CHAPTER IX: COMPLETION; NORMALIZATION THEOREMS

§1: COMPLETION

Let A be a ring, M an A-module, I a directed index set, and $\mathcal{F} = \{M_i \mid i \in I\}$ a family of submodules of M with $M_i \supseteq M_j$ for $i \leq j$. For every $x \in M$ let $\{x + M_i \mid i \in I\}$ be a system of neighborhoods of x and define a topology on M as follows:

$\mathcal{U} \subseteq M$ open $\Longleftrightarrow$ $\forall\, x \in \mathcal{U}$ $\exists\, M_i \in \mathcal{F}$ with $x + M_i \subseteq \mathcal{U}$.

This topology is called a <u>linear topology</u> on M.

(9.1) <u>Remark</u>: (a) $M \times M \xrightarrow{+} M$ is continuous.

(b) For all $a \in A$ multiplication by a: $M \xrightarrow{a\cdot} M$ is continuous.

(c) For $M = A$: $M \times M \xrightarrow{\cdot} M$ is continuous.

(9.2) <u>Remark</u>: (a) Every M_i is open and closed.

(b) $\overline{0} = \bigcap_{i \in I} M_i$

(c) M is separated (Hausdorff) if and only if $\bigcap_{i \in I} M_i = 0$.

<u>Proof</u>: (a) For all $x \in M_i$, $x + M_i \subseteq M_i$ and M_i is open. Moreover, for all $y \in M$ the set $y + M_i$ is open. Then $M - M_i = \bigcup_{y \notin M_i} (y + M_i)$ is open as a union of open sets. Thus M_i is closed.

(b) If $x \in \overline{0}$ then $(x + M_i) \cap \{0\} \neq \emptyset$ for all i and for all $i \in I$ there is an $m_i \in M_i$ with $x = m_i$. Hence $x \in \bigcap_{i \in I} M_i$. Conversely, if $x \in \bigcap_{i \in I} M_i$, then for all $i \in I$ there is an $m_i \in M_i$ with $x = m_i$. Thus $(x + M_i) \cap \{0\} \neq \emptyset$ for all $i \in I$.

(c) "$\Rightarrow$": If M is separated, $\{0\}$ is a closed subset. By (b): $\{0\} = \bigcap_{i \in I} M_i$.

"$\Leftarrow$": Let $x, y \in M$ with $x \neq y$. Since $x - y \neq 0$ there is an $i \in I$ with $x - y \notin M_i$. Since M_i is a submodule of M, $(x + M_i) \cap (y + M_i) = \emptyset$.

(9.3) Remark: Let N be a submodule of M.

(a) The linear topology on M/N defined by $\{M_i+N/N \mid i\in I\}$ is the quotient topology of the topology on M defined by $\{M_i \mid i\in I\}$.

(b) $\overline{N} = \bigcap_{i\in I} (N+M_i)$.

Proof: (a) Let $\pi : M \longrightarrow M/N$ be the natural map and let $\mathcal{U} \subseteq M/N$ be a subset. $\mathcal{U}$ is open in the linear topology $\iff \forall\, x\in\pi^{-1}(\mathcal{U})\ \exists\, i\in I$ with $\pi(x)+\pi(M_i+N)\subseteq \mathcal{U}$

$\iff \forall\, x\in\pi^{-1}(\mathcal{U})\ \exists\, i\in I$ with $x+M_i \subseteq \pi^{-1}(\mathcal{U})$

$\iff \mathcal{U}$ is open in the quotient topology

(b) By (a): $\overline{N}/N = \overline{N/N} = \bigcap_{i\in I}(M_i+N/N)$ where the last equation follows from (9.2)(b).

By (9.2)(b) the module $M/\bigcap_{i\in I} M_i$ is separated. It is called the <u>separated module associated</u> with M.

For all $i,j\in I$ with $i\leq j$ (and $M_i\supseteq M_j$) let $\pi_{ij} : M/M_j \longrightarrow M/M_i$ denote the natural map. Then $\{M/M_i, \pi_{ij}\}$ is an inverse system and $\widehat{M} = \varprojlim M/M_i$ is called the <u>completion</u> of M with respect to $\mathcal{F}$.

By the universal property of inverse limits there is a unique A-linear map $\varphi: M \longrightarrow \widehat{M}$ so that the diagram

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\pi_i"'] & & \widehat{M} \arrow[dl, "\beta_i"] \\ & M/M_i & \end{tikzcd}$$

commutes. Obviously, $\varphi(x) = (x+M_i \mid i\in I)$ and $\ker\varphi = \bigcap_{i\in I} M_i$. In particular, $\varphi: M\longrightarrow \widehat{M}$ is injective if and only if M is separated. If $M=A$, then $\widehat{M}=\widehat{A}$ is a ring and φ is a homomorphism of rings.

