CHAPTER III: FINITELY GENERATED ALGEBRAS OVER A FIELD

§1: ALGEBRAICALLY INDEPENDENT ELEMENTS; TRANSCENDENCE DEGREE

Let $K \subseteq L$ be a field extension. Recall that an element $\alpha \in L$ is called algebraic over K if there is a nonzero polynomial $f(x) \in K[x]$ with $f(\alpha) = 0$. α is algebraic over K if and only if the homomorphism of rings $\varphi : K[x] \longrightarrow L$ defined by $\varphi|_K = id_K$ and $\varphi(x) = \alpha$ is not injective. α is called transcendental or algebraically independent over K if φ is injective.

(3.1) Definition: A subset $\{\alpha_i\}_{i\in I} \subseteq L$ is called algebraically independent over K if the homomorphism of rings $\varphi : K[t_i]_{i\in I} \longrightarrow L$ from the polynomial ring in variables $\{t_i\}_{i\in I}$ over K into L, defined by $\varphi|_K = id_K$ and $\varphi(t_i) = \alpha_i$ for all $i \in I$, is injective. An element $\alpha \in L$ which is algebraically independent over K is called transcendental over K.

(3.2) Remark: The set $\{\alpha_i\}_{i\in I} \subseteq L$ is algebraically independent over K if and only if the set of 'monomials' $\{\prod_{i\in I} \alpha_i^{n_i} \mid n_i \in \mathbb{N}$ with $n_i = 0$ for almost all $i \in I\}$ is linearly independent over K.

(3.3) Definition: The field extension $K \subseteq L$ is called purely transcendental if there is an algebraically independent subset $\{\alpha_i\}_{i\in I} \subseteq L$ such that $L = Q(K[\alpha_i]_{i\in I}) = K(\alpha_i)_{i\in I}$, that is, if L is the smallest field (in L) containing K and $\{\alpha_i\}_{i\in I}$.

(3.4) Remark: If L is purely transcendental over K, then L is isomorphic to the field of quotients of the polynomial ring $K[t_i]_{i\in I}$.

(3.5) Theorem: Let $A = \{\alpha_i\}_{i\in I}$, $B = \{\beta_j\}_{j\in J} \subseteq L$ be subsets. The following are equivalent:

(a) $A \cup B$ is algebraically independent over K and $A \cap B = \emptyset$.

(b) A is algebraically independent over K and B is algebraically independent over $K(A) = K(\alpha_i)_{i\in I}$.

Proof: Let R be a ring, $T = \{t_i\}_{i\in I}$ and $U = \{u_j\}_{j\in J}$ sets of variables over R with $T \cap U = \emptyset$. Then $R[T\cup U] \cong (R[T])[U]$.

(b) $\Rightarrow$ (a): By assumption the homomorphism of rings $\delta: K(A)[u_j]_{j\in J} \longrightarrow L$ with $\delta|_{K(A)} = id_{K(A)}$ and $\delta(u_j) = \beta_j$ for all $j\in J$ is injective. In particular, the restriction of δ:

$$\varphi = \delta|_{K[A][u_j]} : K[A][u_j]_{j\in J} \longrightarrow L$$

is injective. Since A is algebraically independent over K: $K[A] = K[\alpha_i]_{i\in I} \cong K[t_i]_{i\in I}$. This yields an injective homomorphism from the polynomial ring into L:

$$\tau: K[t_i, u_j] \cong (K[t_i])[u_j] \longrightarrow L$$

with $\tau|_K = id_K$, $\tau(t_i) = \alpha_i$ and $\tau(u_j) = \beta_j$ for all $i \in I, j\in J$.

(a) $\Rightarrow$ (b): By a similar argument.

(3.6) Corollary: Let $B \subseteq L$ be an algebraically independent subset and $a \in L - B$. Then: a is transcendental over $K(B)$ if and only if $B\cup\{a\}$ is algebraically independent over K.

(3.7) Proposition: Let $B \subseteq L$ be a nonempty subset. The following are equivalent:

(a) B is algebraically independent over K.

(b) Every element $b\in B$ is transcendental over $K(B - \{b\})$.

Proof: (a) $\Rightarrow$ (b): By (3.6).

(b) $\Rightarrow$ (a): Suppose that B is algebraically dependent over K. Then there is a finite subset $B_0 \subseteq B$ such that B_0 is algebraically dependent over K. Let $M \subseteq B_0$ be a maximal algebraically independent subset of B_0 and let $b \in B_0 - M$. By assumption b is transcendental over $K(B - \{b\})$. Then b is transcendental over the smaller field $K(M) \subseteq K(B - \{b\})$. By (3.6) $M \cup \{b\}$ is algebraically independent over K, a contradiction.

(3.8) Definition: A subset $B \subseteq L$ is called a transcendence basis of L over K if:

(a) B is algebraically independent over K.

(b) L is algebraic over $K(B)$.

(3.9) Theorem: Let $B_0 \subseteq S \subseteq L$ be subsets such that:

(a) L is algebraic over $K(S)$

(b) B_0 is algebraically independent over K.

Then there is a subset $B \subseteq S$ with $B_0 \subseteq B$ which is a transcendence basis of L over K.

Proof: Consider the set: $\mathfrak{M} = \{ C \subseteq S \mid B_0 \subseteq C$ and C algebraically independent over $K \}$. Since $B_0 \in \mathfrak{M}$, $\mathfrak{M} \neq \emptyset$ and $\mathfrak{M}$ is partially ordered by inclusion.

Claim: $\mathfrak{M}$ is inductively ordered.

Pf of Cl: Let $\mathcal{K} = \{C_i\}_{i \in I} \subseteq \mathfrak{M}$ be a chain in $\mathfrak{M}$. We have to show that the set $C = \bigcup_{i \in I} C_i$ is algebraically independent over K. Suppose not. Then there is a finite subset $C_0 \subseteq C$ which is algebraically dependent over K. Since $\mathcal{K}$ is a chain there is an $i \in I$ with $C_0 \in C_i$ and C_i is algebraically dependent over K, a contradiction.

By Zorn's Lemma there is a maximal subset $B \in \mathfrak{M}$. If an element $x \in S - B$ is transcendental over $K(B)$, by (3.6) $B \cup \{x\}$ is transcendental over K, a contradiction to the maximality of B in $\mathfrak{M}$. Thus S is algebraic over $K(B)$ implying that $K(S)$ is algebraic over $K(B)$ and L is algebraic over $K(B)$.

(3.10) Corollary: For every field extension $K \subseteq L$ there is a transcendence basis B of L over K. (L is algebraic over K if and only if $B = \emptyset$.)

(3.11) Proposition: Let $K \subseteq K' \subseteq L$ be an intermediate field and let $B \subseteq L$ be algebraically independent over K. If K' is algebraic over K the set B is algebraically independent over K'.

Proof: Suppose that B is algebraically dependent over K'. Then there is an element $x \in B$

such that x is algebraic over $K'(B-\{x\})$. The field $K'(B-\{x\}) = K(B-\{x\})(K')$ is algebraic over $K(B-\{x\})$. Thus x is algebraic over $K(B-\{x\})$, a contradiction.

