<u>Proof</u>: "$\Longrightarrow$": Suppose that M is injective and let $M \subseteq E$ be an essential extension of M. By (6.25) there is an A-module $N \subseteq E$ with $M \oplus N = E$. Thus $E = M$.

"$\Longleftarrow$": Suppose that M has no proper essential extensions. Let E be an injective A-module with $M \subseteq E$. Consider the set:

$$\mathfrak{M} = \{ N \subseteq E \mid N \text{ a submodule and } N \cap M = 0 \}.$$

Since $(0) \in \mathfrak{M}$, $\mathfrak{M} \neq \emptyset$. $\mathfrak{M}$ is partially ordered by inclusion and every chain in $\mathfrak{M}$ has an upper bound in $\mathfrak{M}$. Thus Zorn's Lemma applies and there is a maximal submodule $N_0 \subseteq E$ with $M \cap N_0 = 0$. The composition of A-linear maps: $\varphi = \pi i : M \xhookrightarrow{i} E \xrightarrow{\pi} E/N_0$ is injective and we may consider M as an A-submodule of E/N_0.

<u>Claim</u>: $M \subseteq E/N_0$ is an essential extension.

<u>Pf of Cl</u>: Every nonzero submodule $\overline{N} \subseteq E/N_0$ is image of a submodule $N \subseteq E$ with $N_0 \subsetneq N$. By the maximality of N_0: $N \cap M \neq 0$ and thus $M \cap \overline{N} \neq 0$.

Since M has no proper essential extensions, φ is surjective and $M \cong E/N_0$. This implies that $E = M + N_0$. Since $M \cap N_0 = 0$ we have that $E = M \oplus N_0$. By (6.24) every direct summand of an injective module is injective.

(6.88) <u>Theorem</u>: Let E be an A-module, $M \subseteq E$ a submodule. The following conditions are equivalent:

(a) E is a maximal essential extension of M, that is, no proper extension of E is an essential extension of M.

(b) E is injective and E is an essential extension of M.

(c) E is injective and there is no proper injective submodule E' of E with $M \subseteq E' \subsetneq E$.

Moreover, for every A-module M an A-module E exists so that conditions (a), (b) or (c) are satisfied.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (6.87) it suffices to show that E has no proper essential extensions. Suppose that $E \subseteq F$ is an essential extension of E and let $N \subseteq F$ be a nonzero submodule. $N_0 = N \cap E$ is a nonzero submodule of E. Thus $N_0 \cap M \neq 0$ and therefore

$N \cap M \neq 0$. F is an essential extension of M. By the maximality of E: $E = F$.

(b) $\Rightarrow$ (c): Suppose that $E' \subseteq E$ is an injective submodule with $M \subseteq E'$. By (6.25) there is a submodule $E'' \subseteq E$ with $E = E' \oplus E''$. Then $E'' \cap M = 0$ and therefore $E'' = 0$ and $E = E'$.

(c) $\Rightarrow$ (a): Consider the set:

$\mathcal{M} = \{N \subseteq E \mid N$ a submodule, $M \subseteq N$, and N is an essential extension of $M\}$.

Since $M \in \mathcal{M}$, $\mathcal{M} \neq \phi$, and $\mathcal{M}$ is partially ordered by inclusion. Let $\mathcal{N} = \{N_i\}_{i \in I} \subseteq \mathcal{M}$ be a chain in $\mathcal{M}$ and put $N_0 = \bigcup_{i \in I} N_i$. N_0 is a submodule of E with $M \subseteq N_0$. Let $U \subseteq N_0$ be a nonzero submodule. Then there is an $i \in I$ with $U \cap N_i \neq 0$. Since N_i is an essential extension of M: $(U \cap N_i) \cap M \neq 0$ and N_0 is an essential extension of M. By Zorn's Lemma $\mathcal{M}$ has a maximal element E'.

<u>Claim</u>: E' is a maximal essential extension of M.

<u>Pf of Cl</u>: Let N be an A-module with $M \subseteq E' \subseteq N$ so that N is an essential extension of M. We have a diagram:

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "i"'] & N \arrow[dl, dotted, "\varphi"] \\ & E & \end{tikzcd}$$

where i is the inclusion of E' into E. Since E is injective, there is an A-linear map $\varphi: N \to E$ with $\varphi|_{E'} = i$. Since N is an essential extension of M, N is an essential extension of E'. If $\ker(\varphi) \neq 0$, then $\ker(\varphi) \cap E' \neq 0$ and i fails to be injective. Thus $\ker(\varphi) = 0$ and $N \cong \varphi(N) \subseteq E$ is an essential extension of M. By the maximality of E': $\varphi(N) = E'$ and E' is a maximal essential extension of M.