For all $i\in I$ the maps $\beta_i : \widehat{M} \longrightarrow M/M_i$ are the restrictions of the projections $\rho_i : \prod_{j\in I} M/M_j \longrightarrow M/M_i$. Let $M_i^* = \ker\beta_i$. Then $\mathcal{F}^* = \{M_i^* \mid i\in I\}$ is a family of submodules of $\widehat{M}$ with $M_i^*\supseteq M_j^*$ for $i\leq j$. Thus $\mathcal{F}^*$ defines a linear topology on $\widehat{M}$.

On the other hand, $\mathcal{F}$ induces the discrete topology on M/M_i. Consider the product topology on $\prod_{i\in I} M/M_i$ and the topology induced by it on the submodule $\hat{M} = \varprojlim M/M_i$. A basic system of neighborhoods of 0 is given by $\{\bigcap_{\text{finite}} \beta^{-1}(0)\} = \{\bigcap_{\text{finite}} M_i\}$. Since I is directed this topology on $\hat{M}$ coincides with the topology defined by $\mathcal{F}^*$.

(9.4) Definition: An A-module M is called complete w.r.t. $\mathcal{F}$ if $\varphi: M \xrightarrow{\cong} \hat{M}$.

(9.5) Proposition:

(a) $\varphi: M \longrightarrow \hat{M}$ is continuous

(b) $\ker\varphi = \bigcap_{i\in I} M_i$

(c) $\operatorname{im}\varphi$ is dense in $\hat{M}$

(d) $\hat{M}$ is complete.

Proof: (a) Let $x \in M$. We have to show that for every $M_i^* \in \mathcal{F}^*$ there is an $M_j \in \mathcal{F}$ so that $\varphi(x+M_j) \subseteq \varphi(x) + M_i^*$. But $\varphi(M_i) \subseteq \ker\beta_i = M_i^*$.

(c) Let $y \in \hat{M}$. We have to show that for every $M_i^* \in \mathcal{F}^*$ there is an $x \in M$ with $\varphi(x) \in y + M_i^*$. Since π_i is surjective there is an $x \in M$ with $\pi_i(x) = \beta_i(y)$. Then $\beta_i(\varphi(x) - y) = 0$ and $\varphi(x) - y \in M_i^*$.

(d) Since π_i are surjective, it follows that β_i are surjective and induce isomorphisms $\hat{M}/M_i^* \cong M/M_i$ that are compatible with the projection maps. Thus there is a natural isomorphism $\hat{\hat{M}} \cong \hat{M}$.

Let M and M' be A-modules with linear topologies given by $\mathcal{F} = \{M_i \mid i\in I\}$, $\mathcal{F}' = \{M_j' \mid j \in I'\}$ and let $f: M \longrightarrow M'$ be an A-linear map which is continuous, that is, for all $j \in I'$ there is an $i(j) \in I$ with $f(M_{i(j)}) \subset M_j'$.

(9.6) Proposition: There exists a unique continuous A-linear map $\hat{f}: \hat{M} \longrightarrow \hat{M}'$ so that the diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \varphi \downarrow & & \downarrow \varphi' \\ \widehat{M} & \xrightarrow{\hat{f}} & \widehat{M'} \end{array}$$ commutes.

Proof: Uniqueness follows since M is dense in $\widehat{M}$ and $\widehat{M'}$ is separated. For existence define $\hat{f}: \varprojlim M/M_i \longrightarrow \varprojlim M'/M'_j$ by sending $(x_i + M_i \mid i \in I)$ to $(f(x_{i(j)}) + M'_j \mid j \in I')$. This is a well defined A-linear map because $f(M_{i(j)}) \subseteq M'_j$ and because for $k \geq j$ in I' there is an $\ell \in I$ with $\ell \geq i(k)$ and $\ell \geq i(j)$, hence $x_{i(k)} + M_{i(k)} = x_\ell + M_{i(k)}$ and $x_{i(j)} + M_{i(j)} = x_\ell + M_{i(j)}$. Thus $f(x_{i(k)}) + M'_k = f(x_{i(j)}) + M'_k$. Since $\hat{f}(M^*_{i(j)}) \subseteq M^*_j$, the map $\hat{f}$ is continuous.

(9.7) Remark: (a) The uniqueness in (9.6) implies that $\widehat{f \circ g} = \hat{f}\,\hat{g}$ for A-linear maps $f: M \longrightarrow M'$ and $g: M'' \longrightarrow M$ and that $\widehat{id_M} = id_{\widehat{M}}$.
(b) Let $M = M'$ and assume that $\mathcal{F}$ and $\mathcal{F}'$ define the same topology on M (i.e. for all $j \in I'$ there is an $i(j) \in I$ with $M_{i(j)} \subseteq M'_j$ and for all $i \in I$ there is an $j(i) \in I'$ with $M'_{j(i)} \subseteq M_i$). Then $f = id: M \xrightarrow{\cong} M'$ is a homeomorphism. Thus by (a), $\hat{f}: \widehat{M} \xrightarrow{\cong} \widehat{M'}$ is an A-linear homeomorphism. In particular, the topology on $\widehat{M}$ only depends on the topology of M.

