§ 4: FINITENESS CONDITIONS

(1.53) Proposition: Let $(\mathcal{M}, \leq)$ be a partially ordered set. The following are equivalent:
(a) Every ascending chain $M_1 \leq M_2 \leq \dots$ of elements of $\mathcal{M}$ is stationary, that is, there is an $n \in \mathbb{N}$ such that $M_n = M_{n+k}$ for all $k \in \mathbb{N}$.
(b) Every nonempty subset of $\mathcal{M}$ has a maximal element.

Proof: (a) $\Rightarrow$ (b): Suppose that (b) is false and let $\mathcal{T} \subseteq \mathcal{M}$ be a nonempty subset which doesn't have a maximal element. Let $M_1 \in \mathcal{T}$. M_1 is not maximal in $\mathcal{T}$, thus there is an element $M_2 \in \mathcal{T}$ with $M_1 \lneq M_2$. Since $M_2 \in \mathcal{T}$ is not maximal, there is an $M_3 \in \mathcal{T}$ with $M_2 \lneq M_3$. Continuing we obtain an infinite sequence:

$$M_1 \lneq M_2 \lneq M_3 \lneq \dots .$$

(b) $\Rightarrow$ (a): Let $M_1 \leq M_2 \leq \dots$ be an ascending chain in $\mathcal{M}$. Apply (b) to the subset: $\mathcal{T} = \{M_1, M_2, \dots \}$.

(1.54) Proposition: Let $(\mathcal{M}, \leq)$ be a partially ordered set. The following are equivalent:
(a) Every descending chain $M_1 \geq M_2 \geq \dots$ of elements of $\mathcal{M}$ is stationary, that is, there is an $n \in \mathbb{N}$ such that $M_n = M_{n+k}$ for all $k \in \mathbb{N}$.
(b) Every nonempty subset of $\mathcal{M}$ has a minimal element.

Proof: similar to the proof of (1.53).

(1.55) Definition: Let M be an A-module. The set $\mathcal{M} = \{N \subseteq M \mid N \text{ a submodule}\}$ of submodules of M is partially ordered by inclusion.
(a) M satisfies the ascending chain condition or M is a Noetherian A-module if $(\mathcal{M}, \subseteq)$ satisfies the conditions of (1.53).
(b) M satisfies the descending chain condition or M is an Artinian A-module if $(\mathcal{M}, \subseteq)$ satisfies the conditions of (1.54).

(c) A is called a <u>Noetherian</u> (<u>Artinian</u>) ring if the A-module A is Noetherian (Artinian).

(1.56) <u>Examples:</u> (a) Every PID is Noetherian. In particular, $\mathbb{Z}$ and $K[x]$, the polynomial ring in one variable over a field K, are Noetherian.
(b) Let $n \in \mathbb{N}$ with $n \neq 0$. $\mathbb{Z}/n\mathbb{Z}$ is Artinian.

<u>Proof</u>: (a) Let A be a PID and $(a_1) \subseteq (a_2) \subseteq \dots$ a chain of ideals of A. The union $I = \bigcup_{i \in \mathbb{N}} (a_i)$ is an ideal of A and thus principal: $I = (a)$. Then $a \in (a_n)$ for some $n \in \mathbb{N}$ and $(a_n) = (a_{n+k})$ for all $k \in \mathbb{N}$.
(b) If $I \subseteq \mathbb{Z}/n\mathbb{Z}$ is an ideal, then $I = d\,\mathbb{Z}/n\mathbb{Z}$ where d is a divisor of n. $\mathbb{Z}/n\mathbb{Z}$ has only finitely many ideals $\Rightarrow$ $\mathbb{Z}/n\mathbb{Z}$ is Artinian (and Noetherian).
<u>Note</u>: $\mathbb{Z}$ is not Artinian. The descending chain of ideals: $(2) \supseteq (4) \supseteq (8) \supseteq \dots$ is not stationary.

(1.57) <u>Lemma</u>: Let M be an A-module, $E \subseteq F \subseteq M$ and $N \subseteq M$ submodules of M. If $E \cap N = F \cap N$ and $E+N/N = F+N/N$, then $E = F$.