(3.12) Theorem: Suppose that B and C are transcendence bases of L over K. If B is finite then C is finite and $|B| = |C|$.

Proof: Let $|B| = n$. It suffices to show: if C is a transcendence basis of L over K then $|C| \leq n = |B|$. The proof is by induction on n.
$n=0$: L is algebraic over K and any nonempty subset $M \subseteq L$ is algebraic over K.
$n-1 \Rightarrow n$: $K \subseteq L$ is not algebraic and any transcendence basis C of L over K is not empty. Let $x \in C$, then x is transcendental over K. Apply (3.9) to $B_0 = \{x\}$ and $S = B \cup \{x\}$. Hence there is a subset $B' \subseteq B$ such that $B' \cup \{x\}$ is algebraically independent over K and L is algebraic over $K(B' \cup \{x\})$. If $x \notin B$ $|B'| \leq n-1$ since $B \cup \{x\}$ algebraically dependent. If $x \in B$, take $B' = B - \{x\}$. Let $C' = C - \{x\}$. The sets C' and B' are algebraically independent over $K' = K(x)$ and L is algebraic over $K'(B')$ or $K'(C')$, that is, B' and C' are transcendence bases of L over K'. By induction hypothesis: $|C'| \leq |B'| \leq n-1$ and $|C'| \leq n$.

(3.13) Definition: Let B be a transcendence basis of L over K. The number of elements in B is called the transcendence degree of L over K. Notation: $|B| = \mathrm{trdeg}_K(L)$.

(3.14) Theorem: Let $K \subseteq L \subseteq F$ be field extensions. Then $\mathrm{trdeg}_K(F) = \mathrm{trdeg}_L(F) + \mathrm{trdeg}_K(L)$.

Proof: Let B be a transcendence basis of L over K and C a transcendence basis of F over L. Obviously, $B \cap C = \emptyset$. We claim that $B \cup C$ is a transcendence basis of F over K. By assumption F is algebraic over $L(C)$ and L is algebraic over $K(B)$. Thus F is algebraic over $K(B \cup C)$. Since C is algebraically independent over L, C is algebraically independent over $K(B)$. By (3.5) $B \cup C$ is algebraically independent over K

§2: HILBERT'S NULLSTELLENSATZ ; DIMENSION

(3.15) Proposition: Let $K \subseteq L$ be a field extension and $\alpha_1, \ldots, \alpha_n \in L$ algebraic over K. Let $x_1, \ldots, x_n$ be variables over K. For all $1 \leq i \leq n$ there is a polynomial $f_i \in K[x_1, \ldots, x_i]$ which is monic in x_i so that: $K(\alpha_1, \ldots, \alpha_n) = K[\alpha_1, \ldots, \alpha_n] \cong K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

Proof: By induction on n. If $n=1$ then there is a monic polynomial $f \in K[x]$ so that: $K(\alpha) = K[\alpha] \cong K[x]/(f)$. f is the minimal polynomial of α over K.

$n-1 \Rightarrow n$: By induction hypothesis: $E = K(\alpha_1, \ldots, \alpha_{n-1}) \cong K[x_1, \ldots, x_{n-1}]/(f_1, \ldots, f_{n-1})$ where $f_i \in K[x_1, \ldots, x_i]$ monic in x_i for all $1 \leq i \leq n-1$. Since α_n is algebraic over E $E(\alpha_n) = E[\alpha_n] = E[x_n]/(g)$ where $g \in E[x_n]$ is the minimal polynomial of α_n over E. Consider the surjective homomorphism of rings:

$$\varphi: K[x_1, \ldots, x_n] \longrightarrow K[\alpha_1, \ldots, \alpha_{n-1}][x_n] = E[x_n]$$

defined by $\varphi|_K = id_K$; $\varphi(x_i) = \alpha_i$ for $1 \leq i \leq n-1$, and $\varphi(x_n) = x_n$. Then there is a polynomial $f_n \in K[x_1, \ldots, x_n]$, which is monic in x_n with $\varphi(f_n) = g$. Then

$$K(\alpha_1, \ldots, \alpha_n) = K[\alpha_1, \ldots, \alpha_n] \cong K[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

(3.16) Definition and Remark: Let A and B be rings. B is called an A-algebra if B is an A-module. If B is an A-algebra the map $\varphi: A \longrightarrow B$ defined by $\varphi(a) = a.1_B$ $\forall a \in A$ is a homomorphism of rings. Conversely, every homomorphism of rings $\varphi: A \longrightarrow B$ defines an A-module structure on B and thus makes B into an A-algebra. If $\varphi: A \longrightarrow B$ is a homomorphism of rings B is called a finitely generated A-algebra if there are elements $b_1, \ldots, b_n \in B$ so that the homomorphism $\psi: A[x_1, \ldots, x_n] \longrightarrow B$ defined by $\psi|_A = \varphi$ and $\psi(x_i) = b_i$ $\forall i = 1, \ldots, n$ is surjective. Note that a finitely generated A-algebra is, in general, not a finitely generated A-module.

(3.17) Theorem: Let $A \subseteq B \subseteq C$ be ring extensions and assume that A is a Noetherian ring,

C is a finitely generated A-algebra and a finitely generated B-module. Then B is finitely generated as an A-algebra.

<u>Proof</u>: First note that we may replace A by any ring $A \subseteq A_0 \subseteq B$ which is finitely generated as an A-algebra. We want to construct a ring A_0 with $A \subseteq A_0 \subseteq B$ such that:

(a) A_0 is a finitely generated A-algebra

(b) C is a finitely generated A_0-module.

Then A_0 is a Noetherian ring and C is a Noetherian A_0-module. B can be considered an A_0-submodule of C. Thus B is a finitely generated A_0-module and a finitely generated A-algebra.

In order to construct A_0 write $C = A[\alpha_1, -, \alpha_n] = B\beta_1 + — + B\beta_m$ for some $\alpha_i, \beta_j \in C$. Then $\alpha_i = \sum_{j=1}^{m} \lambda_{ij}\beta_j$ where $\lambda_{ij} \in B$, $1 \le i \le n$.

Replace A by
$$A' = A[\lambda_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le m] \subseteq B.$$

Thus we may assume $\alpha_i \in A[\beta_1, -, \beta_m]$ and therefore $C = A[\beta_1, -, \beta_m] = B\beta_1 + \dots + B\beta_m$.

There are elements $\mu_{ijk} \in B$ so that for all $1 \le i, j \le m$:
$$\beta_i\beta_j = \sum_{k=1}^{m} \mu_{ijk}\,\beta_k .$$

Put
$$A_0 = A[\mu_{ijk} \mid 1 \le i,j \le m \text{ and } 1 \le k \le m] \subseteq B.$$

Then $C = A_0[\beta_1, -, \beta_m] = A_0\beta_1 + \dots + A_0\beta_m$ is a finitely generated A_0-module.

(3.18) <u>Theorem</u>: Let K be a field and R a finitely generated K-algebra. If R is a field then R is algebraic over K.

