CHAPTER 0: PRELIMINARIES

§1: RINGS

(0.1) Definition: A ring A is a (nonempty) set together with two binary operations, "+" and "·", such that:

(a) A is an abelian group with respect to "+".

(b) A is a semigroup with respect to "·".

(c) The distributive laws hold: For all $a, b, c \in A$:

$$a(b+c) = ab + ac \quad \text{and} \quad (b+c)a = ba + ca.$$

Note: A semigroup is a nonempty set with an associative operation.

Throughout the course we only study commutative rings A with an identity element 1_A, i.e. $(A, \cdot)$ is a commutative semigroup with an identity element $1_A = 1$. In the following a ring A is a commutative ring with identity element. Note that we allow the case where $1 = 0$, i.e. A may be the null ring.

(0.2) Definition: Let A and B be rings, i.e. A and B are commutive rings with identity elements 1_A and 1_B, respectively. A map $\varphi: A \longrightarrow B$ is called a homomorphism of rings if

(a) $\forall\ x, y \in A: \quad \varphi(x+y) = \varphi(x) + \varphi(y)$

(b) $\forall\ x, y \in A: \quad \varphi(xy) = \varphi(x)\varphi(y)$

(c) $\varphi(1_A) = 1_B$.

(0.3) Definition: Let A be a ring.

(a) A subset $B \subseteq A$ is called a subring of A if B is closed under addition and multiplication and if $1_A \in B$.

(b) A subset $I \subseteq A$ is called an ideal of A if:

(i) I is an additive subgroup of A
(ii) $\forall\ x \in I$ and $\forall\ y \in A$: $xy \in I$.

Note: Ideals are much more interesting than subrings! If $I \subseteq A$ is an ideal we can define the quotient ring A/I. The structure of A/I relates to the structure of A and is in many cases simpler.

(0.4) Remark: Let $\varphi : A \longrightarrow B$ be a homomorphism of rings, $I \subseteq B$ an ideal. Then:

(a) $\varphi^{-1}(I) \subseteq A$ is an ideal called the contraction of I.
(b) If $J \subseteq A$ is an ideal its image $\varphi(J)$ is not an ideal of B unless φ is surjective. The smallest ideal of B containing $\varphi(J)$ is called the extension of J.
(c) $\varphi(A) \subseteq B$ is a subring of B.

(0.5) Remark: Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be homomorphisms of rings. The composition $\psi \circ \varphi : A \longrightarrow C$ is a homomorphism of rings.

(0.6) Remark: Let A be a ring, $I \subseteq A$ an ideal. In particular, $(A,+)$ is an abelian group and $I \subseteq A$ is a normal subgroup of A. The quotient group A/I is an abelian group under the operation : $[x]+[y]=[x+y]\ \ \forall\, x,y \in A$. A/I is a commutative ring with identity $1_{A/I} = [1]$ under the multiplication: $[x][y]=[xy]$ for all $x,y \in A$. The canonical map:

$$\begin{aligned} \varepsilon :\ A &\longrightarrow A/I \\ x &\longmapsto [x] \end{aligned}$$

is a surjective homomorphism of rings.

(0.7) Remark: Let A be a ring and $I \subseteq A$ an ideal. The maps:

$$\Phi : \{J \mid J \subseteq A \text{ an ideal with } I \subseteq J\} \longrightarrow \{K \mid K \subseteq A/I \text{ an ideal}\}$$
$$J \longmapsto \varepsilon(J) = \Phi(J)$$

and

$$\Psi : \{K \mid K \subseteq A/I \text{ an ideal}\} \longrightarrow \{J \mid J \subseteq A \text{ an ideal with } I \subseteq J\}$$
$$K \longmapsto \varepsilon^{-1}(K) = \Psi(K)$$

are inverse to each other and order preserving, that is, $J_1 \subseteq J_2 \Rightarrow \Phi(J_1) \subseteq \Phi(J_2)$ and $K_1 \subseteq K_2 \Rightarrow \Psi(K_1) \subseteq \Psi(K_2)$. Conclusion: There is a one-to-one correspondence between the ideals of A which contain I and the ideals of A/I.

(0.8) <u>Definition:</u> Let $\varphi : A \longrightarrow B$ be a homomorphism of rings. The <u>kernel</u> of φ, $\ker(\varphi)$, is defined by:

$$\ker(\varphi) = \varphi^{-1}(0) = \{x \in A \mid \varphi(x) = 0\}.$$

(0.9) <u>Remark:</u> (a) $\ker(\varphi)$ is an ideal of A.
(b) $\ker(\varphi) = (0) \Longleftrightarrow \varphi$ is injective.