<u>Proof</u>: Let $f \in F$. Since $E+N/N = F+N/N$ there are elements $e \in E$ and $n_1, n_2 \in N$ such that $f + n_1 = e + n_2 \Rightarrow f - e = n_2 - n_1 \in F \cap N = E \cap N \Rightarrow f \in E$.

(1.58) <u>Proposition</u>: Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of A-modules. M is Noetherian (Artinian) if and only if M' and M'' are Noetherian (Artinian).

<u>Proof</u>: '$\Rightarrow$': Suppose that M is Noetherian (Artinian). Let $N \subseteq M$ be a submodule. The modules N and M/N are Noetherian (Artinian). The statement follows since M' is isomorphic to a submodule N of M and M'' is isomorphic to M/N.
'$\Leftarrow$': Consider an ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M.

Identify M' with a submodule of M and M'' with the factor module M/M'.
If M' and M'' are Noetherian the chains

$$M_1 \cap M' \subseteq M_2 \cap M' \subseteq \dots \qquad \text{in } M'$$

$$M_1+M'/M' \subseteq M_2+M'/M' \subseteq \dots \qquad \text{in } M'' = M/M'$$

are stationary. There is an $n \in \mathbb{N}$ with $M_n \cap M' = M_{n+k} \cap M'$ and $M_n+M'/M' = M_{n+k}+M'/M'$ for all $k \in \mathbb{N}$. By (1.57): $M_n = M_{n+k}$ for all $k \in \mathbb{N}$.
In the Artinian case a similar argument works for a descending chain of submodules of M.

(1.59) Corollary: Let $M_1, M_2, \dots, M_n$ be A-modules. The following are equivalent:
(a) For all $1 \le i \le n$ M_i is Noetherian (Artinian).
(b) $\bigoplus_{i=1}^{n} M_i$ is Noetherian (Artinian).

Proof: By induction on n. Apply (1.58) to the exact sequence:

$$0 \longrightarrow M_n \longrightarrow \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0 .$$

Note: Let A be Noetherian (Artinian). For all $n \in \mathbb{N}$ A^n is Noetherian (Artinian).

(1.60) Proposition: Let A be a Noetherian (an Artinian) ring. Every finitely generated A-module is Noetherian (Artinian).

Proof: A finitely generated A-module M is a homomorphic image of A^n for some suitable $n \in \mathbb{N}$.

(1.61) Corollary: Let A be a Noetherian (Artinian) ring and $I \subseteq A$ an ideal. A/I is a Noetherian (Artinian) ring.

Proof: A/I is a fin. gen. A-module. Every ideal of A/I is an A-submodule of A/I.

§5: NOETHERIAN RINGS AND MODULES

(1.62) Proposition: An A-module M is Noetherian if and only if every submodule of M is finitely generated.

Proof: "$\Rightarrow$" Suppose that M is Noetherian and that $N \subseteq M$ is a submodule. Consider $\tau = \{ U \subseteq N \mid U \text{ a submodule and } U \text{ is finitely generated} \}$. $\tau \neq \emptyset$ since $(0) \in \tau$. M is Noetherian, thus τ has a maximal element N_0. If $N \neq N_0$ let $x \in N - N_0$. Then $Ax + N_0 \in \tau$ and $N_0 \subsetneq Ax + N_0$, a contradiction.
"$\Leftarrow$": Let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submodules of M. By assumption the submodule $N = \bigcup_{i \in \mathbb{N}} M_i$ is finitely generated. Thus $N = M_n$ for some $n \in \mathbb{N}$ and $M_n = M_{n+k}$ for all $k \in \mathbb{N}$.

(1.63) Hilbert's Basis Theorem: Let A be a Noetherian ring. The polynomial ring $A[x]$ is Noetherian.

