CHAPTER II: PRIMARY DECOMPOSITION

§1: $\mathrm{ASS}_A(M)$ AND $\mathrm{SUPP}_A(M)$

Recall the following definitions from Chapter I: Let M be an A-module. The annihilator of M is the set $\mathrm{ann}_A(M) = \{a \in A \mid am = 0 \ \forall m \in M\}$. For an element $m \in M$ we define the annihilator of m by: $\mathrm{ann}_A(m) = \{a \in A \mid am = 0\} = 0 :_A Am$. $\mathrm{ann}_A(M)$ and $\mathrm{ann}_A(m)$ are ideals of A.

(2.1) Remark: Let M be an A-module. Then $\quad NZD(M) = A - \bigcup_{m \in M-(0)} \mathrm{ann}_A(m)$.

(2.2) Proposition: The maximal elements of the set $\Gamma := \{\mathrm{ann}_A(m) \mid m \in M-(0)\}$ are prime ideals of A. (Note that Γ may not contain a maximal element.)

Proof: Let $P = \mathrm{ann}_A(m)$ be a maximal element of Γ and $a, b \in A$ with $ab \in P$ and $b \notin P$. Then $bm \neq 0$ and $I = \mathrm{ann}_A(bm) \in \Gamma$. Since $P \subseteq I$ by the maximality of P: $I = P$ and $a \in P$.

(2.3) Definition: Let M be an A-module. A prime ideal $P \in \mathrm{Spec}(A)$ for which there exists an $m \in M$ with $P = \mathrm{ann}_A(m)$ is called an associated prime ideal of M. The set of the associated prime ideals of M is denoted by

$$\mathrm{Ass}_A(M) = \{P \in \mathrm{Spec}(A) \mid P \text{ an associated prime ideal of } M\}.$$

(2.4) Remark: (a) Let M be an A-module and $m \in M-(0)$. Then $Am \cong A/\mathrm{ann}_A(m)$ as A-modules. This implies that $P \in \mathrm{Ass}_A(M)$ if and only if there is an A-linear map : $A/P \longrightarrow M$.

(b) For every prime ideal $P \in \mathrm{Spec}(A)$: $\mathrm{Ass}_A(A/P) = \{P\}$.

(2.5) <u>Lemma</u>: Let A be a Noetherian ring and M an A-module. Then
$$M = (0) \Longleftrightarrow \mathrm{Ass}_A(M) = \emptyset.$$

<u>Proof</u>: "$\Longleftarrow$": Suppose that $M \neq (0)$. Then $\Gamma = \{\mathrm{ann}_A(m) \mid m \in M \setminus (0)\}$ is a nonempty set of ideals of A. Since A is Noetherian Γ has a maximal element which is prime by (2.2).

(2.6) <u>Lemma</u>: Let A be a Noetherian ring and M an A-module. Then
$$ZD(M) = \bigcup_{P \in \mathrm{Ass}_A(M)} P$$

<u>Proof</u>: "$\supseteq$": by definition
"$\subseteq$": Let $a \in ZD(M)$. Then $a \in \mathrm{ann}_A(m)$ for some $m \in M - (0)$. $\mathrm{ann}_A(m) \in \Gamma$ and since A is Noetherian there is a maximal element $\mathrm{ann}_A(x) \in \Gamma$ with $\mathrm{ann}_A(m) \subseteq \mathrm{ann}_A(x)$. Then $P = \mathrm{ann}_A(x) \in \mathrm{Ass}_A(M)$.

(2.7) <u>Lemma</u>: Let A be a Noetherian ring and M a nonzero, finitely generated A-module. There is a normal series of M:
$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_{n-1} \subseteq M_n = M$$
such that for all $0 \le i \le n-1$: $M_{i+1}/M_i \cong A/P_i$ for some prime ideal $P_i \subseteq A$.

<u>Proof</u>: By (2.5) there is a prime ideal $P \in \mathrm{Ass}_A(M) \neq \emptyset$. This corresponds to an element $m \in M - (0)$ with $\mathrm{ann}_A(m) = P$. Set $M_1 = Am \cong A/P$ and set $P = P_0$. If $M/M_1 \neq (0)$ replace M by M/M_1 and repeat the argument. This produces an A-submodule $\overline{M_2}$ of M/M_1 with $\overline{M_2} \cong A/P_1$. Let M_2 be the contraction of $\overline{M_2}$ to M. Then $M_1 \subseteq M_2$ and $\overline{M_2} = M_2/M_1 \cong A/P_1$. We obtain an ascending chain of submodules $(0) = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ with factor modules M_{i+1}/M_i isomorphic to A/P_i for some $P_i \in \mathrm{Spec}(A)$. Since A is Noetherian after n steps: $M_n = M$.

(2.8) Remark: The normal series of (2.7) is not uniquely determined by M. For example if $M = \mathbb{Z}$ the series: $(0) \subseteq \mathbb{Z}$ and $(0) \subseteq (6) \subseteq (3) \subseteq \mathbb{Z}$ satisfy the conditions of (2.7).