By (a) $\Rightarrow$ (b) E' is an injective A-module. By (c) $E = E'$.

In order to show the existence of an A-module E with properties (a), (b), (c), let $\widetilde{E}$ be an injective A-module with $M \subseteq \widetilde{E}$. The proof (c) $\Rightarrow$ (a) shows that $\widetilde{E}$ contains a maximal essential extension E of M.

(6.89) <u>Definition</u>: Let M be an A-module. An A-module E that satisfies one of the equivalent conditions of (6.88) is called the <u>injective</u> <u>hull</u> of M.

The following proposition shows that the injective hull is unique up to isomorphism.

(6.90) <u>Proposition</u>: Let M be an A-module and E an injective hull of M.
(a) Let D be an injective A-module with $M \subseteq D$. If $i: M \longrightarrow D$ is the embedding then there is an injective A-linear map $\varphi: E \longrightarrow D$ with $\varphi|_M = i$.
(b) If E' is another injective hull of M then there is an isomorphism $\varphi: E \longrightarrow E'$ with $\varphi|_M = id_M$.

<u>Proof</u>: (a) Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, hook, "i"'] & E \arrow[dl, dotted, "\varphi"] \\ & D & \end{tikzcd}$$

By the injectivity of D there is an A-linear map $\varphi: E \longrightarrow D$ with $\varphi|_M = i$. If $\ker(\varphi) \neq 0$ then $\ker(\varphi) \cap M \neq 0$ since E is an essential extension of M. Thus $\ker(\varphi) = 0$ and φ is injective.
(b) If E' is another injective hull of M, by (a) there is an injective A-linear map $\varphi: E \longrightarrow E'$ with $\varphi|_M = id_M$. Then $im(\varphi)$ is an injective submodule of E' with $M \subseteq im(\varphi)$. By (6.88) $E' = im(\varphi)$ and φ is an isomorphism.

§5: THE TORSION FUNCTOR Γ_I

(6.91) Definition: Let A be a ring, $I \subseteq A$ an ideal, and M an A-module. The set
$$\Gamma_I(M) := \bigcup_{n\in\mathbb{N}} (0 :_M I^n)$$
is called the I-torsion of M.

(6.92) Remark: $\Gamma_I(M)$ is the set of all elements of M which are annihilated by some power of M. Obviously, $(0:_M I) \subseteq (0:_M I^2) \subseteq \dots \subseteq (0:_M I^n) \subseteq \dots$ is an increasing chain of submodules of M which is stationary if M is Noetherian. In any case, $\Gamma_I(M)$ is a submodule of M.

(6.93) Definition: A covariant functor $F: A\text{-mod} \longrightarrow A\text{-mod}$ from the category of A-modules into itself is called A-linear if the map $\mathrm{Hom}_A(M,N) \xrightarrow{\Phi_{M,N}} \mathrm{Hom}_A(FM,FN)$ defined by $\Phi_{M,N}(f) = Ff$ is A-linear for all A-modules M and N. That is, for all $f,g: M \longrightarrow N$ and all $a \in A$: $F(f+g) = Ff + Fg$ and $F(af) = aFf$.

(6.94) Proposition: $\Gamma_I(-)$ is a covariant A-linear functor from the category of A-modules into itself.

Proof: If $f: M \longrightarrow N$ is an A-linear map of A-modules then $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$. Let $\Gamma_I(f)$ be the restriction of f: $\Gamma_I(f) = f|_{\Gamma_I(M)}: \Gamma_I(M) \longrightarrow \Gamma_I(N)$. If $g: N \longrightarrow L$ is another A-linear map then obviously, $\Gamma_I(gf) = \Gamma_I(g)\Gamma_I(f)$. Moreover, $\Gamma_I(\mathrm{id}_M) = \mathrm{id}_{\Gamma_I(M)}$. $\Gamma_I(-)$ is a covariant functor from A-mod into itself. It is easy to verify that $\Gamma_I(-)$ is A-linear.