(0.10) <u>Theorem:</u> Let $\varphi : A \longrightarrow B$ be a homomorphism of rings and $I \subseteq \ker(\varphi)$ an ideal of A. Then there is a unique homomorphism of rings $\psi : A/I \longrightarrow B$ such that the diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & B \\ A/I \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes, i.e. $\psi \circ \varepsilon = \varphi$, where $\varepsilon : A \longrightarrow A/I$ is the canonical map.

(0.11) <u>Corollary:</u> (First Isomorphism Theorem) Let $\varphi : A \longrightarrow B$ be a homomorphism of rings. φ induces an isomorphism of rings:

$$A/\ker(\varphi) \cong \varphi(A) \subseteq B.$$

(Note that $\varphi(A)$ is a subring of B.)

(0.12) Definition: Let A be a ring; $I, J, I_\lambda \subseteq A$, $\lambda \in \Lambda$, ideals of A.

(a) $I + J = \{ a+b \mid a \in I \text{ and } b \in J \}$ the sum of the ideals I and J.

$\sum_{\lambda\in\Lambda} I_\lambda = \{ \sum_{\lambda\in\Lambda} a_\lambda \mid a_\lambda \in I_\lambda$ and all, but finitely many $a_\lambda = 0 \}$ the sum of the ideals I_λ.

$I.J = \{ \sum_{i=1}^{n} a_i b_i \mid n \in \mathbb{N},\ a_i \in I,\ b_i \in J \}$ the product of the ideals I and J.

(b) Let $S \subseteq A$ be a subset. The ideal (S) generated by S is the smallest ideal of A that contains S:

$$(S) = \bigcap_{\substack{J \subseteq A \text{ an ideal} \\ S \subseteq J}} J$$

If $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq I$, we say that I is generated by $\{a_\lambda\}_{\lambda\in\Lambda}$ if I is the smallest ideal which contains $\{a_\lambda\}_{\lambda\in\Lambda}$.

(0.13) Remark: Let A be a ring and $I, J, K \subseteq A$ ideals.

(a) If $I = (a_\lambda)_{\lambda\in\Lambda}$, that is, if I is generated by $\{a_\lambda\}_{\lambda\in\Lambda}$, then

$$I = (a_\lambda)_{\lambda\in\Lambda} = \{ \sum_{\lambda\in\Lambda} b_\lambda a_\lambda \mid b_\lambda \in A \text{ and all, but finitely many } b_\lambda = 0 \}.$$

(b) The operations "$+, \cdot, \cap$" on ideals are commutative and associative. Moreover, the distributive laws hold: $I.(J+K) = IJ + IK$.

(0.14) Definition: Let $A_1, \ldots, A_n$ be rings. The direct product of $A_1, \ldots, A_n$ is defined as:

$$A = \prod_{i=1}^{n} A_i = \{ (a_1, \ldots, a_n) \mid a_i \in A \}.$$

A is a commutative ring by component-wise addition and multiplication with identity element $1_A = (1, 1, \ldots, 1)$.

(0.15) Remark: Notation as in (0.14). For all $1 \leq j \leq n$ there are canonical maps:

$$p_j : A \longrightarrow A_j \qquad\qquad \text{and} \qquad\qquad i_j : A_j \longrightarrow A$$
$$(a_1, \ldots, a_n) \longmapsto a_j \qquad\qquad\qquad\qquad a \longmapsto (0, \ldots, 0, \underset{\substack{\uparrow \\ j^{\text{th}}}}{a}, 0, \ldots, 0)$$

The <u>projection</u> map p_j is a surjective homomorphism of rings. The <u>embedding</u> i_j is injective with the property: $i_j(a+b) = i_j(a) + i_j(b)$ and $i_j(ab) = i_j(a)\, i_j(b)$ for all $a, b \in A_j$. However, i_j is <u>not</u> a homomorphism of rings since the identity element $1 \in A_j$ is not mapped into the identity element of A (if $n \geq 2$ and at least 2 of the rings A_i are not the null ring).

(0.16) <u>Definition</u>: A ring $A \neq \{0\}$ is called an <u>integral</u> <u>domain</u> if whenever $a, b \in A$ with $ab = 0$ then $a = 0$ or $b = 0$.