(2.9) Theorem: Let M be an A-module and $N \subseteq M$ a submodule. Then:

$$\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N).$$

Proof: (1) $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M)$

Pf of (1): $P \in \mathrm{Ass}_A(N) \iff$ there is an injective A-linear map: $A/P \longrightarrow N$.

(1) follows since $N \subseteq M$.

(2) $\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N)$

Pf of (2): Let $P \in \mathrm{Ass}_A(M)$ and $m \in M-(0)$ with $P = \mathrm{ann}_A(m)$. If $m \in N$ then $P \in \mathrm{Ass}_A(N)$. Suppose $m \notin N$ and consider $\overline{m} = m + N \in M/N - (0)$. Then $P \subseteq \mathrm{ann}_A(\overline{m})$.

If $P = \mathrm{ann}_A(\overline{m})$ then $P \in \mathrm{Ass}_A(M/N)$ and we are done.

If $P \subsetneqq \mathrm{ann}_A(\overline{m})$ then there is an $a \in \mathrm{ann}_A(\overline{m}) - P$ with $a\overline{m} = 0$. This implies $am \in N$. We claim that $\mathrm{ann}_A(m) = \mathrm{ann}_A(am)$. The inclusion "$\subseteq$" is obvious. Let $b \in \mathrm{ann}_A(am)$. Then $bam = 0$ and $ab \in \mathrm{ann}_A(m) = P$. Since P is prime and $a \notin P$: $b \in P$. Thus $P = \mathrm{ann}_A(am)$ with $am \in N$ and $P \in \mathrm{Ass}_A(N)$.

(2.10) Corollary: Let M be an A-module. Suppose that M has a normal series $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ with factor modules $M_{i+1}/M_i \cong A/P_i$ with $P_i \in \mathrm{Spec}(A)$ for all $0 \leq i \leq n-1$. Then $\mathrm{Ass}_A(M) \subseteq \{P_0, P_1, \text{---}, P_{n-1}\}$.

Proof: By induction on the length n of the normal series. If $n = 1$ then $0 = M_0 \subseteq M_1 = M$ and $M \cong A/P_0$. Thus $\mathrm{Ass}_A(M) = \{P_0\}$.

$n-1 \Rightarrow n$: By induction hypothesis: $\mathrm{Ass}_A(M_{n-1}) \subseteq \{P_0, \text{---}, P_{n-2}\}$. Apply (2.9) to $N = M_{n-1}$ and $M = M$. Then $\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(M_{n-1}) \cup \mathrm{Ass}_A(M/M_{n-1}) \subseteq$

$$\subseteq \{P_0, \text{---}, P_{n-2}\} \cup \{P_{n-1}\} = \{P_0, \text{---}, P_{n-1}\}.$$

(2.11) <u>Corollary</u>: Let A be a Noetherian ring and M a finitely generated A-module. Then $|\mathrm{Ass}_A(M)| < \infty$.

<u>Proof</u>: (2.7) and (2.10).

(2.12) <u>Theorem</u>: Let A be a Noetherian ring, $S \subseteq A$ a multiplicative subset, and M an A-module. Then: $\mathrm{Ass}_{S^{-1}A}(S^{-1}M) = \{ S^{-1}P \mid P \in \mathrm{Ass}_A(M) \text{ and } P \cap S = \emptyset \}$.

<u>Proof</u>: "$\supseteq$": Let $P \in \mathrm{Ass}_A(M)$ with $P \cap S = \emptyset$. Then $P = \mathrm{ann}_A(m)$ for some $m \in M$. Consider $Q = \mathrm{ann}_{S^{-1}A}(\frac{m}{1}) \subseteq S^{-1}A$. Obviously, $S^{-1}P \subseteq Q$. Let $\frac{a}{s}\frac{m}{1} = 0$. Then there is a $t \in S$ so that $tam = 0$. This implies $ta \in P$ and $t \notin P$. Since P is prime $a \in P$ and thus $\frac{a}{s} \in S^{-1}P$.

"$\subseteq$": Let $Q \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$. Then $Q = \mathrm{ann}_{S^{-1}A}(\frac{m}{s})$ for some $\frac{m}{s} \in S^{-1}M$. Moreover, $Q = S^{-1}P$ for some $P \in \mathrm{Spec}(A)$. Since A is Noetherian the ideal P is finitely generated. Suppose $P = (p_1, \ldots, p_r)$. Then $\frac{p_i}{1}\frac{m}{s} = 0$ for all $1 \leq i \leq r$ and there is a $t \in S$ with $tp_i m = 0$ for all $1 \leq i \leq r$. Since $\frac{m}{s} \neq 0$, $tm \neq 0$ and $P \subseteq \mathrm{ann}_A(tm) \neq A$. On the other hand, if $atm = 0$ for some $a \in A$ then $\frac{a}{1}\frac{tm}{ts} = 0$ and $\frac{a}{1} \in Q = \mathrm{ann}_{S^{-1}A}(\frac{m}{s})$. Therefore $a \in P$ and $P = \mathrm{ann}_A(tm) \in \mathrm{Ass}_A(M)$.