<u>Proof</u>: Suppose that R is a field which is not algebraic over K. The field extension $K \subseteq R$ is finitely generated and there is a transcendence basis $\{y_1, -, y_t\} \subseteq R$ of R over K. With $T = K(y_1, -, y_t)$: $K \subseteq T \subseteq R$ and T is purely transcendental over K and R is finite algebraic over T. This means: K is a Noetherian ring, R a finitely generated

K-algebra and a finitely generated T-module. By (3.17) T is a finitely generated K-algebra. Hence there are elements $f_i, g_i \in K[y_1, \dots, y_t]$, $g_i \neq 0$, such that

$$T = K(y_1, \dots, y_t) = Q(K[y_1, \dots, y_t]) = K[f_1/g_1, \dots, f_n/g_n].$$

Since $K[y_1, \dots, y_t]$ is not a field there is a g_i with $g_i \notin K$. Set $h = \prod_{i=1}^{n} g_i + 1$. Then $h \notin K$ and h is not a unit in $K[y_1, \dots, y_t]$. But $1/h \in T$ and therefore:

$$1/h = {\sum_{(i_1, \dots, i_n) \in \mathbb{N}^n}}' a_{(i_1, \dots, i_n)} \left(\frac{f_1}{g_1}\right)^{i_1} \cdots \left(\frac{f_n}{g_n}\right)^{i_n} \qquad \text{(finite sum)}$$

where $a_{(i_1, \dots, i_n)} \in K$. Since the sum is finite there is an $r \in \mathbb{N}$ so that:

$$(*) \qquad (g_1 \cdots g_n)^r = \Big({\sum_{(i_1, \dots, i_n)}}' a_{(i_1, \dots, i_n)} f_1^{i_1} \cdots f_n^{i_n} g_1^{r-i_1} \cdots g_n^{r-i_n}\Big) h \in K[y_1, \dots, y_t].$$

$K[y_1, \dots, y_t]$ is isomorphic to the polynomial ring $K[x_1, \dots, x_t]$, in particular, $K[y_1, \dots, y_t]$ is a factorial domain. By $(*)$ h divides $(g_1 \cdots g_n)^r$ which implies that there is a prime element p in $K[y_1, \dots, y_t]$ which divides h and $(g_1 \cdots g_n)$, a contradiction.

(3.19) <u>Theorem</u> (Hilbert's Nullstellensatz I) Let K be a field and $m \subseteq K[x_1, \dots, x_n]$ a maximal ideal in the polynomial ring. Then

(a) $K[x_1, \dots, x_n]/m$ is an algebraic extension field of K.

(b) m is generated by n elements.

(c) If K is algebraically closed there are elements $a_i \in K$ with $m = (x_1 - a_1, \dots, x_n - a_n)$.

<u>Proof</u>: (a) $K[x_1, \dots, x_n]/m$ is a field and a finitely generated K-algebra. By (3.18) $K[x_1, \dots, x_n]/m$ is algebraic over K.

(b) By (a) and (3.15).

(c) If K is algebraically closed the embedding $K \hookrightarrow K[x_1, \dots, x_n]/m$ is surjective. Thus there are $a_i \in K$ with $x_i \equiv a_i \pmod{m}$ implying $m = (x_1 - a_1, \dots, x_n - a_n)$.

(3.20) <u>Theorem</u>: (Hilbert's Nullstellensatz II) Let K be a field, R a finitely generated K-algebra, and $I \subseteq R$ an ideal. Then

$$\operatorname{rad}(I) = \bigcap_{I \subseteq m;\ m \text{ max. id.}} m.$$

Proof: We may assume $I=(0)$ and have to show $nil(R) = Jrad(R)$. The last statement is equivalent to: If $f \notin nil(R)$ then there is a maximal ideal $m \subseteq R$ with $f \notin m$. If $f \notin nil(R)$ then $f^n \neq 0$ for all $n \in \mathbb{N}$ and the localization $R_f = S^{-1}R$ where $S = \{1, f, \dots, f^n, \dots\}$ is not the null ring. Let $n \subseteq R_f$ be a maximal ideal and $m = i_{R,f}^{-1}(n)$ where $i_{R,f}: R \longrightarrow R_f$ is the canonical map. We claim that m is a maximal ideal of R. In order to see this first note that $R_f = i_{R,f}(R)[1/f]$ and R_f is a finitely generated K-algebra. By (3.18) R_f/n is an algebraic extension field of K. Consider the injective homomorphisms: $K \hookrightarrow R/m \hookrightarrow R_f/n = L$. Since $K \subseteq L$ is an algebraic field extension any intermediate ring $K \subseteq T \subseteq L$ is a field. Thus R/m is a field and m is maximal in R.

(3.21) Definition: Let R be a ring. A finite chain of $n+1$ distinct prime ideals of R: $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_{n-1} \subsetneq P_n$ is called a prime ideal chain of length n in R.

(3.22) Definition: Let R be a ring and $M \neq 0$ a finitely generated R-module.

(a) Let $P \subseteq R$ be a prime ideal. The height of P is defined by:
$\mathrm{ht}\,P = \sup\{n \mid \exists$ a prime ideal chain of length n inside P: $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n \subseteq P\}$.

(b) Let $I \subseteq R$ be an ideal. The height of I is defined by:
$\mathrm{ht}\,I = \inf\{\mathrm{ht}\,P \mid P \subseteq R$ a prime ideal with $I \subseteq P\}$.

(c) The (Krull) dimension of R is defined by: $\dim R = \sup\{\mathrm{ht}\,P \mid P$ a prime ideal of $R\}$.

(d) The (Krull) dimension of M is defined by: $\dim_R(M) := \dim(R/\mathrm{ann}_R(M))$.

(e) Let $P \subseteq R$ be a prime ideal. $\dim(P) := \dim(R/P)$ is called the dimension of P.

(3.23) Remark: Let R be a ring and $P \subseteq R$ a prime ideal. Then $\mathrm{ht}\,P = \dim(R_P) = \mathrm{ht}\,PR_P$.

(3.24) Note: Let K be a field and R a finitely generated K-algebra and a domain. The field of quotients $L = Q(R)$ is a finitely generated field extension of K. We define:
$\mathrm{trdeg}_K(R) := \mathrm{trdeg}_K(L)$.

(3.25) <u>Theorem</u>: Let K be a field and R a finitely generated K-algebra and a domain.
Then: $\dim(R) = \operatorname{trdeg}_K(R)$.

<u>Proof</u>: Set $d = \dim(R)$ and $t = \operatorname{trdeg}_K(R)$. We want to show: $d \leq t$ and $t \leq d$.

(a) $d \leq t$.