(9.8) Theorem: Let $N \subseteq M$ be a submodule. Then the subspace topology on N coincides with the linear topology given by $\{M_i \cap N \mid i \in I\}$.
(a) $0 \longrightarrow \widehat{N} \xrightarrow{\hat{\iota}} \widehat{M} \xrightarrow{\hat{\pi}} \widehat{M/N}$ is exact. Furthermore $\hat{\iota}(\widehat{N})$ is the closure of $\varphi(N)$ in $\widehat{M}$.
(b) If $I = \mathbb{N}$, then $0 \longrightarrow \widehat{N} \xrightarrow{\hat{\iota}} \widehat{M} \xrightarrow{\hat{\pi}} \widehat{M/N} \longrightarrow 0$ is exact. In particular, $\widehat{M/N} \cong \widehat{M}/\widehat{N}$.

Proof: (a) There are natural exact sequences $0 \longrightarrow N/M_i \cap N \longrightarrow M/M_i \longrightarrow M/N + M_i \longrightarrow 0$. Apply $\varprojlim$ and use the fact that $\varprojlim$ is left exact (7.83). This shows that the sequence of (a) is exact. To show that $\hat{\iota}(\widehat{N}) = \overline{\varphi(N)}$ notice that $\varphi(N)$ is dense in $\widehat{N}$, hence $\varphi(N)$ is dense in $\hat{\iota}(\widehat{N})$. On the other hand 0 is closed in $\widehat{M/N}$, hence

$\hat{\iota}(\hat{N}) = \hat{\pi}^{-1}(0)$ is closed in $\widehat{M}$.

(b) The inverse systems $\{N/M_i \cap N \mid i \in \mathbb{N}\}$, $\{M/M_i \mid i \in \mathbb{N}\}$, and $\{M/N+M_i \mid i \in \mathbb{N}\}$ satisfy the assumptions of (7.86). In particular, for $j \geq i$: $N/M_j \cap N \longrightarrow N/M_i \cap N$ is surjective. Thus $0 \longrightarrow \widehat{N} \longrightarrow \widehat{M} \longrightarrow \widehat{M/N} \longrightarrow 0$ is exact.

<u>Cauchy sequences</u>

In the following let M be an A-module and $\mathcal{F} = \{M_i \mid i \in \mathbb{N}\}$ a set of submodules of M with $M_j \subseteq M_i$ whenever $j \geq i$. For $x, y \in M$ define the <u>distance</u> of x and y by

$$d(x,y) = \inf\{2^{-n} \mid x - y \in M_n\}.$$

Then for all $x, y, z \in M$ the following conditions are satisfied:

(a) $d(x,y) \in \mathbb{R}$, $d(x,y) \geq 0$, $d(x,x) = 0$

(b) $d(x,y) = d(y,x)$

(c) $d(x,y) \leq d(x,z) + d(z,y)$

d defines a premetric on M. If additionally, $\bigcap_{i \in \mathbb{N}} M_i = 0$, then

(d) $d(x,y) = 0 \Leftrightarrow x = y$

and d defines a metric on M. The topology defined by d is the linear topology of M defined by $\mathcal{F}$.

An element $(x_n) \in \prod_{n \in \mathbb{N}} M$ is a <u>Cauchy sequence</u> of M (w.r.t. $\mathcal{F}$) if for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ with $x_i - x_j \in M_n$ for all $i, j \geq N$. Define

$$\mathcal{C}_{\mathcal{F}}(M) = \{(x_n) \in \textstyle\prod_{n \in \mathbb{N}} M \mid (x_n) \text{ is a Cauchy sequence w.r.t. } \mathcal{F}\}.$$

$\mathcal{C}_{\mathcal{F}}(M)$ is an A-submodule of $\prod_{n \in \mathbb{N}} M$. (If M is a ring, $\mathcal{C}_{\mathcal{F}}(M)$ is a subring of $\prod M$.)
An element $(y_n) \in \prod_{n \in \mathbb{N}} M$ is called a <u>zero sequence</u> of M if for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ so that $y_i \in M_n$ for all $i \geq N$. Let

$$\mathcal{Z}_{\mathcal{F}}(M) = \{(y_n) \in \textstyle\prod_{n \in \mathbb{N}} M \mid (y_n) \text{ is a zero sequence w.r.t. } \mathcal{F}\}.$$

$\mathcal{Z}_{\mathcal{F}}(M)$ is a submodule of $\mathcal{C}_{\mathcal{F}}(M)$. Define the $*$-<u>completion</u> of M w.r.t. $\mathcal{F}$ by:

$$M^* = (M, \mathcal{F})^* = \mathcal{C}_{\mathcal{F}}(M) / \mathcal{Z}_{\mathcal{F}}(M).$$

(9.9) Properties of the $*$-completion

(a) Let $\psi: M \longrightarrow M^*$ be defined by $\psi(x) = (x)_{n\in\mathbb{N}} + Z_{\mathcal{F}}(M)$. Then ψ is an A-linear map with $\ker(\psi) = \bigcap_{n\in\mathbb{N}} M_n$.