(2.13) <u>Definition</u>: Let M be an A-module. The <u>support</u> of M is the set of prime ideals: $\mathrm{Supp}_A(M) = \{ P \subseteq A \mid P \text{ a prime ideal and } M_P \neq 0 \} \subseteq \mathrm{Spec}(A)$.

<u>Note</u>: For any ring A: $\mathrm{Supp}_A(A) = \mathrm{Spec}(A)$.

(2.14) <u>Proposition</u>: Let M be a finitely generated A-module. Then

$$\mathrm{Supp}_A(M) = \{ P \subseteq A \mid P \text{ a prime ideal and } \mathrm{ann}_A(M) \subseteq P \} = V(\mathrm{ann}_A(M)).$$

<u>Proof</u>: "$\subseteq$": Let $P \in \mathrm{Spec}(A)$ with $\mathrm{ann}_A(M) \not\subseteq P$. Then there is an element

$t \in \operatorname{ann}_A(M) - P$. Hence $tm = 0$ for all $m \in M$ and $M_p = 0$.

"$\supseteq$": M is a finitely generated A-module: $M = Am_1 + \ldots + Am_n$.

This implies: $\operatorname{ann}_A(M) = \bigcap_{i=1}^{n} \operatorname{ann}_A(m_i) \supseteq \prod_{i=1}^{n} \operatorname{ann}_A(m_i)$.

Let $P \in \operatorname{Spec}(A)$ with $\operatorname{ann}_A(M) \subseteq P$. Then for some $1 \le i \le n$: $\operatorname{ann}_A(m_i) \subseteq P$. We claim that $\frac{m_i}{1} \neq 0$ in M_P. Suppose $\frac{m_i}{1} = 0$ in M_P then there is a $t \in A - P$ with $tm_i = 0$ implying $t \in \operatorname{ann}_A(m_i)$, a contradiction. This shows that $M_P \neq 0$ and $P \in \operatorname{Supp}(M)$.

(2.15) __Definition__: Let A be a ring and $I \subseteq A$ an ideal. The set $V(I) = \{P \in \operatorname{Spec}(A) \mid I \subseteq P\}$ is called the __variety__ of I.

(2.16) __Proposition__: Let A be a Noetherian ring and M an A-module. Suppose that M has a normal series: $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ so that for all $0 \le i \le n-1$: $M_{i+1}/M_i \cong A/P_i$ for some $P_i \in \operatorname{Spec}(A)$. Then

$$\operatorname{Ass}_A(M) \subseteq \{P_0, \ldots, P_{n-1}\} \subseteq \operatorname{Supp}_A(M)$$

and all three sets have the same minimal elements.

__Proof__: By (2.10): $\operatorname{Ass}_A(M) \subseteq \{P_0, \ldots, P_{n-1}\}$. In order to show $P_i \in \operatorname{Supp}_A(M)$ consider $A/P_i \cong M_{i+1}/M_i$. Since $(A/P_i)_{P_i} \neq 0$ and $(M_{i+1}/M_i)_{P_i} \cong (M_{i+1})_{P_i}/(M_i)_{P_i}$ we obtain $(M_{i+1})_{P_i} \neq 0$. Since $(M_{i+1})_{P_i} \subseteq M_{P_i}$ the statement $M_{P_i} \neq 0$ follows. It remains to show that the three sets have the same minimal elements. Obviously, it suffices to show that every minimal element of $\operatorname{Supp}_A(M)$ is contained in $\operatorname{Ass}_A(M)$. Let $P \in \operatorname{Supp}_A(M)$ be minimal. Hence the A_P-module M_P is nonzero. Every prime ideal $Q \in \operatorname{Spec}(A_P)$ is of the form $Q = qA_P$ for some prime ideal $q \subseteq P$. Note that the localization $(M_P)_Q$ is isomorphic to the localization M_q. Thus by the minimality of P: $(M_P)_Q = 0$ for all $Q \in \operatorname{Spec}(A_P)$ with $Q \neq PA_P$ and therefore: $\operatorname{Supp}_{A_P}(M_P) = \{PA_P\}$. Since M has a normal series with prime factors, the A_P-module M_P has a normal series with prime factors. Let $0 \subseteq N_1 \subseteq \cdots \subseteq N_r = M_P$ be such a series with $N_1 \cong A_P/Q_1$ for some $Q_1 \in \operatorname{Spec}(A_P)$. Then $\operatorname{Ass}_{A_P}(N_1) = \operatorname{Ass}_{A_P}(A_P/Q_1) = \{Q_1\} \subseteq \operatorname{Ass}_{A_P}(M_P)$. This shows that $\operatorname{Ass}_{A_P}(M_P) \neq \emptyset$. Since $\operatorname{Ass}_{A_P}(M_P) \subseteq \operatorname{Supp}_{A_P}(M_P)$, $\operatorname{Ass}_{A_P}(M_P) = \{PA_P\}$ and by (2.12): $P \in \operatorname{Ass}_A(M)$.