Let $(0) = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_d \subsetneq R$ be a chain of prime ideals of maximal length d in R.
We claim: $\operatorname{trdeg}_K(R) \gneq \operatorname{trdeg}_K(R/P_1) \gneq \dots \gneq \operatorname{trdeg}_K(R/P_d)$.
This means, we have to show: if $Q \subsetneq P$ are prime ideals of R then $\operatorname{trdeg}_K(R/Q) \gneq \operatorname{trdeg}_K(R/P)$
We may assume that $Q = (0)$ and need to show for all prime ideals $P \subseteq R$ with $P \neq (0)$:
$\operatorname{trdeg}_K(R) \gneq \operatorname{trdeg}_K(R/P)$.
For a prime ideal $P \subseteq R$ with $P \neq (0)$ let $r = \operatorname{trdeg}_K(R/P)$. Choose $a_1, \dots, a_r \in R$ so that $\{\bar{a}_1, \dots, \bar{a}_r\} \subseteq R/P$ is a transcendence basis of $Q(R/P)$ over K. This implies that $a_1, \dots, a_r \in R$ are algebraically independent over K and that $K[a_1, \dots, a_r] \cap P = (0)$.
Note that $K[a_1, \dots, a_r]$ is (isomorphic to) the polynomial ring over K in r variables.
Let $S = K[a_1, \dots, a_r] - (0)$. Then $L = S^{-1}(K[a_1, \dots, a_r]) = Q(K[a_1, \dots, a_r]) \subseteq S^{-1}R$.
Since R is a finitely generated K-algebra, $S^{-1}R$ is a finitely generated L-algebra and a domain. Since $S \cap P = \emptyset$, $S^{-1}P \subseteq S^{-1}R$ is a proper, non zero prime ideal of $S^{-1}R$.
If $Q(R)$ is algebraic over L every intermediate ring $L \subseteq A \subseteq Q(R)$ is a field and $S^{-1}R$ is a field, contradiction. Thus $\operatorname{trdeg}_L(S^{-1}R) > 0$ and

$$\operatorname{trdeg}_K(S^{-1}R) = \operatorname{trdeg}_K(L) + \operatorname{trdeg}_L(S^{-1}R) = r + \operatorname{trdeg}_L(S^{-1}R) > r.$$

(b) $d \geq t$

By induction on t. If $t = 0$, then $R = Q(R)$ is a field. Let $t > 0$ and let $a \in R$ be transcendental over K. Set $L = K(a)$ and $S = K[a] - (0) \subseteq R$. R is a finitely generated $K[a]$-algebra and $S^{-1}R$ is a finitely generated $S^{-1}K[a] = L$-algebra.
Moreover, $\operatorname{trdeg}_L(S^{-1}R) = t-1$ and by induction hypothesis: $\dim S^{-1}R \geq t-1$. Let $0 = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_{t-1} \subsetneq S^{-1}R$ be a chain of prime ideals of length $t-1$ in $S^{-1}R$.
Consider its contraction to R:

$$0 = Q_0 \subsetneq Q_1 = P_1 \cap R \subsetneq Q_2 = P_2 \cap R \subsetneq \dots \subsetneq Q_{t-1} = P_{t-1} \cap R \subsetneq R$$

Since $Q_{t-1} \cap S = \emptyset$ we obtain that $Q_{t-1} \cap K[a] = (0)$. This implies that $\bar{a} \in R/Q_{t-1}$ is transcendental over K. By (3.18) R/Q_{t-1} is not a field. Thus there is a prime ideal $P \subseteq R/Q_{t-1}$ with $P \neq (0)$. This shows that $\dim R \geq t$.

(3.26) <u>Corollary</u>: Let K be a field and $K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. Then $\dim K[x_1, \ldots, x_n] = n$.

§3: ALGEBRAIC VARIETIES

(3.27) Definition: Let K be a field. The set $\mathbb{A}^n_K = \mathbb{A}^n = K^n = \{(a_1,\dots,a_n) \mid a_i \in K\}$ is called the affine n-space. An element $P=(a_1,\dots,a_n) \in \mathbb{A}^n$ is called a point and a_i are the coordinates of P.

(3.28) Definition: (a) Let $T \subseteq K[x_1,\dots,x_n]$ be a subset of the polynomial ring. The set $Z(T) = \{P \in \mathbb{A}^n_K \mid f(P)=0 \text{ for all } f \in T\} \subseteq \mathbb{A}^n_K$ is the zero set of T in $\mathbb{A}^n_K$.
(b) A subset $V \subseteq \mathbb{A}^n_K$ is called an affine variety (or algebraic variety) if there is a subset $T \subseteq K[x_1,\dots,x_n]$ with $V = Z(T)$.

(3.29) Examples: (a) Linear varieties: Let $T = \{f_1,\dots,f_m\} \subseteq K[x_1,\dots,x_n]$ where the f_i are linear polynomials. The investigation of $Z(T)$ is part of linear algebra.
(b) Hypersurfaces: Let $f \in K[x_1,\dots,x_n] - K$ be a polynomial. The set $Z(f)$ is called a hypersurface in $\mathbb{A}^n_K$. If $n=3$, $Z(f)$ is called a surface. If K is not algebraically closed (hyper)surfaces may degenerate to a single point, a curve or the empty set. For example, if $K=\mathbb{R}$, then $Z(x_1^2+x_2^2+x_3^2) = \{(0,0,0)\}$ and $Z(x^2+y^2+1) = \emptyset$.
Hypersurfaces of order 2 (called quadrics) are described by an equation:

$$f = \sum_{i,k=1}^{n} a_{ik} x_i x_k + \sum_{i=1}^{n} b_i x_i + c = 0 \quad \text{where } a_{ik}, b_i, c \in K.$$

(c) Plane algebraic curves are hypersurfaces in $\mathbb{A}^2_K$. For example:

$$(x_1^2+x_2^2+4x_2)^2 - 16(x_1^2+x_2^2) = 0.$$

(d) If $T = \{x_1x_3, x_2x_3\}$ then $Z(T) \subseteq \mathbb{A}^3_{\mathbb{R}}$ is the (x_1,x_2)-plane together with the x_3-axis.
(e) Algebraic groups: For every $A \in GL_n(K)$ consider the point $(A, \det(A^{-1})) \in \mathbb{A}^{n^2+1}_K$. The set of all these points can be identified with the hypersurface:

$$H = Z\left(\det(x_{ik})_{i,k=1,\dots,n}\, Y - 1\right) \subseteq \mathbb{A}^{n^2+1}_K.$$

Matrix multiplication provides a group operation on H:

$$\begin{array}{ccc} H \times H & \longrightarrow & H \\ [(A,\det(A^{-1})),(B,\det(B^{-1}))] & \longmapsto & (AB, \det((AB)^{-1})). \end{array}$$

Varieties which are provided with a group operation are called algebraic groups.

(3.30) Remark: Let $T \subseteq K[x_1,\dots,x_n]$ be a subset and $I=(T)$ the ideal generated by T. Then $Z(T)=Z(I)$. Every affine variety is the zero set of an ideal in $K[x_1,\dots,x_n]$. Since $K[x_1,\dots,x_n]$ is Noetherian $Z(T)$ is the zero set of finitely many polynomials.

(3.31) Proposition: (a) If $Y_1=Z(T_1), Y_2=Z(T_2) \subseteq \mathbb{A}^n_K$ are affine varieties so is $Y_1 \cup Y_2$.
(b) If $\{Y_i = Z(T_i)\}_{i\in I}$ is a set of affine varieties the intersection $\bigcap_{i\in I} Y_i$ is an affine variety.
(c) $\emptyset$ and $\mathbb{A}^n_K$ are affine varieties.