Proof: Suppose that $A[x]$ is not Noetherian and let $I \subseteq A[x]$ be an ideal which is not finitely generated. Let $f_1 \in I - (0)$ be an element of minimal degree. Since $I \neq (f_1)$, there is an element $f_2 \in I - (f_1)$ of minimal degree. Continue to choose elements $f_n \in I$ so that $f_{i+1} \in I - (f_1, \text{—}, f_i)$ of minimal degree. Let $k_i = \deg f_i$ and let a_i be the leading coefficient of f_i. By construction $k_1 \leq k_2 \leq k_3 \leq \dots$. Let $J_n = (a_1, \text{—}, a_n) \subseteq A$. Then $J_1 \subseteq J_2 \subseteq \dots$ is an ascending chain of ideals of A. Since A is Noetherian there is an $r \in \mathbb{N}$ so that $J_r = J_{r+k}$ for all $k \in \mathbb{N}$. Hence $a_{r+1} = \sum_{i=1}^{r} b_i a_i$ for some $b_i \in A$. Consider

$$g = f_{r+1} - \sum_{i=1}^{r} b_i f_i x^{k_{r+1} - k_i}.$$

Obviously, $g \in I$, $\deg g < \deg f_{r+1}$ and $g \notin (f_1, \text{—}, f_r)$, a contradiction.

(1.64) Corollary: Let A be a Noetherian ring and $I \subseteq A[x_1, \ldots, x_n]$ an ideal in the polynomial ring over A. The ring $B = A[x_1, \ldots, x_n]/I$ is Noetherian, that is, every finitely generated algebra over a Noetherian ring is Noetherian.

(1.65) Examples of non-Noetherian rings:

(a) Let K be a field. The polynomial ring $A = K[\{x_i\}_{i \in \mathbb{N}}]$ in infinitely many variables is not Noetherian. $(x_1) \subseteq (x_1, x_2) \subseteq \ldots \subseteq (x_1, \ldots, x_n) \subseteq \ldots$ is an ascending, non-stationary chain of ideals.

(b) The ring of entire functions $A = \{ f : \mathbb{C} \longrightarrow \mathbb{C} \mid f \text{ analytic on } \mathbb{C} \}$ is not Noetherian. Define for all $n \in \mathbb{N}$:

$$I_n = \{ f \in A \mid f(z) = 0 \ \forall z \in \mathbb{N} \text{ with } z > n \}.$$

$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is an increasing, non-stationary chain of ideals in A. (Weierstraß factorization theorem)

(c) The ring of continuous functions on $[0,1]$: $A = \{ f : [0,1] \longrightarrow \mathbb{R} \mid f \text{ continuous} \}$ is not Noetherian. For all $n \in \mathbb{N}$ let

$$I_n = \{ f \in A \mid f(x) = 0 \ \forall x \in [0, \tfrac{1}{n}] \}.$$

$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is an ascending, non-stationary chain of ideals of A.

(1.66) Definition: Let A be a ring, M an A-module, $N, N' \subseteq M$ submodules and $I \subseteq A$ an ideal. Define:

$(N : N') = (N : N')_A := \{ a \in A \mid aN' \subseteq N \}$

$(N : I) = (N : I)_M := \{ x \in M \mid Ix \subseteq N \}$

$(0 : M) = (0 : M)_A = \operatorname{ann}(M)$ is called the annihilator of M. M is called a faithful A-module if $\operatorname{ann}(M) = 0$.

(1.67) Remark: $(N : N')_A$ is an ideal of A and $(N : I)_M$ is a submodule of M.

(1.68) Theorem (Cohen): Let A be a ring. If all prime ideals of A are finitely generated then A is Noetherian.

<u>Proof</u>: Consider the set:

$$\mathcal{M} = \{ I \subseteq A \mid I \text{ a non-finitely generated ideal} \}.$$

If A is not Noetherian, $\mathcal{M} \neq \emptyset$ and $\mathcal{M}$ is partially ordered by inclusion. In order to verify that Zorn's Lemma applies, let $\mathcal{K} \subseteq \mathcal{M}$ be a chain. The ideal

$$J = \bigcup_{I \in \mathcal{K}} I$$

is not finitely generated. (If $J = (f_1, \ldots, f_n)$ then $J = I$ for some $I \in \mathcal{K}$.)
Hence $\mathcal{M}$ has a maximal element I.

<u>Claim</u>: I is a prime ideal.