(6.95) Proposition: Let A be a Noetherian ring and $I, J \subseteq A$ ideals. Then:

(a) For all A-modules M: $\Gamma_I(\Gamma_J(M)) = \Gamma_{I+J}(M)$

(b) $\Gamma_I(-) = \Gamma_J(-)$ if and only if $\mathrm{rad}(I) = \mathrm{rad}(J)$.

Proof: (a) "$\subseteq$": Let $x \in \Gamma_I(\Gamma_J(M))$. Then $x \in \Gamma_J(M)$ and $I^r x = 0$ for some $r \in \mathbb{N}$. Thus $J^s x = 0$ and $I^r x = 0$ for some $r, s \in \mathbb{N}$ implying that $(I+J)^{r+s} x = 0$. Thus $x \in \Gamma_{I+J}(M)$.
"$\supseteq$": If $x \in \Gamma_{I+J}(M)$ then $(I+J)^r x = 0$ for some $r \in \mathbb{N}$. In particular, $I^r x = 0 = J^r x$. Thus $x \in \Gamma_I(\Gamma_J(M))$.
(b) "$\Rightarrow$": $\Gamma_I(A/\mathrm{rad}(J)) = \Gamma_J(A/\mathrm{rad}(J)) = A/\mathrm{rad}(J)$. Thus $I \subseteq \mathrm{rad}(J)$. Similarly, $J \subseteq \mathrm{rad}(I)$
"$\Leftarrow$": $\mathrm{rad}(I) = \mathrm{rad}(J)$ implies that $I^n \subseteq J$ for some $n \in \mathbb{N}$. Thus $\Gamma_J(M) \subseteq \Gamma_I(M)$ for every A-module M. Similarly, $\Gamma_I(M) \subseteq \Gamma_J(M)$.

(6.96) Proposition: The I-torsion functor $\Gamma_I(-)$ is left exact.

Proof: Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence of A-modules. We have to show that $0 \to \Gamma_I(L) \xrightarrow{\Gamma_I(f)} \Gamma_I(M) \xrightarrow{\Gamma_I(g)} \Gamma_I(N)$ is exact. Obviously, $\Gamma_I(f)$ is injective and $\Gamma_I(g)\Gamma_I(f) = \Gamma_I(gf) = 0$. It remains to show that $\ker(\Gamma_I(g)) \subseteq \mathrm{im}(\Gamma_I(f))$.
Let $x \in \ker(\Gamma_I(g))$. Then $g(x) = 0$ and there is an $r \in \mathbb{N}$ with $I^r x = 0$. Let $y \in L$ with $f(y) = x$. For all $a \in I^r$: $f(ay) = ax = 0$. Since f is injective $ay = 0$ for all $a \in I^r$.
Hence $y \in \Gamma_I(L)$ and $x \in \mathrm{im}(\Gamma_I(f))$.

(6.97) Proposition: Let A be a Noetherian ring and M an A-module. Then:
(a) $\Gamma_I(\Gamma_I(M)) = \Gamma_I(M)$
(b) $\Gamma_I(M/\Gamma_I(M)) = 0$
(c) If M is a finitely generated A-module then $\Gamma_I(M) = 0$ if and only if I contains a NZD of M (an M-regular element).

Proof: (a), (b) trivial
(c) "$\Rightarrow$": If $\Gamma_I(M) = 0$ then $I \nsubseteq P$ for all $P \in \mathrm{Ass}_A(M)$. Since $\mathrm{Ass}_A(M)$ is a finite set we have $I \nsubseteq \bigcup_{P \in \mathrm{Ass}_A(M)} P$. Every element $a \in I - \bigcup_{P \in \mathrm{Ass}_A(M)} P$ is a NZD on M
"$\Leftarrow$". If $a \in I$ is a NZD on M, then $am \neq 0$ for all $m \in M - (0)$. Thus $a^n m \neq 0$ for all $n \in \mathbb{N}$ and $m \in M - (0)$.

(6.98) <u>Remark</u>: Together with the Hom-functors $Hom_A(M,-)$ and $Hom_A(-,M)$, and the tensor functor $M \otimes_A -$, the torsion functor $\Gamma_I(-)$ is an important tool in commutative algebra. Its derived functors (Chapter $\overline{VII}$) are the local cohomology functors $H^i_I(-)$. For a deeper understanding of these functors more investigations into $\Gamma_I(-)$ are needed. For example, a basic result states that $\Gamma_I(-)$ is a direct limit of certain Hom-functors. This yields that its derived functors $H^i_I(-)$ are direct limits of derived Hom-functors.