(2.17) <u>Corollary</u>: Let A be a Noetherian ring and $I \subseteq A$ an ideal. There are only finitely many minimal prime ideals $P_1, \ldots, P_r$ containing I. In particular, A has only finitely many minimal prime ideals.

<u>Proof</u>: Apply (2.16) to the A-module $M = A/I$. Then $\operatorname{Supp}_A(A/I) = V(I)$.

§2: PRIMARY DECOMPOSITION

(2.18) Lemma: Let A be a Noetherian ring, $I \subseteq A$ an ideal and M a finitely generated A-module. The following are equivalent:

(a) There is an $m \in M-(0)$ with $Im = 0$.

(b) For all $a \in I$ there is an $m \in M-(0)$ with $am = 0$.

(c) There is a prime ideal $P \in \mathrm{Ass}_A(M)$ with $I \subseteq P$.

Proof: (b) $\Rightarrow$ (c): (b) implies that $I \subseteq ZD(M)$. By (2.6): $I \subseteq \bigcup_{P \in \mathrm{Ass}_A(M)} P$. By (2.11) the set $\mathrm{Ass}_A(M)$ is finite. Thus $I \subseteq P$ for some $P \in \mathrm{Ass}_A(M)$.

(c) $\Rightarrow$ (a): Let $P \in \mathrm{Ass}_A(M)$ with $I \subseteq P$. Since $P = \mathrm{ann}_A(m)$ for some $m \in M-(0)$: $Im = 0$.

(2.19) Lemma: Let A be a Noetherian ring, $I \subseteq A$ an ideal and M a finitely generated A-module. The following are equivalent:

(a) For some integer $n > 0$: $I^n M = 0$

(b) $I \subseteq \bigcap_{P \in \mathrm{Ass}_A(M)} P$

(c) $\mathrm{Supp}_A(M) \subseteq V(I)$.

Proof: (a) $\Leftrightarrow$ (b): By (2.11): $\mathrm{Ass}_A(M)$ is a finite set. Moreover, by (2.16) the minimal prime ideals of $\mathrm{Ass}_A(M)$ and $\mathrm{Supp}_A(M)$ are identical. Thus:

$$\bigcap_{P \in \mathrm{Ass}_A(M)} P = \bigcap_{\substack{P \in \mathrm{Ass}_A(M)\\ P \text{ min}}} P = \bigcap_{\substack{P \in \mathrm{Supp}_A(M)\\ P \text{ min}}} P = \bigcap_{P \in \mathrm{Supp}_A(M)} P := K$$

By (2.14): $\mathrm{Supp}_A(M) = V(\mathrm{ann}_A(M))$. This implies that $\mathrm{rad}(\mathrm{ann}_A(M)) = K$. Since I is finitely generated: (b) $\Leftrightarrow I \subseteq K = \mathrm{rad}(\mathrm{ann}_A(M)) \Leftrightarrow I^n \subseteq \mathrm{ann}_A(M)$ for some $n \in \mathbb{N} \Leftrightarrow$ (a).

(c) $\Leftrightarrow$ (a): Since $\mathrm{Supp}_A(M) = V(\mathrm{ann}_A(M))$: (c) $\Leftrightarrow V(\mathrm{ann}_A(M)) \subseteq V(I) \Leftrightarrow$ $\mathrm{rad}(I) \subseteq \mathrm{rad}(\mathrm{ann}_A(M)) \Leftrightarrow I^n \subseteq \mathrm{ann}_A(M)$ for some $n \in \mathbb{N}$, since I is finitely generated.

(2.20) <u>Lemma</u>: Let A be a Noetherian ring, $a \in A$, and M a finitely generated A-module. The following are equivalent:

(a) The A-linear map: $t_a : M \longrightarrow M$ with $t_a(m) = am$, $\forall m \in M$, is nilpotent.

(b) $a \in \bigcap_{P \in \mathrm{Ass}_A(M)} P$.

<u>Proof</u>: (a) $\Rightarrow$ (b): Since M is finitely generated and t_a nilpotent there is an $n \in \mathbb{N}$ with $a^n m = 0$ for all $m \in M$. Thus: $a^n \in \mathrm{ann}_A(M) \subseteq \mathrm{rad}(\mathrm{ann}_A(M)) = \bigcap_{P \in \mathrm{Ass}_A(M)} P$.

(b) $\Rightarrow$ (a): $a \in \bigcap_{P \in \mathrm{Ass}_A(M)} P = \mathrm{rad}(\mathrm{ann}_A(M))$ implies $a^n \in \mathrm{ann}_A(M)$ for some $n \in \mathbb{N}$.

(2.21) <u>Proposition</u>: Let A be a Noetherian ring and M a finitely generated A-module. The following are equivalent:

(a) For all $a \in A$ the A-linear map: $t_a : M \longrightarrow M$ is either injective or nilpotent.

(b) $|\mathrm{Ass}_A(M)| = 1$.