<u>Pf of Cl</u>: If I is not prime there are elements $a, b \in A$ with $ab \in I$ and $a \notin I$, $b \notin I$. Therefore $I + (a) \notin \mathcal{M}$ and $I + (b) \notin \mathcal{M}$. Thus there are $u_1, \ldots, u_n \in I$ such that $I + (b) = (u_1, \ldots, u_n, b)$. Since $(I : (b)) = \{ x \in A \mid xb \in I \} \supseteq I + (a)$ we also have that $(I : (b)) \notin \mathcal{M}$ and therefore $(I : (b)) = (v_1, \ldots, v_m)$ for some $v_i \in A$.
But then: $I = (u_1, \ldots, u_n, bv_1, \ldots, bv_m)$. Obviously, $(u_1, \ldots, u_n, bv_1, \ldots, bv_m) \subseteq I$. To verify the other inclusion let $z \in I$. Then $z \in I + (b)$ and there are elements $d_i \in A$ and $y \in A$ such that

$$z = \sum_{i=1}^{n} d_i u_i + by$$

$\Rightarrow$ $by \in I$ and $y \in (I : (b))$. Thus $y = \sum_{i=1}^{m} c_i v_i$ for some $c_i \in A$.

(1.69) <u>Corollary</u>: Let A_1 and A_2 be Noetherian rings. The product $A_1 \times A_2$ is a Noetherian ring.

<u>Proof</u>: Every prime ideal of $A_1 \times A_2$ is of the form $P \times A_2$ or $A_1 \times Q$ where $P \in \operatorname{Spec}(A_1)$ and $Q \in \operatorname{Spec}(A_2)$.

(1.70) <u>Proposition</u>: Let A be a ring and M a Noetherian A-module. Then $A/\operatorname{ann}(M)$ is a Noetherian ring.

<u>Proof</u>: We may replace A by $\overline{A} = A/\mathrm{ann}(M)$ and view M as an $\overline{A}$-module. Thus we can assume that M is a faithful A-module. Let $M = \sum_{i=1}^{n} Am_i$ and consider the map:

$$\begin{aligned} \varphi : A &\longrightarrow M^n \\ a &\longmapsto (am_1, \text{—}, am_n) \end{aligned}$$

φ is A-linear and injective since $\mathrm{ann}(M) = 0$. Thus A is isomorphic to a submodule of the Noetherian A-module M^n. A is Noetherian.

(1.71) <u>Theorem</u> (Formanek, 1975) Let A be a ring and B a finitely generated, faithful A-module. Suppose that the set

$$\mathcal{M} = \{ IB \mid I \text{ an ideal of } A \}$$

satisfies the ascending chain condition when partially ordered by inclusion. Then A is a Noetherian ring.

<u>Proof</u>: By (1.70) it suffices to show that there is a faithful Noetherian A-module. Suppose that B is not a Noetherian A-module. Since B is a finitely generated A-module (1.60) implies that A is not a Noetherian ring. Consider the following subset of $\mathcal{M}$:

$$\mathcal{N} = \{ IB \mid I \subseteq A \text{ an ideal and } B/IB \text{ not Noetherian} \}.$$

Since $(0) \in \mathcal{N}$, $\mathcal{N} \neq \emptyset$ and by assumption $\mathcal{N}$ has a maximal element IB, where $I \subseteq A$ is an ideal. Replace

A by $A/\mathrm{ann}(B/IB)$ and B by B/IB.

Thus we may assume:

(*) B is a finitely generated, non-Noetherian, faithful A-module and for every ideal $J \subseteq A$ with $J \neq (0)$, the factor module B/JB is Noetherian.

Consider the following set of submodules of B:

$$\Gamma := \{ N \subseteq B \mid N \text{ a submodule and } B/N \text{ is faithful over } A \}.$$

If $B = Ab_1 + \dots + Ab_n$, then

(**) $N \in \Gamma \iff \forall\, a \in A - (0) : \{ab_1, \dots, ab_n\} \not\subseteq N$.