(b) Let M and M' be A-modules with linear topologies $\mathcal{F} = \{M_i \mid i\in\mathbb{N}\}$ and $\mathcal{F}' = \{M_i' \mid i\in\mathbb{N}\}$ and let $f: M \longrightarrow M'$ be a continuous A-linear map, that is, for all $j\in\mathbb{N}$ there is an $i(j)\in\mathbb{N}$ with $f(M_{i(j)}) \subseteq M_j'$. f induces an A-linear map $f^*: M^* \longrightarrow M'^*$.

Proof: Home work

(9.10) Theorem: There is an isomorphism $f: M^* = (M,\mathcal{F})^* \longrightarrow \widehat{M} = (M,\mathcal{F})^\wedge$ so that the diagram

$$\begin{tikzcd} & M^* \arrow[d, "f"] \\ M \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & \widehat{M} \end{tikzcd}$$

commutes, where ψ is defined as above and $\varphi: M \longrightarrow \widehat{M} = \varprojlim M/M_i$ is the natural map defined by $\varphi(x) = (x + M_i)$.

Proof: Define a map $\tilde{f}: C_{\mathcal{F}}(M) \longrightarrow \widehat{M}$ as follows: If $x = (x_n) \in C_{\mathcal{F}}(M)$ choose for all $n\in\mathbb{N}$ an integer $N(n)\in\mathbb{N}$ so that $x_i - x_j \in M_n$ for all $i,j \geq N(n)$. Define $\tilde{f}(x) = (x_{N(n)} + M_n)$ and notice that $\tilde{f}(x)$ is independent of the choice of $N(n)$, in particular, every zero sequence is mapped into $0_{\widehat{M}} = (M_n)$. $\tilde{f}$ is A-linear and factors through an A-linear map $f: M^* \longrightarrow \widehat{M}$.

In order to show that f is an isomorphism we define an A-linear map $g: \widehat{M} \longrightarrow M^*$. For $y = (y_n + M_n) \in \widehat{M}$ let $y_n \in M$ denote a representative of $y_n + M_n$. Then $(y_n) \in \prod_{n\in\mathbb{N}} M$ is a Cauchy sequence. If y_n' denotes another representative of $y_n + M_n$ in M, then $(y_n) - (y_n')$ is a zero sequence. Thus by defining $g(y) = (y_n) + Z_{\mathcal{F}}(M)$ the A-linear map g is well defined. Moreover, g is inverse to f.

§2: I-ADIC COMPLETIONS

Let A be a ring, $I \subseteq A$ an ideal, and M an A-module. The linear topology defined by $\{I^i M \mid i \in \mathbb{N}\}$ is called the I-adic topology on M. The I-adic completion $\widehat{M} = (M,I)^\wedge$ of M is the completion of M with respect to the I-adic topology: $\widehat{M} = \varprojlim M/I^iM$. The module M is called I-adically complete if $\widehat{M} \cong M$. The I-adic completion $\widehat{A}$ of A is a ring and $\widehat{M}$ is an $\widehat{A}$-module via $(a_i + I^i)(x_i + I^iM) = (a_i x_i + I^iM)$. If $f: M \longrightarrow M'$ is an A-linear map, then f is continuous in the I-adic topology since $f(I^iM) \subseteq I^iM'$. Thus $\widehat{f}$ exists and is defined by $\widehat{f}(x_i + I^iM) = (f(x_i) + I^iM')$. In particular, $\widehat{f}$ is $\widehat{A}$-linear.

(9.11) Remark: Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and $N \subseteq M$ a submodule. By the Artin-Rees theorem there is an $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$: $I^nM \cap N = I^{n-n_0}(I^{n_0}M \cap N)$. The I-adic topology on N coincides with the linear topology on N defined by $\{I^iM \cap N \mid i \in \mathbb{N}\}$. Thus (9.8) yields the following result:

(9.12) Proposition: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and $0 \to M' \to M \to M'' \to 0$ an exact sequence of finitely generated A-modules. The induced sequence of the I-adic completions $0 \longrightarrow \widehat{M'} \longrightarrow \widehat{M} \longrightarrow \widehat{M''} \longrightarrow 0$ is exact.