<u>Proof</u>: (a) $\Leftrightarrow$ $\forall a \in A$: $a \in \bigcap_{P \in \mathrm{Ass}_A(M)} P$ or $a \in NZD(M) \underset{(2.6)}{=} A - \bigcup_{P \in \mathrm{Ass}_A(M)} P$

$\Leftrightarrow$ $A = \bigcap_{P \in \mathrm{Ass}_A(M)} P \cup \left(A - \bigcup_{P \in \mathrm{Ass}_A(M)} P\right) \Leftrightarrow \bigcup_{P \in \mathrm{Ass}_A(M)} P \subseteq \bigcap_{P \in \mathrm{Ass}_A(M)} P \Leftrightarrow |\mathrm{Ass}_A(M)| = 1$.

(2.22) <u>Remark</u>: If (a) or (b) in (2.21) are satisfied then $\mathrm{Ass}_A(M) = \{P\}$ where $P = \{a \in A \mid t_a \text{ is nilpotent}\}$. This follows from (2.6): $ZD(M) = \bigcup_{P \in \mathrm{Ass}_A(M)} P = P$.

(2.23) <u>Definition</u>: Let A be a Noetherian ring, M a finitely generated A-module, and $N \subseteq M$ a submodule. N is called <u>primary</u> in M if $|\mathrm{Ass}_A(M/N)| = 1$. More precisely, if $\mathrm{Ass}_A(M/N) = \{P\}$ then N is called P-<u>primary</u> in M.

(2.24) <u>Proposition</u>: Let A be a Noetherian ring.

(a) A prime ideal $P \subseteq A$ is P-primary.

(b) Let $P \subseteq A$ be a prime ideal and $Q \subseteq A$ a P-primary ideal. Then there is an

$n \in \mathbb{N}$ with $P^n \subseteq Q$ and $Q \subseteq P$.

(c) An ideal $Q \subseteq A$ is primary if and only if for all $a, b \in A$ with $ab \in Q$ and $a \notin Q$ there is an $n \in \mathbb{N}$ with $b^n \in Q$.

(d) If $Q \subseteq A$ is primary then Q is P-primary where $P = \operatorname{rad}(Q)$.

<u>Proof</u>: (a) By (2.4): $\operatorname{Ass}_A(A/P) = \{P\}$.

(b) Let $Q \subseteq A$ be a P-primary ideal. Then $\operatorname{Ass}_A(A/Q) = \{P\}$ and by (2.16) P is the only minimal prime ideal containing Q. Thus $\operatorname{rad}(Q) = P$. Since P is finitely generated there is an $n \in \mathbb{N}$ with $P^n \subseteq Q$.

(c) "$\Rightarrow$": Suppose that $Q \subseteq A$ is primary. Let $a, b \in A$ with $ab \in Q$ and $a \notin Q$. The map: $t_b : A/Q \longrightarrow A/Q$ is either nilpotent or injective. Since $t_b(\bar{a}) = \overline{ab} = 0$ and $\bar{a} \neq 0$ in A/Q, t_b is nilpotent and $b^n \in Q$ for some $n \in \mathbb{N}$.

"$\Leftarrow$": Let $b \in A$ so that $t_b : A/Q \longrightarrow A/Q$ is not injective. Then there is an $a \in A - Q$ with $t_b(\bar{a}) = 0$ or equivalently: $ab \in Q$. By assumption $b^n \in Q$ for some $n \in \mathbb{N}$ and t_b is nilpotent.

(d) follows from (b).

(2.25) <u>Remark</u>: For a prime ideal $P \subseteq A$, in general, the ideals P^n for $n \geq 2$ are <u>not</u> P-primary. If P is maximal, however, we have the following result:

(2.26) <u>Lemma</u>: Let A be a Noetherian ring, $m \subseteq A$ a maximal ideal and $Q \subseteq A$ an ideal with $\operatorname{rad}(Q) = m$. Q is m-primary.

<u>Proof</u>: Since $\operatorname{rad}(Q) = m$ and m maximal: $\operatorname{Ass}_A(A/Q) = \{m\}$.

(2.27) <u>Theorem</u>: Let A be a Noetherian ring, M an A-module and $\Phi \subseteq \operatorname{Ass}_A(M)$ a subset. Then there is a submodule $N \subseteq M$ with $\operatorname{Ass}_A(M/N) = \Phi$ and $\operatorname{Ass}_A(N) = \operatorname{Ass}_A(M) - \Phi$.

Proof: Consider the set:

$$\mathfrak{M} = \{L \subseteq M \mid L \text{ a submodule and } \mathrm{Ass}_A(L) \subseteq \mathrm{Ass}_A(M) - \Phi\}.$$

$\mathfrak{M}$ is partially ordered by inclusion. Since $\mathrm{Ass}_A(0) = \emptyset$, $(0) \in \mathfrak{M}$ and $\mathfrak{M} \neq \emptyset$.
Let $\mathcal{K} \subseteq \mathfrak{M}$ be a chain. Then $L_0 = \bigcup_{L \in \mathcal{K}} L$ is a submodule of M. If $P \in \mathrm{Ass}_A(L_0)$ then $P = \mathrm{ann}_A(m)$ for some $m \in L_0$. Since $m \in L$ for some $L \in \mathcal{K}$, $P \in \mathrm{Ass}_A(L)$ and therefore $P \in \mathrm{Ass}_A(M) - \Phi$. Thus $L_0 \in \mathfrak{M}$ and $\mathfrak{M}$ is inductively ordered.
By Zorn's Lemma there is a maximal element $N \in \mathfrak{M}$.