Proof: (a) Define $T=T_1T_2=\{fg \mid f\in T_1 \text{ and } g\in T_2\}$. We claim that $Y_1\cup Y_2 = Z(T_1T_2)$.
"$\subseteq$": obvious. "$\supseteq$": If $P\notin Y_1\cup Y_2$ then there are $f\in T_1$ and $g\in T_2$ with $f(P)\neq 0$ and $g(P)\neq 0$. Then $(fg)(P)=f(P)g(P)\neq 0$ and $P\notin Z(T_1T_2)$.
(b) $\bigcap_{i\in I} Y_i = Z(\bigcup_{i\in I} T_i)$
(c) $Z(1)=\emptyset$ and $Z(0)=\mathbb{A}^n_K$.

(3.32) Definition: Define the Zariski-topology on $\mathbb{A}^n_K$ as follows: a subset $Y\subseteq \mathbb{A}^n_K$ is closed if and only if $Y=Z(T)$ is an affine variety. By (3.31) this defines a topology on $\mathbb{A}^n_K$ with open sets $U=\mathbb{A}^n_K - Y$ where Y is an affine variety.

(3.33) Example: Let $K=\mathbb{C}$. The closed sets of $\mathbb{C}=\mathbb{A}^1_{\mathbb{C}}$ are the varieties $Z(I)$ where $I\subseteq \mathbb{C}[x]$ an ideal. Since $\mathbb{C}[x]$ is a PID: $I=(f)$ for some $f\in\mathbb{C}[x]$. Then $f=(x-a_1)\cdots(x-a_n)$ and $Z(f)=\{a_1,\dots,a_n\}$. In the Zariski-topology the closed set of $\mathbb{A}^1_{\mathbb{C}}$ are $\emptyset, \mathbb{C}$, and the finite subsets of $\mathbb{C}$.

(3.34) Definition: Let $Y\subseteq \mathbb{A}^n_K$ be a subset. The set
$$\mathcal{I}(Y)=\{f\in K[x_1,\dots,x_n] \mid f(P)=0 \ \forall P\in Y\}\subseteq K[x_1,\dots,x_n]$$
is called the ideal of Y.

(3.35) Proposition: Let K be a field.

(a) $T_1 \subseteq T_2 \subseteq K[x_1, \dots, x_n] \Rightarrow Z(T_1) \supseteq Z(T_2)$

(b) If $I \subseteq K[x_1, \dots, x_n]$ is an ideal then $Z(I) = Z(\sqrt{I}) = Z(\mathrm{rad}(I))$.

(c) $Y_1, Y_2 \subseteq \mathbb{A}^n_K \Rightarrow \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$

(d) $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n_K \Rightarrow \mathcal{J}(Y_1) \supseteq \mathcal{J}(Y_2)$

(e) If $Y \subseteq \mathbb{A}^n_K$ then $\mathcal{J}(Y) = \mathrm{rad}(\mathcal{J}(Y))$, that is, $\mathcal{J}(Y)$ is a reduced ideal.

(f) If $Y \subseteq \mathbb{A}^n_K$ then $Z(\mathcal{J}(Y)) = \overline{Y}$ where $\overline{Y}$ denotes the closure of Y in the Zariski-topology. In particular, if $Y = Z(I)$ is an affine variety then $Z(\mathcal{J}(Y)) = Y$.

Proof: (a), (c), (d), and (e) are obvious.

(b) By (a): $Z(\mathrm{rad}(I)) \subseteq Z(I)$. Let $P \in Z(I)$ and $f \in \mathrm{rad}(I)$. Then $f^r \in I$ for some $r \in \mathbb{N}$ and $f^r(P) = f(P)^r = 0$. Hence $f(P) = 0$ and $P \in Z(\mathrm{rad}(I))$.

(f) Obviously, $Y \subseteq Z(\mathcal{J}(Y))$. Since $Z(\mathcal{J}(Y))$ is closed in $\mathbb{A}^n_K$: $\overline{Y} \subseteq Z(\mathcal{J}(Y))$.
Let $W \subseteq \mathbb{A}^n_K$ be a closed subset with $Y \subseteq W$. Then $W = Z(I) \supseteq Y$. By (c) $\mathcal{J}(W) = \mathcal{J}(Z(I)) \subseteq \mathcal{J}(Y)$. Thus $I \subseteq \mathcal{J}(Y)$ and by (a): $W = Z(I) \supseteq Z(\mathcal{J}(Y))$. Hence $\overline{Y} = Z(\mathcal{J}(Y))$.

(3.36) Remark: We have order-reversing maps:

$$\{\text{affine varieties in } \mathbb{A}^n_K\} \underset{Z}{\overset{\mathcal{J}}{\rightleftarrows}} \{\text{reduced ideals in } K[x_1, \dots, x_n]\}.$$

By (3.35): $\quad Z \circ \mathcal{J} = \mathrm{id}_{\{\text{aff. var.}\}}$.

Question: $\quad (*) \quad \mathcal{J} \circ Z = \mathrm{id}_{\{\text{red. id.}\}} \quad ?$

Is there a 1-1 correspondence between the set of reduced ideals of $K[x_1, \dots, x_n]$ and the set of affine varieties of $\mathbb{A}^n_K$? In general the answer is no, but if K is algebraically closed then $(*)$ holds.

(3.37) Theorem: (Hilbert's Nullstellensatz) Let K be an algebraically closed field and $I \subseteq K[x_1, \dots, x_n]$ an ideal. Then $\mathcal{J}(Z(I)) = \mathrm{rad}(I)$.

Proof: Obviously, $\mathrm{rad}(I) \subseteq \mathcal{J}(Z(I))$. In order to show the other inclusion let $\mathcal{J} = \mathcal{J}(Z(I))$ and $R = K[x_1, \ldots, x_n]$. By (3.35)(f): $Z(\mathcal{J}) = Z(I)$ and therefore:

$$(*) \quad (a_1, \ldots, a_n) \in Z(\mathcal{J}) \iff (a_1, \ldots, a_n) \in Z(I).$$

Apply Taylor's formula to conclude: if $f \in R$ then $f(a_1, \ldots, a_n) = 0$ if and only if $f \in (x_1 - a_1, \ldots, x_n - a_n)$. Therefore $(*)$ implies:

$$\mathcal{J} \subseteq (x_1 - a_1, \ldots, x_n - a_n) \iff I \subseteq (x_1 - a_1, \ldots, x_n - a_n) \quad \text{and}$$

$$\{(x_1 - a_1, \ldots, x_n - a_n) \mid \mathcal{J} \subseteq (x_1 - a_1, \ldots, x_n - a_n)\} = \{(x_1 - a_1, \ldots, x_n - a_n) \mid I \subseteq (x_1 - a_1, \ldots, x_n - a_n)\}.$$

By Theorem (3.19) every maximal ideal of R is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ since K is algebraically closed and

$$\{m \subseteq R \mid m \text{ a maximal ideal and } \mathcal{J} \subseteq m\} = \{m \subseteq R \mid m \text{ a maximal ideal and } I \subseteq m\}.$$

By Theorem (3.20): $\displaystyle\bigcap_{m \text{ max};\, \mathcal{J} \subseteq m} m = \mathrm{rad}(\mathcal{J}) = \mathcal{J} = \bigcap_{m \text{ max};\, I \subseteq m} m = \mathrm{rad}(I).$

(3.38) Proposition: Every algebraic variety is an intersection of finitely many hypersurfaces.