Since B is a faithful A-module, $(0) \in \Gamma$ and Γ is partially ordered by inclusion. We want to show that Γ is inductively ordered. Let $\mathcal{K} \subseteq \Gamma$ be a chain and set $N = \bigcup_{K \in \mathcal{K}} K$. N is a submodule of B. If $N \notin \Gamma$ there is an $a \in A-(0)$ with $\{ab_1, \dots, ab_n\} \subseteq N$ by $(**)$. But then $\{ab_1, \dots, ab_n\} \subseteq K$ for some $K \in \mathcal{K}$ and $K \notin \Gamma$, a contradiction. By Zorn's Lemma Γ has a maximal element $N_0 \in \Gamma$. Replace B by B/N_0. Then:

(a) B is a faithful A-module and not-Noetherian. (If B is Noetherian by (1.70) A would be Noetherian. But A is assumed to be not Noetherian)

(b) $(*)$ implies: For every ideal $I \subseteq A$ with $I \neq (0)$ the A-module B/IB is Noetherian.

(c) By the maximality of N_0 for every nonzero submodule $N \subseteq B$ the factor module B/N is not faithful over A.

We want to show that conditions (a), (b), and (c) imply that every submodule of B is finitely generated. By (1.62) this yields the contradiction that B is a Noetherian A-module.

Let $N \subseteq B$ be a nonzero submodule. By (c) B/N is not faithful over A and there is an element $a \in A-(0)$ with $aB \subseteq N$ and B/aB Noetherian by (b). Thus N/aB is a finitely generated A-module. Since B is a finitely generated A-module the module aB is finitely generated. Thus N is finitely generated and B is Noetherian, a contradiction.

(1.72) <u>Theorem</u>: (Eagin - Nagata, 1968) Let B be a Noetherian ring and $A \subseteq B$ a subring such that B is a finitely generated A-module. Then A is a Noetherian ring.

<u>Proof</u>: Obviously, B is a faithful A-module. Apply (1.71).

§6: ARTINIAN RINGS AND MODULES

Let M be an A-module. A finite chain of submodules $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ is called a <u>normal</u> series of M. We are interested in normal series of M so that the factor modules M_i/M_{i-1} have special properties:

(1.73) <u>Definition</u>: Let M be an A-module.

(a) Two normal series of M: $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ and $(0) = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = M$ are called <u>equivalent</u> if:

(i) $n = r$

(ii) There is a permutation $\sigma \in S_n$ such that $M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}$ for all $1 \le i \le n$.

The modules M_i/M_{i-1} are called the factors of the series.

(b) The series $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_s = M$ is called a <u>refinement</u> of the series $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ if for all $0 \le i \le n$ there is an $0 \le j \le s$ with $M_i = P_j$.

(1.74) <u>Theorem</u> (Schreier): Let M be an A-module. Any two normal series of M have equivalent refinements.

<u>Proof</u>: MTH 819

(1.75) <u>Definition</u>: An A-module M is called <u>simple</u> if M does not contain a proper submodule.

(1.76) <u>Definition</u>: Let M be an A-module. A normal series of M: $(0) = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ is called a <u>composition series</u> if for all $1 \le i \le n$ the factor module M_i/M_{i-1} is a simple A-module.

<u>Note</u>: Not every A-module admits a composition series. The $\mathbb{Z}$-module $\mathbb{Z}$ has

no composition series while $\mathbb{Z}/n\mathbb{Z}$ for $n \neq 0$ has a composition series.

(1.77) <u>Definition</u>: An A-module M is called of <u>finite length</u> if there exists a composition series of M: $(0)=M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$. r is called the <u>length</u> of M, denoted $\ell_A(M) = r$. We put $\ell_A(M) = \infty$ if there is no composition series of M.

(1.78) <u>Proposition</u>: Let M be an A-module of finite length.
(a) Any normal series of M has a refinement which is a composition series.
(b) Any two composition series of M are equivalent.

<u>Proof</u>: (a) Let $(0)=M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$ be a normal series of M and $(0)=N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_s = M$ a composition series of M. By (1.74) both series have equivalent refinements. There is no proper refinement of a composition series, and a series which is equivalent to a composition series is a composition series.
(b) immediately from (1.74).