Let A be a ring, $I \subseteq A$ an ideal, and M an A-module. The map $\lambda: \widehat{A} \times M \longrightarrow \widehat{M}$ defined by $\lambda((a_i + I^i), m) = (a_i m + I^iM)$ is A-bilinear. Thus λ induces an A-linear map $\tau: \widehat{A} \otimes_A M \longrightarrow \widehat{M}$ defined by $\tau((a_i + I^i) \otimes m) = (a_i m + I^iM)$. Note that τ is also $\widehat{A}$-linear.

(9.13) Theorem: Let A be a Noetherian ring and M a finitely generated A-module. The $\widehat{A}$-linear map $\tau: \widehat{A} \otimes_A M \longrightarrow \widehat{M}$ is an isomorphism.

Proof: A standard induction argument shows that for all $n \in \mathbb{N}-(0)$: $\widehat{A}^n \cong \widehat{A} \otimes_A A^n \cong \widehat{A^n}$. Let M be a finitely generated A-module. Then there are $m, n \in \mathbb{N}$ with $A^m \to A^n \to M \to 0$ exact. Since the tensor product is right exact, the sequence $\widehat{A} \otimes_A A^m \to \widehat{A} \otimes_A A^n \to \widehat{A} \otimes_A M \to 0$ is exact. By (9.12) the sequence $\widehat{A^m} \to \widehat{A^n} \to \widehat{M} \to 0$ is exact. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \widehat{A} \otimes_A A^m & \longrightarrow & \widehat{A} \otimes_A A^n & \longrightarrow & \widehat{A} \otimes_A M & \longrightarrow & 0 \\ \downarrow \varphi_m & & \downarrow \varphi_n & & \downarrow \tau & & \\ \widehat{A^m} & \longrightarrow & \widehat{A^n} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

φ_m and φ_n are isomorphisms. Thus τ is an isomorphism by the 5-lemma.

(9.14) Theorem: Let A be a Noetherian ring and $I \subseteq A$ an ideal. The I-adic completion $(A, I)^\wedge = \widehat{A}$ is flat over A.

Proof: By homework it suffices to show: If $J \subseteq A$ is an ideal, then the induced map $J \otimes_A \widehat{A} \longrightarrow J\widehat{A}$ is injective. This follows by (9.12) and (9.13) since $\widehat{J} \longrightarrow \widehat{A}$ is injective.

(9.15) Proposition: Let A be a Noetherian ring, $I, J, K \subseteq A$ ideals, and let '$\wedge$' denote the I-adic completion. Then:

(a) $\widehat{J} \cong J \otimes_A \widehat{A} \cong J\widehat{A}$

(b) $(JK)^\wedge = (JK, I)^\wedge \cong \widehat{J}\widehat{K}$.

Proof: (a) By (9.13) $\widehat{J} \cong J \otimes_A \widehat{A}$ and since $\widehat{A}$ is flat over A, $J \otimes_A \widehat{A} \cong J\widehat{A}$.
(b) By (a): $(JK)^\wedge \cong JK\widehat{A} = (J\widehat{A})(K\widehat{A}) \cong \widehat{J}\widehat{K}$.

In the following we identify $\widehat{J}$ and $J\widehat{A}$ and consider $\widehat{J}$ an ideal of $\widehat{A}$.

(9.16) Corollary: Under the assumptions of (9.15): $\widehat{A/J} \cong \widehat{A}/\widehat{J} = \widehat{A}/J\widehat{A}$.

Proof: Completing the exact sequence: $0 \to \mathfrak{J} \to A \to A/\mathfrak{J} \to 0$ yields an exact sequence $0 \to \hat{\mathfrak{J}} \to \hat{A} \to \widehat{A/\mathfrak{J}} \to 0$.

For all $n \in \mathbb{N}$ there is a natural (A/I)-linear map $I^n/I^{n+1} \to \hat{I}^n/\hat{I}^{n+1}$ which extends to a homomorphism of graded rings:

$$\zeta: gr_I(A) = \bigoplus_{n\in\mathbb{N}} I^n/I^{n+1} \longrightarrow gr_{\hat{I}}(\hat{A}) = \bigoplus_{n\in\mathbb{N}} \hat{I}^n/\hat{I}^{n+1}.$$

(9.17) Proposition: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and let '^' denote the I-adic completions. The graded homomorphism

$$\zeta: gr_I(A) \longrightarrow gr_I(\hat{A})$$

is an isomorphism.

Proof: Let $m < n$ and consider the exact sequence $0 \to I^n \to I^m \to I^m/I^n \to 0$. Completing yields an exact sequence $0 \to \widehat{I^n} \to \widehat{I^m} \to \widehat{I^m/I^n} \to 0$ and $\widehat{I^m/I^n} \cong \hat{I}^m/\hat{I}^n$. Since $I^k(I^m/I^n) = 0$ for all $k \geq n-m$, the A-module I^m/I^n is I-adically complete. Thus $\widehat{I^m/I^n} \cong I^m/I^n \cong \hat{I}^m/\hat{I}^n$.