(0.17) <u>Definition</u>: Let A be a ring and $S \subseteq A$ a subset. S is called a <u>multiplicative</u> <u>subset</u> of A if $1 \in S$ and $\forall\, a, b \in S \Rightarrow ab \in S$.

(0.18) <u>Definition</u> and <u>Proposition</u>: Let A be a ring and $P \subseteq A$ an ideal. The following conditions are equivalent:

(a) A/P is an integral domain.

(b) $P \neq A$ and $\forall\, a, b \in A$: $ab \in P \Rightarrow a \in P$ or $b \in P$.

(c) $P \neq A$ and $\forall$ ideals $I, J \subseteq A$: $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

(d) $A - P$ is a multiplicative subset of A.

An ideal $P \subseteq A$ which satisfies one of the above conditions is called a <u>prime</u> <u>ideal</u> of A.

<u>Proof</u>: (a) $\Rightarrow$ (b): A/P an integral domain $\Rightarrow A/P \neq \{0\} \Rightarrow P \neq A$.
Let $a, b \in A$ with $ab \in P \Rightarrow ab + P = (a+P)(b+P) = 0 + P$ in $A/P \Rightarrow a \in P$ or $b \in P$.
(b) $\Rightarrow$ (c): Suppose $I \nsubseteq P$ and $J \nsubseteq P \Rightarrow \exists\, a \in I - P$ and $b \in J - P \Rightarrow$ $ab \notin P \Rightarrow IJ \nsubseteq P$.

(c) $\Rightarrow$ (b): Set $I = (a)$ and $J = (b)$. Then $ab \in P \Leftrightarrow IJ \subseteq P$.

(b) $\Rightarrow$ (a): Let $a+P,\ b+P \in A/P$ with $(a+P)(b+P) = 0+P. \Rightarrow ab \in P \Rightarrow$ $a \in P$ or $b \in P \Rightarrow a+P = 0+P$ or $b+P = 0+P$.

(b) $\Leftrightarrow$ (d): trivial.

(0.19) <u>Examples</u>: (a) Let A be a factorial domain and $p \in A$ a prime element. Then $(p) \subseteq A$ is a prime ideal.

(b) Let K be a field and x, y, z variables over K. The ideals (x, y), (y, z) and (x, y, z) are prime ideals of $A = K[x, y, z]$.

(0.20) <u>Definition</u>: Let A be a ring and $m \subseteq A$ an ideal with $m \neq A$. m is a <u>maximal</u> <u>ideal</u> of A if for every ideal $I \subseteq A$ with $m \subseteq I$ either $m = I$ or $I = A$.

(0.21) <u>Proposition</u>: Let A be a ring and $m \subseteq A$ an ideal. The following are equivalent:

(a) A/m is a field.

(b) $m \subseteq A$ is a maximal ideal.

<u>Proof</u>: (a) $\Rightarrow$ (b): The only ideals of the field A/m are (0) and A/m. By (0.7) the only ideals of A containing m are m and A.

(b) $\Rightarrow$ (a): Let $a+m \in A/m$ with $a+m \neq 0+m$. Then $a \notin m$ and $m+(a) = A \Rightarrow \exists\ n \in m$ and $b \in A$ with $n + ab = 1 \Rightarrow (a+m)(b+m) = 1+m$.

§2: MODULES

(0.22) Definition: Let A be a ring. An A-module M is an abelian (additive) group $(M,+)$ together with a map: $\varphi: A\times M \longrightarrow M$

$$(a,m) \longmapsto \varphi(a,m) = am$$

such that:

(a) $\forall\, a\in A$, $\forall\, m_i\in M$: $a(m_1+m_2) = am_1 + am_2$

(b) $\forall\, a_i\in A$, $\forall\, m\in M$: $(a_1+a_2)m = a_1m + a_2m$

(c) $\forall\, a_i\in A$, $\forall\, m\in M$: $(a_1a_2)m = a_1(a_2m)$

(d) $\forall\, m\in M$: $1m = m$.

(0.23) Remark: Let M be an abelian group and

$$\mathrm{End}(M) = \{\tau: M \longrightarrow M \mid \tau \text{ is a homomorphism of groups}\}$$

be the set of all endomorphisms of M. $\mathrm{End}(M)$ is a noncommutative ring under the operations $(\tau+\sigma)(m) = \tau(m)+\sigma(m)$ and $(\tau\sigma)(m) = \tau(\sigma(m))$ $\forall\, m\in M$. M is an A-module if and only if there is a homomorphism of rings:

$$\Phi: A \longrightarrow \mathrm{End}(M)$$

(with $\Phi(1_A) = \mathrm{id}_M$).