Claim 1: $\mathrm{Ass}_A(M/N) \subseteq \Phi$.

Pf of Cl.1: Let $P \in \mathrm{Ass}_A(M/N)$. Then there is an injective A-linear map $\varphi: A/P \to M/N$. The image of φ corresponds to a submodule $N' \subseteq M$ with $N \subsetneq N'$ and $N'/N \cong A/P$. Thus $\mathrm{Ass}_A(N'/N) = \{P\}$. By (2.9): $\mathrm{Ass}_A(N') \subseteq \mathrm{Ass}_A(N) \cup \{P\}$. Since N is maximal in $\mathfrak{M}$, $N' \notin \mathfrak{M}$, and therefore: $P \in \Phi$.

Claim 2: $\mathrm{Ass}_A(M/N) = \Phi$ and $\mathrm{Ass}_A(N) = \mathrm{Ass}_A(M) - \Phi$.

Pf of Cl.2: We know: $\mathrm{Ass}_A(M/N) \subseteq \Phi$ and $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M) - \Phi$.
On the other hand by (2.9): $\mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N)$. This forces:
$\mathrm{Ass}_A(N) = \mathrm{Ass}_A(M) - \Phi$ and $\mathrm{Ass}_A(M/N) = \Phi$.

(2.28) Definition: Let A be a Noetherian ring, M an A-module and $N \subseteq M$ a submodule. A finite family of primary submodules $\{Q_1, \ldots, Q_n\}$ with

$$N = \bigcap_{i=1}^{n} Q_i$$

is called a primary decomposition of N in M.

(2.29) Example: Let $A = M = \mathbb{Z}$ and $N = (n)$ for some $n \in \mathbb{Z} - (0)$. The integer n has a prime decomposition: $n = (\pm 1)\, p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ where the p_i are distinct positive prime numbers. Then

$$(n) = (p_1)^{\alpha_1} \cap \cdots \cap (p_r)^{\alpha_r}$$

is a primary decomposition of the ideal (n) in $\mathbb{Z}$.

(2.30) Theorem: (Existence of primary decompositions) Let A be a Noetherian ring, M a finitely generated A-module, and $N \subseteq M$ a submodule of M. There are finitely many primary submodules $Q_1, \ldots, Q_r$ of M such that:

(a) $N = Q_1 \cap \ldots \cap Q_r$

(b) For all $1 \leq i \leq r$ Q_i is P_i-primary

(c) $\mathrm{Ass}_A(M/N) = \{P_1, \ldots, P_r\}$.

Proof: First note that $N = \bigcap_{i=1}^{r} Q_i$ is a primary decomposition of N in M if and only if $(0) = \bigcap_{i=1}^{r} Q_i/N$ is a primary decomposition of (0) in M/N. We may assume that $N = (0)$ and $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$. By (2.27) there are submodules Q_i of M such that: $\mathrm{Ass}_A(M/Q_i) = \{P_i\}$ and $\mathrm{Ass}_A(Q_i) = \{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_r\}$. By definition the submodules Q_i are P_i-primary in M. Set $L = Q_1 \cap \ldots \cap Q_r$. Since $L \subseteq Q_i$ for all $1 \leq i \leq r$ $\mathrm{Ass}_A(L) \subseteq \mathrm{Ass}_A(Q_i)$. But $\bigcap_{i=1}^{r} \mathrm{Ass}_A(Q_i) = \emptyset$ and $\mathrm{Ass}_A(L) = \emptyset$. This shows: $L = (0)$.

(2.31) Lemma: Let A be a ring and $M_1, \ldots, M_r$ A-modules. Then

$$\mathrm{Ass}_A\Big(\bigoplus_{i=1}^{r} M_i\Big) = \bigcup_{i=1}^{r} \mathrm{Ass}_A(M_i).$$

Proof: By induction on r. For the induction step consider the exact sequences:

$$0 \longrightarrow M_r \longrightarrow \bigoplus_{i=1}^{r} M_i \longrightarrow \bigoplus_{i=1}^{r-1} M_i \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{i=1}^{r-1} M_i \longrightarrow \bigoplus_{i=1}^{r} M_i \longrightarrow M_r \longrightarrow 0$$

This shows: $\mathrm{Ass}_A(\bigoplus_{i=1}^{r} M_i) \subseteq \mathrm{Ass}_A(\bigoplus_{i=1}^{r-1} M_i) \cup \mathrm{Ass}_A(M_r)$ and $\mathrm{Ass}_A(\bigoplus_{i=1}^{r-1} M_i) \subseteq \mathrm{Ass}_A(\bigoplus_{i=1}^{r} M_i)$.
By induction hypothesis: $\mathrm{Ass}_A(\bigoplus_{i=1}^{r} M_i) = \bigcup_{i=1}^{r} \mathrm{Ass}_A(M_i)$.