(9.18) Corollary: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and '^' the I-adic completion. Then for all $n \in \mathbb{N}$: $A/I^n \cong \hat{A}/\hat{I}^n \cong \hat{A}/I^n\hat{A}$.

Proof: By the proof of (9.17) with $m = 0$.

(9.19) Corollary: Let A be a Noetherian ring, $I \subseteq A$ an ideal and $\hat{A}$ the I-adic completion of A. The canonical homomorphism of rings: $\hat{A} \to (\hat{A}, I)^\wedge = (\hat{A}, \hat{I})^\wedge$ is an isomorphism.

Proof: By (9.18): $(\hat{A}, \hat{I})^\wedge = \varprojlim \hat{A}/\hat{I}^n = \varprojlim \hat{A}/I^n\hat{A} = (\hat{A}, I)^\wedge \cong \varprojlim A/I^n = \hat{A}$.

(9.20) <u>Corollary</u>: Let A be a Noetherian ring, $I \subseteq A$ an ideal, and '^' the I-adic completion. Then $\hat{I} \subseteq \mathcal{J}\mathrm{rad}(\hat{A})$.

<u>Proof</u>: We have to show that for all $\hat{z} = (z_i + I^i) \in \hat{I}$ the element $1 - \hat{z} = (1 - z_i + I^i)$ is a unit in $\hat{A}$. Note that $\hat{z} \in \hat{I}$ implies that $z_i \in I$ for all i and consider $\hat{y} = (1 + z_i + \dots + z_i^{i-1} + I^i)$. Then $(1-\hat{z})\hat{y} = ((1-z_i)(1+z_i + \dots + z_i^{i-1}) + I^i) = (1 - z_i^i + I^i) = (1 + I^i)$ and $1-\hat{z}$ is a unit.

(9.21) <u>Corollary</u>: Let A be a Noetherian ring, $m \subseteq A$ a maximal ideal of A, and $\hat{A} = (A, m)^\wedge$ the m-adic completion of A. Then

(a) $\hat{A}$ is a (quasi) local ring with maximal ideal $m\hat{A} = \hat{m}$.

(b) The mA_m-adic completion of A_m is isomorphic to $\hat{A}$, i.e. $\hat{A} \cong \widehat{A_m} = (A_m, mA_m)^\wedge$.

<u>Proof</u>: (a) By (9.18) $\hat{A}/m\hat{A} \cong A/m$, thus $m\hat{A}$ is a maximal ideal of $\hat{A}$. By (9.20) $\hat{A}$ is (quasi) local.

(b) For all $n \in \mathbb{N}$: $A/m^n = A_m/m^n A_m$.

(9.22) <u>Proposition</u>: Let A be a Noetherian ring and $I \subseteq \mathcal{J}\mathrm{rad}(A)$ an ideal in the Jacobson radical of A. Then:

(a) The I-adic completion $\hat{A} = (A, I)^\wedge$ is faithfully flat over A.

(b) The natural morphism $\varphi: A \longrightarrow (A, I)^\wedge$ is injective.

<u>Proof</u>: (a) By (9.14) $\hat{A}$ is flat over A. Since $A/I \cong \hat{A}/I\hat{A}$ and $I \subseteq \mathcal{J}\mathrm{rad}(A)$ for all maximal ideals $m \subseteq A$, $m\hat{A} \neq \hat{A}$. Thus $\hat{A}$ is faithfully flat over A.

(b) Since $I \subseteq \mathcal{J}\mathrm{rad}(A)$: $\ker(\varphi) = \bigcap_{n \in \mathbb{N}} I^n = (0)$.

(9.23) <u>Example</u>: Let A be a ring; $M = B = A[x_1, \dots, x_n]$ the polynomial ring in n variables over A and $I = (x_1, \dots, x_n)$. Then $\hat{B} = (B, I)^\wedge = A[\![x_1, \dots, x_n]\!]$.

<u>Proof</u>: Define $\varphi: A[\![x_1,\dots,x_n]\!] \longrightarrow \hat{B} = \varprojlim B/I^i$ by $\varphi\left(\sum_{(i)\in\mathbb{N}^n} a_{(i)} x_1^{i_1}\cdots x_n^{i_n}\right) = \left(\sum_{|(i)|<j} a_{(i)} x_1^{i_1}\cdots x_n^{i_n} + I^j\right)$ where $|(i)| = i_1+\dots+i_n$. φ is an isomorphism of rings.

(9.24) <u>Remark</u>: A similar argument as in the proof of Hilbert's basis theorem shows: If A is a Noetherian ring then the power series ring $A[\![x_1,\dots,x_n]\!]$ is Noetherian.

(9.25) <u>Theorem</u>: Let A be a Noetherian ring, $I\subseteq A$ an ideal, and $\hat{A}$ the I-adic completion.
(a) If $I=(a_1,\dots,a_n)$ then $\hat{A}\cong A[\![x_1,\dots,x_n]\!]/(x_1-a_1,\dots,x_n-a_n)$.
(b) $\hat{A}$ is Noetherian.