(0.24) Definition: Let M and N be A-modules.

(a) a map $\varphi: M\longrightarrow N$ is an A-module homomorphism or an A-linear map if:

(i) $\forall\, m, m'\in M$: $\varphi(m+m') = \varphi(m)+\varphi(m')$

(ii) $\forall\, a\in A, \forall\, m\in M$: $\varphi(am) = a\varphi(m)$

(b) $\mathrm{Hom}_A(M,N) = \{\varphi: M\longrightarrow N \mid \varphi \text{ } A\text{-linear}\}$

denotes the set of all A-linear maps from M to N.

(0.25) Remark: Let M, N, and L be A-modules.

(a) $\varphi \in \mathrm{Hom}_A(M,N)$ and $\psi \in \mathrm{Hom}_A(N,L) \implies \psi\circ\varphi \in \mathrm{Hom}_A(M,L)$

(b) Let $\varphi_1, \varphi_2 \in \mathrm{Hom}_A(M,N)$ and define for all $m \in M$ and all $a \in A$:

$$\left.\begin{aligned} (\varphi_1+\varphi_2)(m) &:= \varphi_1(m) + \varphi_2(m) \\ (a\varphi_1)(m) &:= a\,\varphi_1(m) \end{aligned}\right\} \implies \varphi_1+\varphi_2,\ a\varphi_1 \in \mathrm{Hom}_A(M,N).$$

This defines an addition and scalar multiplication on $\mathrm{Hom}_A(M,N)$. Under these operations $\mathrm{Hom}_A(M,N)$ is an A-module.

(c) $\mathrm{Hom}_A(M,M)$ is a ring (noncommutative) under multiplication the composition of maps.

(0.26) <u>Definition</u>: Let M be an A-module.

(a) a subset $N \subseteq M$ is an A-<u>submodule</u> of M if

(i) N is a subgroup of $(M,+)$.

(ii) $\forall\, a \in A$ and $\forall\, n \in N$: $an \in N$.

(b) Let $N \subseteq M$ be an A-submodule. The factor group M/N is an A-module under the operation: $a(m+N) = am + N$ for $\forall\, a \in A$, $m \in M$. M/N is called the <u>factor</u> or <u>quotient</u> <u>module</u> of M by N.

(0.27) <u>Remark</u>: Let A be a ring and M and N A-modules.

(a) A is naturally an A-module. The A-submodules of A are exactly the ideals of A.

(b) Let $L \subseteq M$ be an A-submodule. The canonical map:

$$\begin{aligned} \varphi:\ M &\longrightarrow M/L \\ m &\longmapsto m+L \end{aligned}$$

is A-linear.

(c) Let $\varphi: M \longrightarrow N$ be an A-linear map. The <u>kernel</u> of φ $\ker(\varphi) = \{m \in M \mid \varphi(m) = 0\}$ is a submodule of M and the <u>image</u> of φ $\mathrm{im}(\varphi) = \varphi(M)$ is a submodule of N.

(d) The module $M/\mathrm{im}(\varphi) = \mathrm{coker}(\varphi)$ is called the <u>cokernel</u> of φ.

(e) Let $\varphi : M \longrightarrow N$ be an A-linear map. Then
φ is injective $\Longleftrightarrow \ker(\varphi) = 0$
φ is surjective $\Longleftrightarrow \operatorname{im}(\varphi) = N \Longleftrightarrow \operatorname{coker}(\varphi) = 0$
(f) Let $N \subseteq M$ be an A-submodule. Then there is a 1-1 correspondence between the submodules L of M with $N \subseteq L$ and the submodules of M/N.

(0.28) <u>Proposition:</u> Let $\varphi : M \longrightarrow N$ be a linear map of A-modules and $\mathcal{U} \subseteq M$ a submodule with $\mathcal{U} \subseteq \ker(\varphi)$. There is exactly one A-linear map $\overline{\varphi} : M/\mathcal{U} \longrightarrow N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\text{can.}"'] & N \\ M/\mathcal{U} \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

commutes. Moreover, $\ker(\overline{\varphi}) = \ker(\varphi)/\mathcal{U}$.