(1.79) <u>Proposition</u>: Let M be an A-module and $U \subseteq M$ a submodule.
(a) $\ell_A(M) < \infty \iff \ell_A(U) < \infty$ and $\ell_A(M/U) < \infty$
(b) If $\ell_A(M) < \infty$ then $\ell_A(M) = \ell_A(U) + \ell_A(M/U)$
(c) Suppose that $\ell_A(M) < \infty$. Then $U \neq M \iff \ell_A(U) < \ell_A(M)$ and $U \neq 0 \iff \ell_A(M/U) < \ell_A(M)$.

<u>Proof</u>: (a) "$\Leftarrow$": Consider composition series
$(0) = U_0 \subsetneq U_1 \subsetneq \dots \subsetneq U_r$ of U and $(0) = \overline{V_0} \subsetneq \overline{V_1} \subsetneq \dots \subsetneq \overline{V_s} = M/U$ of M/U.
Let $\varphi: M \longrightarrow M/U$ be the canonical map. For all $0 \le i \le s$ put $V_i = \varphi^{-1}(\overline{V_i})$.
Then $U \subseteq V_i$ for all $0 \le i \le s$ and $V_i/U = \overline{V_i}$. Thus:

$$V_i/V_{i-1} = V_i+U/V_{i-1}+U \cong (V_i+U/U)/(V_{i-1}+U/U) \cong \overline{V_i}/\overline{V_{i-1}}$$

$$(0) = \mathcal{U}_0 \subsetneq \mathcal{U}_1 \subsetneq \dots \subsetneq \mathcal{U}_r = \mathcal{U} = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_s = M$$

is a composition series of M.

"$\Rightarrow$": Consider the normal series of M: $(0) = M_0 \subsetneq M_1 = \mathcal{U} \subsetneq M$. By (1.78) this series has a refinement which is a composition series:

$$\underbrace{(0) = M_0 = \mathcal{U}_0 \subsetneq \mathcal{U}_1 \subsetneq \dots \subsetneq \mathcal{U}_r = M = \mathcal{U}}_{(*)} \underbrace{= V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_s = M}_{(**)}.$$

$(*)$ is a composition series of $\mathcal{U}$. From $(**)$ we obtain the composition series of $M/\mathcal{U}$:

$$(0) = \overline{V_0} = V_0/\mathcal{U} \subsetneq \overline{V_1} = V_1/\mathcal{U} \subsetneq \dots \subsetneq \overline{V_s} = V_s/\mathcal{U} = M/\mathcal{U}.$$

(b) and (c) follow from (a).

(1.80) <u>Remark</u>: Length is an additive function, i.e., let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then $\ell_A(M) = \ell_A(M') + \ell_A(M'')$.

<u>Proof</u>: Identify M' with a submodule $\mathcal{U}$ of M and M'' with $M/\mathcal{U}$.

<u>Recall</u>: An A-module M satisfies the a.c.c. (ascending chain condition) if every ascending chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ is stationary. M satisfies the a.c.c. $\Leftrightarrow$ M is Noetherian $\Leftrightarrow$ every nonempty set of submodules of M contains a maximal element. Similarly, an A-module M satisfies the d.c.c. (descending chain condition) if every descending chain of submodules $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is stationary. M satisfies the d.c.c. $\Leftrightarrow$ M is Artinian $\Leftrightarrow$ every nonempty set of submodules of M contains a minimal element.

(1.81) <u>Proposition</u>: Let A be a ring and M an A-module. The following are equivalent:

(a) $\ell_A(M) < \infty$

(b) M satisfies the a.c.c. and the d.c.c.

<u>Proof</u>: (a) $\Rightarrow$ (b): Every normal series of submodules of M has length $\leq \ell_A(M)$.

(b) $\Rightarrow$ (a): The a.c.c. implies that every nonempty set of submodules has a maximal element. Let $M_1 \subsetneq M$ be a maximal submodule $\mathcal{U}$ with $\mathcal{U} \neq M$; then let $M_2 \subsetneq M_1$ be a maximal submodule $V \subseteq M_1$ with $V \neq M_1$. This yields a descending chain of submodules $M \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$. By the d.c.c. the chain is stationary. By construction the factor modules are simple. The finite chain corresponds to a composition series of M.