(2.32) Proposition: Let A, M, and N be as in (2.30) and suppose that $N = Q_1 \cap \ldots \cap Q_r$ is a primary decomposition of N in M with Q_i P_i-primary. Then $\mathrm{Ass}_A(M/N) \subseteq \{P_1, \ldots, P_r\}$.

Proof: The A-linear map $\varphi: M/N \longrightarrow \bigoplus_{i=1}^{r} M/Q_i$ defined by $\varphi(m+N) = (m+Q_1, \ldots, m+Q_r)$ is injective. Thus $\mathrm{Ass}_A(M/N) \subseteq \mathrm{Ass}_A(\bigoplus_{i=1}^{r} M/Q_i) = \bigcup_{i=1}^{r} \mathrm{Ass}_A(M/Q_i) = \{P_1, \ldots, P_r\}$.

(2.33) Remark: Primary decompositions are not unique. For example, let A be a Noetherian ring and $P \subsetneq Q$ prime ideals of A. P and $P \cap Q$ are primary decompositions of P.

(2.34) Lemma: Let A be a Noetherian ring, M a finitely generated A-module and $Q_1, \ldots, Q_n \subseteq M$ P-primary submodules. $Q = Q_1 \cap \ldots \cap Q_n$ is a P-primary submodule of M.

Proof: Consider again the injective A-linear map: $M/Q \longrightarrow \bigoplus_{i=1}^{n} M/Q_i$. Then

$$\emptyset \neq \mathrm{Ass}_A(M/Q) \subseteq \bigcup_{i=1}^{n} \mathrm{Ass}_A(M/Q_i) = \{P\}.$$

(2.35) Definition: Let A be a Noetherian ring, M a finitely generated A-module and $N \subseteq M$ a submodule. A primary decomposition $N = \bigcap_{i=1}^{n} Q_i$ of N in M is called a shortest primary decomposition if:

(a) Q_i is P_i-primary and $P_i \neq P_j$ whenever $i \neq j$.

(b) For all $1 \leq i \leq n$: $\bigcap_{j=1, j\neq i}^{n} Q_j \nsubseteq Q_i$.

(2.36) Remark: Under the assumptions of (2.35) every submodule N has a shortest primary decomposition.

(2.37) Proposition: Assumptions as in (2.35). Let $N = \bigcap_{i=1}^{n} Q_i$ be a shortest primary decomposition of N in M. Then:

(a) $\mathrm{Ass}_A(Q_i/N) = \bigcup_{j=1, j\neq i}^{n} \mathrm{Ass}_A(M/Q_j)$ for all $1 \leq i \leq n$.

(b) $\mathrm{Ass}_A(M/N) = \bigcup_{i=1}^{n} \mathrm{Ass}_A(M/Q_i)$.

Proof: (b) By (2.32): $\mathrm{Ass}_A(M/N) \subseteq \{P_1, \ldots, P_n\}$. For $1 \leq i \leq n$ set $L_i = \bigcap_{j=1, j\neq i}^{n} Q_j$.

Then $L_i \cap Q_i = N$ and $L_i \neq N$ since $N = \bigcap Q_i$ is a shortest primary decomposition.
Thus $(0) \neq L_i/N = L_i/L_i \cap Q_i \cong L_i + Q_i/Q_i \hookrightarrow M/Q_i$. This implies that $\mathrm{Ass}_A(L_i/N) = \{P_i\}$ and since $L_i/N \subseteq M/N$: $P_i \in \mathrm{Ass}_A(M/N)$.

(a) Obviously, $N = \bigcap_{j=1, j\neq i}^{n} (Q_j \cap Q_i)$. Consider the injective A-linear map:

$$\varphi : Q_i/N \longrightarrow \bigoplus_{j=1, j\neq i}^{n} \left(Q_i/Q_j \cap Q_i \right)$$

This shows: $\mathrm{Ass}_A(Q_i/N) \subseteq \bigcup_{j=1, j\neq i}^{n} \mathrm{Ass}_A(Q_i/Q_j \cap Q_i)$.

Since $Q_i \nsubseteq Q_j$: $0 \neq Q_i/Q_i \cap Q_j \cong Q_i + Q_j/Q_j \subseteq M/Q_j$ and $\mathrm{Ass}_A(Q_i/Q_i \cap Q_j) = \{P_j\}$.
This shows: $\mathrm{Ass}_A(Q_i/N) \subseteq \bigcup_{j=1, j\neq i}^{n} \mathrm{Ass}_A(M/Q_j)$.