(0.29) <u>Remark:</u> (1st Isomorphism Theorem) If $\mathcal{U} = \ker(\varphi)$ then $\overline{\varphi}$ is injective and $M/\ker(\varphi) \cong \operatorname{im}(\varphi)$.

(0.30) <u>Examples:</u> (a) Let $A = K$ be a field. The K-modules are exactly the K-vector spaces.
(b) Every abelian group $(M,+)$ is a $\mathbb{Z}$-module by: $\forall n \in \mathbb{Z}, m \in M$:
$n = 0 : \ 0m = 0$; $n > 0 : \ nm = \underbrace{m + \dots + m}_{n\text{-times}}$; $n < 0 : \ nm = \underbrace{(-m) + \dots + (-m)}_{(-n)\text{-times}}$
(c) Let A be a ring; I an index set. Consider the set:
$A^{(I)} = \{ f : I \longrightarrow A \mid f \text{ a map with } f(i) = 0 \text{ for all, but finitely many } i \in I \}$
and define addition and scalar multiplication on $A^{(I)}$ by:
$\forall f, g \in A^{(I)}, \forall a \in A, \forall i \in I$: $\quad (f+g)(i) = f(i) + g(i)$
$(af)(i) = af(i)$.
$A^{(I)}$ is an A-module under these operations. Modules of this form are called <u>free</u> A-modules. Usually we write elements of $A^{(I)}$ as sequences

$(a_i)_{i\in I} \in A^{(I)}$ where $a_i = 0$ for almost all $i \in I$. If I is a finite set with $|I| = n$ we write $A^{(I)} = A^n$.

(0.31) <u>Remark</u>: Let M be an A-module and $M_i \subseteq M$, $i \in I$, submodules of M.

(a) $\sum_{i\in I} M_i = \{ \sum_{i\in I} m_i \mid m_i \in M_i \text{ and } m_i = 0 \text{ for almost all } i \in I \}$ is a submodule of M, called the <u>sum</u> of M_i.

(b) $\bigcap_{i\in I} M_i$ is a submodule of M.

(c) $\sum_{i\in I} M_i = \bigcap_{\substack{N \subseteq M \text{ a submodule} \\ M_i \subseteq N \ \forall i \in I}} N$ i.e. $\sum M_i$ is the smallest submodule of M which contains M_i for all $i \in I$.

(0.32) <u>Proposition</u>: (<u>More Isomorphism Theorems</u>) Let M be an A-module and $M_1, M_2 \subseteq M$ submodules.

(a) If $M_2 \subseteq M_1$ then $(M/M_2)/(M_1/M_2) \cong M/M_1$

(b) $(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$.

<u>Proof</u>: (a) Define $\varphi : M/M_2 \longrightarrow M/M_1$ by $\varphi(m + M_2) = m + M_1$. φ is a well defined, A-linear map with $\ker(\varphi) = M_1/M_2$. By the 1st Isomorphism Theorem (0.29): $(M/M_2)/(M_1/M_2) \cong \operatorname{im}(\varphi) = M/M_1$.

(b) Consider the map:
$$\psi : M_2 \longrightarrow (M_1 + M_2)/M_1$$
$$m \longrightarrow m + M_1.$$
ψ is a surjective A-linear map with $\ker(\psi) = M_1 \cap M_2$. The statement follows again with (0.29).

(0.33) <u>Definition</u>: Let M be an A-module and $\{x_i\}_{i\in I} \subseteq M$.

(a) $\{x_i\}_{i\in I}$ is called a <u>system of generators</u> of M if $\sum_{i\in I} (Ax_i) = M$.

(b) $\{x_i\}_{i\in I}$ is called <u>linearly independent</u> if whenever $a_i, b_i \in A$ with $\sum'_{i\in I} a_i x_i = \sum'_{i\in I} b_i x_i$ then $a_i = b_i$ for all $i \in I$. ($\sum'$ indicates finite sums, i.e.

$a_i = 0$ and $b_i = 0$ for all but finitely many $i \in I$.)

(c) $\{x_i\}_{i\in I}$ is called a <u>basis</u> of M if $\{x_i\}_{i\in I}$ is a linearly independent system of generators of M.

(d) M is called a <u>finitely generated</u> or <u>finite</u> (!) A-module if M has a finite system of generators.