Proof: If $Y \subseteq \mathbb{A}^n_K$ is an algebraic variety then $Y = Z(I)$ for some ideal $I \subseteq K[x_1, \ldots, x_n]$. Since $I = (f_1, \ldots, f_m)$ is finitely generated, $Y = Z(f_1, \ldots, f_m) = Z(f_1) \cap \ldots \cap Z(f_m)$.

(3.39) Proposition: Let K be a field. Every decreasing chain of varieties $Y_1 \supseteq Y_2 \supseteq \ldots$ in $\mathbb{A}^n_K$ is stationary.

Proof: Since $K[x_1, \ldots, x_n]$ is Noetherian the ascending chain of ideals $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y_2) \subseteq \ldots$ is stationary, i.e., there is an $N \in \mathbb{N}$ such that $\mathcal{J}(Y_N) = \mathcal{J}(Y_{N+k})$ for all $k \in \mathbb{N}$. Then $Z(\mathcal{J}(Y_N)) = Y_N = Z(\mathcal{J}(Y_{N+k})) = Y_{N+k}$.

(3.40) Definition: A topological space X is called irreducible if whenever $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$ then $X = A_1$ or $X = A_2$. A subset $X' \subseteq X$ is irreducible if X' is irreducible in the induced topology.

(3.41) Lemma: Let X be a topological space. The following are equivalent:
(a) X is irreducible
(b) If $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ are nonempty open subsets then $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$.
(c) Any nonempty open subset of X is dense in X.

Proof: Homework

(3.42) Lemma: Let X be a topological space and $X' \subseteq X$ a subset. The following are equivalent:
(a) X' is irreducible
(b) If $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ are open subsets with $X' \cap \mathcal{U}_i \neq \emptyset$ then $X' \cap \mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$.
(c) The closure $\overline{X'}$ of X' is irreducible.

Proof: (b) $\Leftrightarrow$ (c): follows from: if $\mathcal{U} \subseteq X$ is an open subset then $\mathcal{U} \cap \overline{X'} \neq \emptyset \Leftrightarrow \mathcal{U} \cap X' \neq \emptyset$.

(3.43) Definition: An irreducible component of a topological space is a maximal irreducible subset.

(3.44) Proposition: (a) Irreducible components are closed.
(b) Every irreducible subset is contained in an irreducible component.
(c) A topological space is the union of its irreducible components.

Proof: (a) follows from (3.42). Since every point is irreducible it suffices to show (b).
Let $X' \subseteq X$ be an irreducible subset. Consider the set:

$$\mathcal{M} = \{ Y \subseteq X \mid X' \subseteq Y \text{ and } Y \text{ is irreducible} \}.$$

$\mathcal{M}$ is partially ordered by inclusion. Since $X' \in \mathcal{M}$, $\mathcal{M} \neq \emptyset$. Let $\mathcal{K} = \{Y_\lambda\}_{\lambda \in \Lambda}$ be a chain in $\mathcal{M}$. Let $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$. Obviously, $X' \subseteq Y$. We claim that Y is irreducible. Let $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ be open subsets with $\mathcal{U}_i \cap Y \neq \emptyset$. Then there are $\lambda, \mu \in \Lambda$ with $\mathcal{U}_1 \cap Y_\lambda \neq \emptyset$ and $\mathcal{U}_2 \cap Y_\mu \neq \emptyset$. Suppose $Y_\lambda \subseteq Y_\mu$, then $\mathcal{U}_1 \cap Y_\mu \neq \emptyset$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \cap Y_\mu \neq \emptyset$ since Y_μ irreducible. Thus $\mathcal{U}_1 \cap \mathcal{U}_2 \cap Y \neq \emptyset$ and Y is irreducible.

By Zorn's Lemma $\mathcal{M}$ has a maximal element and X' is contained in an irreducible component of X.

(3.45) Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ with the ordinary topology are Hausdorff spaces. Their irreducible components are points.

(3.46) Definition: A topological space X is called Noetherian if every descending chain $A_1 \supseteq A_2 \supseteq \ldots$ of closed subsets $A_i \subseteq X$ is stationary, or equivalently, the set of closed subsets of X satisfies the d.c.c.

(3.47) Remark: (a) For every field K the affine space $\mathbb{A}^n_K$ is Noetherian in the Zariski-topology. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are Noetherian in the Zariski-topology. They are not Noetherian in the ordinary topology.

(b) Let X be a Noetherian topological space. The a.c.c. holds for the set of open subsets of X, or equivalently, every chain of open subsets $U_1 \subseteq U_2 \subseteq \ldots$ is stationary.

(3.48) Proposition: A Noetherian topological space has only finitely many irreducible components. No irreducible component is contained in the union of the others.

Proof: Let X be a Noetherian topological space. Consider the set:

$$\mathcal{M} = \left\{ V \subseteq X \;\middle|\; \begin{array}{l} V \text{ is closed and } V \text{ is not a union of finitely} \\ \text{many irreducible subsets of } X \end{array} \right\}$$

Suppose that $\mathcal{M} \neq \emptyset$. Since X is Noetherian, every nonempty set of closed subsets of X has a minimal element. Let $Y \in \mathcal{M}$ be a minimal element of $\mathcal{M}$. Y is not irreducible. Thus there are closed subsets $Y_i \subseteq Y$ with $Y = Y_1 \cup Y_2$ and $Y_i \neq Y$. By the minimality of Y every Y_i is the union of finitely many irreducible subsets. Thus Y is the union of finitely many irreducible subsets, a contradiction. Hence $\mathcal{M} = \emptyset$ and $X = X_1 \cup \ldots \cup X_n$ with $X_i \subseteq X$ irreducible. We may assume that the X_i are irreducible components of X. Let Y be an irreducible component of X. Then $Y = \bigcup_{i=1}^{n} (X_i \cap Y)$ and hence $Y = X_i \cap Y$ for some i. This shows $Y = X_i$ and $X_1, X_2, \ldots, X_n$ are the only irreducible components of X. A similar argument shows that no component can be contained in the union of the other components.

(3.49) Corollary: An affine variety $Y \subseteq \mathbb{A}^n_K$ has only finitely many irreducible components $Y_1, \ldots, Y_s$ and $Y = Y_1 \cup \ldots \cup Y_s$.

(3.50) Proposition: Let K be a field and $Y \subseteq \mathbb{A}^n_K$ an affine variety. Y is irreducible if and only if $\mathcal{J}(Y) \subseteq K[x_1, \ldots, x_n]$ is a prime ideal.