(1.82) <u>Proposition</u>: Every prime ideal of an Artinian ring A is maximal.

<u>Proof</u>: Let $P \subseteq A$ be a prime ideal. The ring $B = A/P$ is Artinian and a domain. We want to show that B is a field. Let $b \in B-(0)$ and consider the descending chain of ideals: $(b) \supseteq (b^2) \supseteq \ldots \supseteq (b^n) \supseteq (b^{n+1}) \supseteq \ldots$. Since B is Artinian there is an $r \in \mathbb{N}$ such that $(b^r) = (b^{r+1})$. Thus there is an $a \in B$ with $b^r = ab^{r+1}$. Since B is a domain: $1 = ab$.

(1.83) <u>Proposition</u>: Let A be an Artinian ring. Then A has only finitely many maximal ideals.

<u>Proof</u>: Suppose that A has infinitely many maximal ideals. Take an infinite countable set $\{m_i\}_{i \in \mathbb{N}}$ of maximal ideals of A. Consider the set of ideals:

$$\mathfrak{M} = \{ m_1 \cap \cdots \cap m_r \mid r \in \mathbb{N} \}.$$

Since A is Artinian, $\mathfrak{M}$ has a minimal element $m_1 \cap \ldots \cap m_r$. Then $m_1 \cap \ldots \cap m_r = m_1 \cap \ldots m_r \cap m_{r+1}$ and therefore: $m_{r+1} \supseteq m_1 \cap \ldots \cap m_r$. Thus $m_{r+1} = m_i$ for some $1 \leq i \leq r$.

(1.84) <u>Proposition</u>: Let A be an Artinian ring. The nilradical $\mathrm{nil}(A)$ is nilpotent, that is, there is an $n \in \mathbb{N}$ with $(\mathrm{nil}(A))^n = (0)$.

<u>Proof</u>: A satisfies the d.c.c. Thus there is an $k \in \mathbb{N}$ with $\mathrm{nil}(A)^k = \mathrm{nil}(A)^{k+t}$

for all $t \in \mathbb{N}$. Suppose $\mathrm{nil}(A)^k \neq (0)$ and set $I = \mathrm{nil}(A)^k$. Consider

$$\mathcal{M} = \{ J \subseteq A \mid J \text{ an ideal and } JI \neq (0) \}.$$

Since $A \in \mathcal{M}$, $\mathcal{M} \neq \emptyset$, and $\mathcal{M}$ has a minimal element $J_0 \in \mathcal{M}$. $J_0 I \neq (0)$ implies that there is an element $x \in J_0$ with $xI \neq (0)$. By the minimality of J_0: $J_0 = (x)$. By assumption $I^t = I$ for all $t \in \mathbb{N}$, thus $(xI)I = xI^2 = xI \neq (0)$. This implies that $xI = (x)$. Let $y \in I$ with $x = xy$, then $x = xy = xy^2 = \dots = xy^n$. But $y \in \mathrm{nil}(A)$ and thus $x = 0$, a contradiction. Hence $I = (0)$.

Let A be an Artinian ring. By (1.82) and (1.83) we know that every prime ideal of A is maximal and that A has only finitely many maximal ideals. Let $\mathcal{M} = \{m_1, \dots, m_r\}$ be the set of maximal (prime) ideals of A. By (1.14):

$$\mathrm{nil}(A) = \bigcap_{i=1}^{r} m_i = \mathrm{Jrad}(A).$$

Since $m_1, \dots, m_r$ are mutually comaximal by the Chinese remainder theorem (1.8):

$$\mathrm{nil}(A) = \bigcap_{i=1}^{r} m_i = \prod_{i=1}^{r} m_i .$$

By (1.84) there is an $k \in \mathbb{N}$ such that:

$$\mathrm{nil}(A)^k = \Big(\prod_{i=1}^{r} m_i\Big)^k = \prod_{i=1}^{r} m_i^k = (0).$$

This shows that in an Artinian ring the zero ideal is a (finite) product of maximal ideal. We make use of this fact in order to show that an Artinian ring is Noetherian.