<u>Proof</u>: (a) Obviously, $A\cong A[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n)$. Let $K=(x_1,\dots,x_n)$, then $K+(x_1-a_1,\dots,x_n-a_n) = I+(x_1-a_1,\dots,x_n-a_n)$ and the K-adic topology of B equals the I-adic topology. Thus by (9.16) and (9.23): $(A,I)^\wedge \cong (B,K)^\wedge \cong (A[x_1,\dots,x_n],K)^\wedge/(x_1-a_1,\dots,x_n-a_n)(A[x_1,\dots,x_n],K)^\wedge \cong A[\![x_1,\dots,x_n]\!]/(x_1-a_1,\dots,x_n-a_n)$.
(b) By (a) and (9.24).

(9.26) <u>Corollary</u>: Let (A,m,k) be a local Noetherian ring and $I\subseteq m$ an ideal. Then $\dim A = \dim (A,I)^\wedge$.

<u>Proof</u>: Let $\hat{A}=(A,I)^\wedge$. For all $n\in\mathbb{N}$: $\hat{A}/I^n\hat{A}\cong A/I^n$. $\hat{A}$ is a local Noetherian ring with maximal ideal $\hat{m}=m\hat{A}$. Thus for all $n\in\mathbb{N}$: $\hat{A}/\hat{m}^n \cong A/m^n$ and A and $\hat{A}$ have the same Hilbert-Samuel polynomial, hence $\dim A = \dim\hat{A}$.

(9.27) <u>Corollary</u>: Let (A,m,k) be a local Noetherian ring, $I\subseteq m$ an ideal and $\hat{A}=(A,I)^\wedge$. A is regular if and only if $\hat{A}$ is regular.

<u>Proof</u>: Since $A\longrightarrow\hat{A}$ is faithfully flat, by (8.63)(a) if $\hat{A}$ is regular so is A. The converse follows with (8.63)(b) since $\hat{A}/m\hat{A} = \hat{A}/\hat{m} \cong A/m = k$.

(9.28) Corollary: Let (A,m,k) be a local Noetherian ring, $I \subseteq m$ an ideal and $\hat{A} = (A,I)^\wedge$. Then depth $\hat{A}$ = depth A. In particular, A is CM if and only if $\hat{A}$ is CM.

Proof: Homework

(9.29) Theorem: Let A be a ring, $I \subseteq A$ an ideal, and M an A-module. Assume that A is I-adically complete and M is I-adically separated. Let $x_1, \dots, x_m \in M$ so that $x_1 + IM, \dots, x_m + IM$ generate the A/I-module M/IM. Then $x_1, \dots, x_m$ generate M. In particular, M is finitely generated.

Proof: Let $y \in M$. For all $j \geq 0$ we construct $a_{ij} \in I^j$, $1 \leq i \leq m$, so that with $s_{ij} = \sum_{\nu=0}^{j} a_{i\nu}$ one has $y - \sum_{i=1}^m s_{ij} x_i \in I^{j+1}M$. Then $s_i = \lim_{j\to\infty} s_{ij} \in A$ and $y - \sum_{i=1}^m s_i x_i \in \bigcap I^{j+1}M$. Since M is I-separated, $\bigcap I^j M = 0$, and $y = \sum_{i=1}^m s_i x_i$.

In order to construct a_{ij}, let $j=0$. Then there are $a_{i0} \in A$ so that $y - \sum_{i=1}^m a_{i0} x_i \in IM$. Assume $a_{i\nu}$ for $0 \leq \nu \leq j-1$ have been constructed so that $y - \sum_{i=1}^m s_{ij-1} x_i \in I^j M$. Since $M = \sum_{i=1}^m A x_i + IM$, there are $a_{ij} \in I^j$ so that $y - \sum_{i=1}^m s_{ij-1} x_i - \sum_{i=1}^m a_{ij} x_i \in I^{j+1} M$.

(9.30) Theorem (Chevalley's Theorem) Let (A,m) be a complete Noetherian ring and let $\mathcal{F} = \{I_i \mid i \in \mathbb{N}\}$ be a filtration of A with $\bigcap_{i=0}^\infty I_i = 0$. Then for every integer j there exists an $i(j)$ so that $I_{i(j)} \subseteq m^j$. (i.e. the m-adic topology is the unique weakest topology that is separated.)

Proof: Homework

§3: COHEN'S STRUCTURE THEOREMS

(9.31) Theorem: (Hensel's Lemma) Let (A, m, k) be a local ring that is m-adically complete and let $F \in A[x]$ be a monic polynomial. If $F + mA[x] = gh$ in $k[x]$ where g and h are monic polynomials in $k[x]$ with $\gcd(g,h) = 1$, then there are monic polynomials G and H in $A[x]$ with $F = GH$.