Proof: "$\Rightarrow$": Suppose that Y is irreducible and let $f, g \in K[x_1, \ldots, x_n]$ with $fg \in \mathcal{J}(Y)$. Let $H = Z(f)$ and $L = Z(g)$, then $H \cup L = Z(fg) \supseteq Y = Z(\mathcal{J}(Y))$. Thus $Y = Y \cap (H \cup L) = (Y \cap H) \cup (Y \cap L)$. Since Y is irreducible, $Y \subseteq H$ or $Y \subseteq L$. This implies: $\mathcal{J}(H) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(L) \subseteq \mathcal{J}(Y)$. Hence $f \in \mathcal{J}(Y)$ or $g \in \mathcal{J}(Y)$.

"$\Leftarrow$": Suppose that $\mathcal{J}(Y)$ is a prime ideal. Let $Y_1, Y_2 \subseteq Y$ be affine varieties with $Y = Y_1 \cup Y_2$. Then $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$. Since $\mathcal{J}(Y)$ is prime, $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) \subseteq \mathcal{J}(Y)$. Therefore $\mathcal{J}(Y_1) = \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) = \mathcal{J}(Y)$ and $Y_1 = Z(\mathcal{J}(Y_1)) = Z(\mathcal{J}(Y)) = Y$ or $Y_2 = Z(\mathcal{J}(Y_2)) = Z(\mathcal{J}(Y)) = Y$.

(3.51) Definition: Let $Y \subseteq \mathbb{A}^n_K$ be an affine variety. The ring $A(Y) = K[x_1, \ldots, x_n]/\mathcal{J}(Y)$ is called the coordinate ring of Y or the affine K-algebra of Y.

(3.52) Remark: Let $Y \subseteq \mathbb{A}^n_K$ be an affine variety with coordinate ring $A(Y) = K[x_1, \ldots, x_n]/\mathcal{J}(Y)$. For all $F \in K[x_1, \ldots, x_n]$ define $\varphi_F : Y \longrightarrow K$ by $\varphi_F(P) = F(P)$. Consider the set $D(Y) = \{\varphi : Y \to K \mid \exists\, F \in K[x_1, \ldots, x_n] \text{ with } \varphi = \varphi_F\}$. $D(Y)$ is a ring under the obvious operations and the map $\Phi : K[x_1, \ldots, x_n] \longrightarrow D(Y)$ with $\Phi(F) = \varphi_F$ is a surjective homomorphism of rings with kernel $\mathcal{J}(Y)$. Thus $D(Y) \cong A(Y)$.

(3.53) Definition: Let X be a topological space and $Y \subseteq X$ a closed irreducible subset.

(a) The (Krull) dimension of X is defined by:

$\dim(X) = \sup\{n \in \mathbb{N} \mid \exists$ a chain of nonempty closed irreducible subsets in $X: X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n\}$.

(b) If $Y \neq \emptyset$, the codimension of Y in X is defined by:

$\operatorname{codim}_X(Y) = \sup\{n \in \mathbb{N} \mid \exists$ a chain of closed irreducible subsets $X_i \subseteq X: Y = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n\}$.

(c) If $A \subseteq X$ is an arbitrary nonempty closed subset the codimension of A in X is defined by:

$\operatorname{codim}_X(A) = \sup\{\operatorname{codim}_X Y \mid Y$ an irreducible component of $A\}$

(d) $\dim(\emptyset) := -1$ and $\operatorname{codim}_X(\emptyset) = \infty$.

(3.54) Proposition: Let X be a topological space and $Y \subseteq X$ a closed irreducible subset.

(a) If $\{X_\lambda\}_{\lambda \in \Lambda}$ is the set of irreducible components of X then: $\dim(X) = \sup_{\lambda \in \Lambda}\{\dim(X_\lambda)\}$.

(b) If $A_i \subseteq X$ are closed subsets with $X = A_1 \cup \ldots \cup A_n$ then: $\dim(X) = \sup\{\dim A_i \mid 1 \leq i \leq n\}$.

(c) $\dim(Y) + \operatorname{codim}_X(Y) \leq \dim(X)$ (if $Y \neq \emptyset$).

(d) If X is irreducible with $\dim(X) < \infty$ then: $\dim(Y) < \dim(X) \Longleftrightarrow Y \neq X$.

Proof: (b) If $Y \subseteq X$ is irreducible, then $Y = (A_1 \cap Y) \cup \ldots \cup (A_n \cap Y)$ implies that $Y = A_j \cap Y$ for some $1 \leq j \leq n$ and thus $Y \subseteq A_j$. Apply (a).

(c) Start with a chain of closed irreducible subsets ending in Y and continue with a chain starting in Y.

(3.55) Proposition: Let K be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an affine variety. Then: $\dim(Y) = \dim A(Y)$.

Proof: 1. case: Y is irreducible.

Then $Y = Z(P)$ for a prime ideal $P \subseteq K[x_1, \ldots, x_n]$. $\dim(Y)$ is the supremum of the length n of chains: $(*)$ $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = Y$ where the Y_i are closed irreducible subsets of $\mathbb{A}^n_K$ (resp. Y). Then $Y_i = Z(P_i)$ for prime ideals $P_i \subseteq K[x_1, \ldots, x_n]$. By Hilbert's Nullstellensatz $\mathcal{J}(Y_i) = \mathcal{J}(Z(P_i)) = P_i$ and $(*)$ corresponds to a chain of prime ideals of

length n: $P = P_n \subsetneq P_{n-1} \subsetneq — \subsetneq P_0$. This shows that $\dim(Y) \leq \dim A(Y)$.
Conversely, let $P = Q_m \subsetneq Q_{m-1} \subsetneq — \subsetneq Q_0$ be a chain of prime ideals in $K[x_1, —, x_n]$ starting at P. By Hilbert's Nullstellensatz $Z(Q_i) \neq Z(Q_j)$ for $i \neq j$. This yields a chain $Z(Q_0) \subsetneq Z(Q_1) \subsetneq — \subsetneq Z(Q_m) = Y$ of irreducible closed subsets of length m. Thus also: $\dim A(Y) \leq \dim(Y)$.

<u>2. case</u>: $Y \subseteq \mathbb{A}^n_K$ an arbitrary affine variety.

Then $Y = Y_1 \cup — \cup Y_r$ where $Y_i \subseteq \mathbb{A}^n_K$ are irreducible affine varieties. For some prime ideals $P_i \subseteq K[x_1, —, x_n]$: $Y_i = Z(P_i)$ and $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup — \cup Y_r) = \bigcap_{i=1}^r \mathcal{J}(Y_i) = \bigcap_{i=1}^r P_i$.
This shows that the set $\{P_1, —, P_r\}$ contains all prime ideals which are minimal over $\mathcal{J}(Y)$ and

$$\begin{aligned} \dim A(Y) &= \sup\{\dim P_i \mid 1 \leq i \leq r\} \\ &= \sup\{\dim A(Y_i) \mid 1 \leq i \leq r\} \\ &= \sup\{\dim(Y_i) \mid 1 \leq i \leq r\} && \text{by case 1} \\ &= \dim(Y) && \text{by (3.54)} \end{aligned}$$

(3.56) <u>Corollary</u>: Let K be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an irreducible algebraic variety. Then $\dim(Y) = \dim A(Y) = \operatorname{trdeg}_K(A(Y))$.