(0.34) <u>Remark</u>: An A-module M is free $\iff$ $\exists$ an index set I such that $M \cong A^{(I)} \iff M$ has a basis.

Let $\{M_i\}_{i\in I}$ be a family of A-modules. Set

$$\bigoplus_{i\in I} M_i = \left\{ f : I \longrightarrow \dot{\bigcup_{i\in I}} M_i \;\middle|\; \begin{array}{l} f \text{ is a map with } f(i) \in M_i \text{ and } f(i) = 0 \text{ for} \\ \text{all but finitely many } i \in I \end{array} \right\}$$

Usually we write the elements of $\bigoplus M_i$ as 'sequences' $(m_i)_{i\in I}$ where $m_i \in M_i$ and $m_i = 0$ for almost all $i \in I$. $(m_i)_{i\in I}$ represents the map: $f : I \longrightarrow \dot{\bigcup} M_i$ with $f(i) = m_i \ \forall \ i \in I$.

$\bigoplus_{i\in I} M_i$ is an A-module under the operations:

$$(m_i) + (n_i) = (m_i + n_i)$$
$$a(m_i) = (am_i) \quad \forall \ a \in A.$$

$\bigoplus_{i\in I} M_i$ is called the direct sum of $\{M_i\}_{i\in I}$.

(0.35) <u>Remark</u>: If $M_i = A$ for all $i \in I$, then $\bigoplus_{i\in I} A = A^{(I)}$.

(0.36) <u>Proposition</u>: Let N be an A-module and $M_1, \ldots, M_n \subseteq N$ submodules. Suppose:

(a) $\sum_{i=1}^{n} M_i = N$

(b) $\forall \ 2 \leq i \leq n : \ M_i \cap (M_1 + \cdots + M_{i-1}) = (0)$.

Then $\quad N \cong \bigoplus_{i=1}^{n} M_i$

Proof: By induction on n. $n=1$: trivial

$n-1 \Rightarrow n$: Consider the A-linear map $\varphi: \bigoplus_{i=1}^{n} M_i \longrightarrow N$ defined by $\varphi(m_1, \ldots, m_n) = m_1 + \ldots + m_n$. By (a) φ is surjective. Suppose $\varphi(m_1, \ldots, m_n) = \sum_{i=1}^{n} m_i = 0 \Rightarrow m_n = -\sum_{i=1}^{n-1} m_i \in M_n \cap (M_1 + \cdots + M_{n-1}) = (0)$. Thus $m_n = 0$ and $\sum_{i=1}^{n-1} m_i = 0$. Apply the induction hypothesis to $N' = \sum_{i=1}^{n-1} M_i \subseteq N$. Since $N' \cong \bigoplus_{i=1}^{n-1} M_i$ we obtain $m_i = 0$ for all $i = 1, \ldots, n-1$. This shows that φ is injective.

Let $\{M_i\}_{i\in I}$ be a family of A-modules. For all $i \in I$ there are A-linear maps: $\varkappa_i : M_i \longrightarrow \bigoplus_{i\in I} M_i$ and $p_i : \bigoplus_{i\in I} M_i \longrightarrow M_i$ defined by $\varkappa_i(m) = (m_j)$ where $m_j = 0$ if $i \neq j$ and $m_i = m$ and

$$p_i((m_j)) = m_i.$$

p_i is surjective and is called the projection onto M_i. $\varkappa_i$ is injective. We consider M_i a submodule of $\bigoplus_{i\in I} M_i$ via $\varkappa_i$. Note that $p_i \circ \varkappa_i = \mathrm{id}_{M_i}$.

(0.37) Proposition: Let $\{M_i\}_{i\in I}$ be a family of A-modules and N an A-module.

(a) Suppose that for every $i \in I$ there is given an A-linear map $f_i : M_i \longrightarrow N$. Then there is exactly one A-linear map $f : \bigoplus_{i\in I} M_i \longrightarrow N$ with $f \circ \varkappa_i = f_i$ $\forall\, i \in I$ (or $f|_{M_i} = f_i$, considering M_i a submodule of $\bigoplus M_i$.)

(b) Suppose for all $i \in I$ there is an A-linear map $g_i : N \longrightarrow M_i$ such that for all $n \in N$: $g_i(n) = 0$ for all but finitely many $i \in I$. Then there is a unique A-linear map $g : N \longrightarrow \bigoplus_{i\in I} M_i$ with $p_i \circ g = g_i$ for all $i \in I$.

Proof: Homework