"$\supseteq$": Consider the exact sequence: $0 \longrightarrow Q_i/N \longrightarrow M/N \longrightarrow M/Q_i \longrightarrow 0$.
By (b): $\mathrm{Ass}_A(M/N) = \{P_1, \ldots, P_n\} \subseteq \mathrm{Ass}_A(Q_i/N) \cup \mathrm{Ass}_A(M/Q_i) = \mathrm{Ass}_A(Q_i/N) \cup \{P_i\}$.
Therefore: $\{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\} \subseteq \mathrm{Ass}_A(Q_i/N)$.

(2.38) Theorem: Let A be a Noetherian ring, M a finitely generated A-module and $N \subseteq M$ a submodule. A primary decomposition $N = \bigcap_{i=1}^{n} Q_i$ is a shortest primary decomposition of N in M if and only if for all $1 \leq i \leq n$:

(a) $\mathrm{Ass}_A(M/Q_i) \subseteq \mathrm{Ass}_A(M/N)$

(b) $\mathrm{Ass}_A(M/Q_i) \neq \mathrm{Ass}_A(M/Q_j)$ if $i \neq j$.

Proof: "$\Longrightarrow$": By definition of a shortest primary decomposition and (2.37).
"$\Longleftarrow$": Define $\mathrm{Ass}_A(M/Q_i) = \{P_i\}$. It remains to show that for all $1 \leq i \leq n$: $\bigcap_{j=1, j\neq i}^{n} Q_j \nsubseteq Q_i$. Suppose the contrary: $\bigcap_{j=1, j\neq i}^{n} Q_j \subseteq Q_i$ for some $1 \leq i \leq n$.
Then: $N = \bigcap_{j=1, j\neq i}^{n} Q_j$ and the A-linear map: $M/N \longrightarrow \bigoplus_{j=1, j\neq i}^{n} M/Q_j$ is injective. This implies: $\mathrm{Ass}_A(M/N) \subseteq \{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\}$ and $\mathrm{Ass}_A(M/Q_i) = \{P_i\} \nsubseteq \mathrm{Ass}_A(M/N)$, a contradiction.

(2.39) <u>Summary</u>: Let A be a Noetherian ring, M a finitely generated A-module, and $N \subseteq M$ a submodule. N has a shortest primary decomposition in M: $N = \bigcap_{i=1}^{n} Q_i$. By (2.37): $\mathrm{Ass}_A(M/N) = \bigcup_{i=1}^{n} \mathrm{Ass}_A(M/Q_i) = \{P_1, \dots, P_n\}$ and the prime ideals P_i with $\mathrm{Ass}_A(M/Q_i) = \{P_i\}$ are unique. In general, the <u>primary components</u> Q_i are not unique – at least not all of them.

(2.40) <u>Theorem</u>: Let A be a Noetherian ring, M a finitely generated A-module, and $N \subseteq M$ a submodule with a shortest primary decomposition $N = \bigcap_{i=1}^{n} Q_i$. Assume that $\mathrm{Ass}_A(M/Q_i) = \{P_i\}$ and let P be a minimal prime in $\mathrm{Ass}_A(M/N)$. Then $P = P_j$ for some $1 \leq j \leq n$ and $Q_j = i_{M,P_j}^{-1}(N_{P_j})$ where $i_{M,P_j} : M \longrightarrow M_{P_j}$ is the canonical map. The primary components of N belonging to minimal prime ideals of $\mathrm{Ass}_A(M/N)$ are unique.

<u>Proof</u>: Let $P \in \mathrm{Ass}_A(M/N)$ be minimal. We may assume $P = P_1$ and set $Q = Q_1$. The shortest primary decomposition of $N = \bigcap_{i=1}^{n} Q_i$ in M corresponds to the shortest primary decomposition of $(0) = \bigcap_{i=1}^{n} Q_i/N$ in M/N. Using the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{i_{M,P}} & M_P \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ M/N & \xrightarrow{i_{M/N,P}} & (M/N)_P = M_P/N_P \end{array}$$

we may assume that $N = 0$. By (2.37): $\mathrm{Ass}_A(Q) = \{P_2, \dots, P_n\}$. By (2.12) and the minimality of P: $\mathrm{Ass}_{A_P}(M_P) = \{PA_P\}$ and $\mathrm{Ass}_{A_P}(Q_P) = \emptyset$ (since $P_i \cap (A-P) \neq \emptyset$ for all $2 \leq i \leq n$). Thus $Q_P = 0$ in M_P and $Q \subseteq i_{M,P}^{-1}(Q_P) = i_{M,P}^{-1}(0)$.
Let $U = i_{M,P}^{-1}(Q_P) = i_{M,P}^{-1}(0)$. Since $U/Q \subseteq M/Q$: $\mathrm{Ass}_A(U/Q) \subseteq \mathrm{Ass}_A(M/Q) = \{P\}$.
If $U/Q \neq 0$, $\mathrm{Ass}_A(U/Q) = \{P\}$ and by (2.12) $\mathrm{Ass}_{A_P}(U_P/Q_P) = \{PA_P\}$.
But $U_P = Q_P = 0$ and $\mathrm{Ass}_{A_P}(U_P/Q_P) = \emptyset$, a contradiction. Thus $U = Q = i_{M,P}^{-1}(0)$.