(1.85) <u>Theorem</u>: Let A be a ring in which the zero ideal is product of (finitely many) maximal ideals. Then A is Noetherian if and only if A is Artinian. In particular, every Artinian ring is Noetherian.

For the proof of (1.85) we need:

(1.86) <u>Lemma</u>: Let K be a field and V a K-vector space. The following are equivalent:

(a) $\dim_K(V) < \infty$

(b) $\ell_K(V) < \infty$

(c) V is a Noetherian K-vector space.

(d) V is an Artinian K-vector space.

<u>Proof</u>: Homework

<u>Proof</u> of (1.85): Let $m_i \subseteq A$, $1 \le i \le n$, be maximal ideals with $(0) = m_1 \cdots m_n$. Consider the chain of ideals $A \supseteq m_1 \supseteq m_1 m_2 \supseteq \cdots \supseteq m_1 \cdots m_n = (0)$. The factor modules $M_1 = A/m_1$ and $M_i = m_1 \cdots m_{i-1}/m_1 \cdots m_i$ are $K_i = A/m_i$-vector spaces. By (1.86): the K_i-modules M_i are Noetherian if and only if the M_i are Artinian. Hence the M_i are Noetherian A-modules if and only if they are Artinian A-modules. Put $J_i = m_1 m_2 \cdots m_i$ and consider the exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & A & \longrightarrow & M_1 & \longrightarrow & 0 \\
0 & \longrightarrow & J_2 & \longrightarrow & J_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & J_{r-2} & \longrightarrow & J_{r-1} & \longrightarrow & M_{r-2} & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & J_{r-1} & \xrightarrow{=} & M_{r-1} & \longrightarrow & 0
\end{array}
$$

A Noetherian $\Rightarrow$ J_1, M_1 Noetherian

$\Rightarrow$ J_1, J_2, M_1, M_2 Noetherian

$\vdots$

$\Rightarrow$ $J_1, \ldots, J_{r-1}, M_1, \ldots, M_{r-1}$ Noetherian

$\Rightarrow$ $M_1, \ldots, M_{r-2}, M_{r-1} = J_{r-1}$ Noetherian

$\Rightarrow$ $M_1, \ldots, M_{r-2}, M_{r-1} = J_{r-1}$ Artinian

$\Rightarrow$ $J_{r-2}, J_{r-1}, M_1, \ldots, M_{r-2}$ Artinian

$\vdots$

$\Rightarrow$ A Artinian.

A similar argument shows: A Artinian $\Rightarrow$ A Noetherian

(1.87) Corollary: Let A be a ring. A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

Proof: "$\Rightarrow$": (1.82) and (1.85)
"$\Leftarrow$": Suppose that A is Noetherian and every prime ideal of A is maximal. This implies that every prime ideal of A is maximal and minimal. We show in the next chapter (2.17) that a Noetherian has only finitely many minimal primes. Let $\mathfrak{M} = \{m_1, \dots, m_n\}$ be the set of maximal (minimal) ideals of A.
Then:
$$nil(A) = \bigcap_{i=1}^{n} m_i = \prod_{i=1}^{n} m_i.$$

Since $nil(A)$ is a finitely generated ideal there is a $k \in \mathbb{N}$ such that:
$$nil(A)^k = \prod_{i=1}^{n} m_i^k = (0).$$
By (1.85) A is Artinian.

(1.88) Corollary: Every Artinian ring is isomorphic to a (finite) product of local Artinian rings.

Proof: Let A be an Artinian ring, $\mathfrak{M} = \{m_1, \dots, m_n\}$ the finite set of maximal ideals of A. We know that there is a $k \in \mathbb{N}$ so that:
$$(0) = \prod_{i=1}^{n} m_i^k.$$

Since the ideals $m_1^k, \dots, m_n^k$ are mutually comaximal by the Chinese Remainder Theorem:
$$A \xrightarrow{\cong} \prod_{i=1}^{n} A/m_i^k.$$

The rings A/m_i^k are local Artinian with maximal ideal m_i/m_i^k.