<u>Proof</u>: By (3.25).

§4: THE SPECTRUM OF A RING

(3.57) Definition: For a ring A define:

(a) the spectrum of A: $\quad \mathrm{Spec}(A) := \{P \subseteq A \mid P \text{ a prime ideal}\}$

(b) the maximal spectrum of A: $\quad m\text{-}\mathrm{Spec}(A) = \{m \subseteq A \mid m \text{ a maximal ideal}\}$

(c) Let $I \subseteq A$ be an ideal and $X = \mathrm{Spec}(A)$ or $X = m\text{-}\mathrm{Spec}(A)$. The set $V(I) = \{P \in X \mid I \subseteq P\}$ is called the variety of I.

In the following we assume that $X = \mathrm{Spec}(A)$ or $X = m\text{-}\mathrm{Spec}(A)$.

(3.58) Remark: Let A be a ring.

(a) If $I_1, I_2 \subseteq A$ are ideals then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.

(b) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals in A then $\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$.

(c) $V(A) = \emptyset$ and $V(0) = X$.

The topology on X defined by: $Y \subseteq X$ closed if and only if $Y = V(I)$ for an ideal $I \subseteq A$, is called the Zariski topology on X. Note that the Zariski topology on $m\text{-}\mathrm{Spec}(A)$ is induced by the Zariski topology on $\mathrm{Spec}(A)$.

(3.59) Definition: Let $T \subseteq X$ be a subset. The ideal $\mathcal{J}(T) = \bigcap_{P\in T} P$ if $T \neq \emptyset$ and $\mathcal{J}(T) = A$ if $T = \emptyset$ is called the (vanishing) ideal of T in A.

(3.60) Proposition: (a) For any subset $T \subseteq X$: $V(\mathcal{J}(T)) = \overline{T}$ where $\overline{T}$ is the closure of T in X.

(b) Let $X = \mathrm{Spec}(A)$ and $I \subseteq A$ an ideal. Then $\mathcal{J}(V(I)) = \mathrm{rad}(I)$. Thus there is a 1-1 correspondence between the closed subsets of $X = \mathrm{Spec}(A)$ and the reduced ideals of A.

Proof: (a) Since $T \subseteq V(\mathcal{J}(T))$: $\overline{T} \subseteq V(\mathcal{J}(T))$. Let $V(I)$ be a closed subset of X with $T \subseteq V(I)$. Then $I \subseteq \bigcap_{P\in T} P = \mathcal{J}(T)$ and $V(I) \supseteq V(\mathcal{J}(T))$. Thus $\overline{T} \supseteq V(\mathcal{J}(T))$.

(b) $\quad \mathcal{J}(V(I)) = \bigcap_{P\in V(I)} P = \bigcap_{P \text{ prime};\, I\subseteq P} P = \mathrm{rad}(I)$.

(3.61) Proposition: If A is a Noetherian ring $\mathrm{Spec}(A)$ and $m\text{-}\mathrm{Spec}(A)$ are Noetherian topological spaces.

Proof: The topology on $m\text{-}\mathrm{Spec}(A)$ is induced by the topology of $\mathrm{Spec}(A)$. Thus it suffices to show that $\mathrm{Spec}(A)$ is a Noetherian space. Let $V(I_1) \supseteq V(I_2) \supseteq \dots$ be a descending chain of closed subsets of X. Then $\mathcal{J}(V(I_1)) \subseteq \mathcal{J}(V(I_2)) \subseteq \dots$ is an ascending chain of ideals in A. There is an $n \in \mathbb{N}$ with $\mathcal{J}(V(I_n)) = \mathcal{J}(V(I_{n+k}))$ for all $k \in \mathbb{N}$. Therefore $V(\mathcal{J}(V(I_n))) = V(I_n) = V(I_{n+k}) = V(\mathcal{J}(V(I_{n+k})))$ for all $k \in \mathbb{N}$.

(3.62) Proposition: Let $Y \subseteq X$ be a nonempty closed subset. Y is irreducible if and only if $\mathcal{J}(Y)$ is a prime ideal of A.

Proof: "$\Rightarrow$": Suppose that Y is irreducible and let $fg \in \mathcal{J}(Y)$. Then $Y = V(\mathcal{J}(Y)) \subseteq V(fg)$ and thus $Y = (Y \cap V(f)) \cup (Y \cap V(g))$ since $V(fg) = V(f) \cup V(g)$. This implies $Y \subseteq V(f)$ or $Y \subseteq V(g)$ and $f \in \mathcal{J}(Y)$ or $g \in \mathcal{J}(Y)$.

"$\Leftarrow$": Suppose that $\mathcal{J}(Y)$ is a prime ideal and let $Y = Y_1 \cup Y_2$ with Y_1 and Y_2 closed subsets of X. Then $\mathcal{J}(Y) \subseteq \mathcal{J}(Y_1)$, $\mathcal{J}(Y) \subseteq \mathcal{J}(Y_2)$, and $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$. Since $\mathcal{J}(Y)$ is prime, $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) \subseteq \mathcal{J}(Y)$. Hence: $Y = Y_1$ or $Y = Y_2$.

(3.63) Corollary: Let A be a ring. Then: $\dim(A) = \dim(\mathrm{Spec}(A))$.

(3.64) Corollary: Let A be a ring with the property that every prime ideal of A is the intersection of maximal ideals. Then: $\dim(A) = \dim(m\text{-}\mathrm{Spec}(A))$.

Proof: For rings with the above property: $\mathcal{J}(V(I)) = \bigcap_{P \text{ max.}; I \subseteq P} P = \mathrm{rad}(I)$ for every ideal $I \subseteq A$.

(3.65) Remark: Let K be a field and $A = K[x_1, \dots, x_n]$ the polynomial ring over K.

By (3.20) every prime ideal of A is the intersection of maximal ideals and

$$\dim(A) = n = \dim(m\text{-Spec}(A)).$$

If K is algebraically closed there is a homeomorphism of topological spaces:

$$\begin{aligned} \mathbb{A}^n_K &\xrightarrow{\sim} m\text{-Spec}(A) \\ (a_1, \ldots, a_n) &\longmapsto (x_1 - a_1, \ldots, x_n - a_n). \end{aligned}$$

Similarly, for an affine variety $Y = Z(I) \subseteq \mathbb{A}^n_K$ the topological spaces:

$$\begin{aligned} Y &\xrightarrow{\sim} m\text{-Spec}(A/\mathrm{rad}\, I) \\ (a_1, \ldots, a_n) &\longmapsto (x_1 - a_1, \ldots, x_n - a_n) \end{aligned}$$

are homeomorphic and $\dim(Y) = \dim(m\text{-Spec}(A/\mathrm{rad}(I))) = \dim(\mathrm{Spec}(A/\mathrm{rad}(I))) = \dim A/\mathrm{rad}(I)$.