Proof: Let '$-$' denote images in $k[x]$. We want to construct by induction on j monic polynomials G_j and H_j in $A[x]$ so that $F \equiv G_jH_j \bmod m^{j+1}A[x]$, $\overline{G_0} = g$, $\overline{H_0} = h$, and for $j>0$ $G_j \equiv G_{j-1} \bmod m^j A[x]$ and $H_j \equiv H_{j-1} \bmod m^j A[x]$.

Notice that all G_j have the same degree $\deg g$. Thus $G = \lim_{j\to\infty} G_j$ is a well-defined polynomial in $A[x]$. Likewise for $H = \lim_{j\to\infty} H_j$. Then $F = GH$ and G, H are monic with $\overline{G} = g$ and $\overline{H} = h$.

In order to construct G_j and H_j for $j=0$ let G_0 and H_0 be any monic polynomials with $\overline{G_0} = g$ and $\overline{H_0} = h$. For $j>0$ assume that G_{j-1} and H_{j-1} have been constructed. Then $F - G_{j-1}H_{j-1} = \sum a_i U_i$ where $a_i \in m^j$ and $U_i \in A[x]$ with $\deg U_i < \deg F$. Since $\gcd(g,h) = 1$ there are $p_i, q_i \in k[x]$ with $\overline{U_i} = p_i g + q_i h$. Modifying p_i modulo h (division algorithm) we may assume that $\deg p_i < \deg h$. Then $\deg(q_i h) < \deg F = \deg g + \deg h$ and $\deg q_i < \deg h$. Let $P_i, Q_i \in A[x]$ with $\overline{P_i} = p_i$ and $\overline{Q_i} = q_i$ and with $\deg P_i = \deg p_i$ and $\deg Q_i = \deg q_i$. Then $U_i - P_iG_0 - Q_iH_0 \in mA[x]$. Setting $G_j = G_{j-1} + \sum a_i Q_i$ and $H_i = H_{j-1} + \sum a_i P_i$ we have $F - G_jH_j \equiv \sum a_i U_i - G_0(\sum a_i P_i) - (\sum a_i Q_i)H_0 \bmod m^{j+1}A[x]$ and hence $F - G_jH_j \in m^{j+1}A[x]$. Moreover, $G_j \equiv G_{j-1} \bmod m^j A[x]$ and G_j is monic since $\deg Q_i = \deg q_i < \deg g = \deg G_{j-1}$. Likewise for H_j.

(9.32) Proposition: Let $\varphi: A \to B$ be a homomorphism of rings and let $J \subseteq B$ be an ideal. Assume that B is J-adically complete and that $a_1, \dots, a_n$ are elements of J.

(a) There is a unique homomorphism of rings $\psi: A[[x_1, \dots, x_n]] \to B$ with $\psi|_A = \varphi$ and $\psi(x_i) = a_i$ for $1 \le i \le n$.

(b) If the induced map $\pi\varphi: A \longrightarrow B/J$ is surjective and $J = (a_1, -, a_n)$ then ψ is surjective

<u>Proof</u>: (a) Let $F = \sum b_{(i)} x_1^{i_1} \dots x_n^{i_n} \in A[\![x_1, -, x_n]\!]$. Define $\psi(F) = \sum \varphi(b_{(i)}) a_1^{i_1} \dots a_n^{i_n}$. Since B is J-adically complete, $\psi(F)$ is a well-defined element of B and ψ is a homomorphism of rings with $\psi|_A = \varphi$ and $\psi(x_i) = a_i$ for all $1 \le i \le n$.

In order to prove uniqueness notice that $\psi|_{A[x_1,-,x_n]}$ is uniquely determined and that $A[\![x_1, -, x_n]\!]$ is the $I = (x_1, -, x_n)$-adic completion of $A[x_1, -, x_n]$. Since ψ is continuous ($\psi(I) \subseteq J$), ψ extends uniquely to a homomorphism of the completions (9.6).

(b) Let $T = A[\![x_1, -, x_n]\!]$, $I = (x_1, -, x_n)T$, and consider B as a T-module via ψ. T is I-adically complete, B is I-adically separated, and $1 + IB \in B/IB = B/J$ generates B/J over T/I. By (9.29) $1 \in B$ generates B as a T-module, that is, ψ is surjective.

(9.33) <u>Definition</u>: Let A be a ring, $\varepsilon: \mathbb{Z} \longrightarrow A$ the homomorphism of rings defined by $\varepsilon(1) = 1_A$. If $n \in \mathbb{N}$ with $\ker(\varepsilon) = (n)$, then A is called a ring of <u>characteristic</u> n. Notation: $\mathrm{char}(A) = n$.

(9.34) <u>Remark</u>: (a) If A is a domain, $\mathrm{char}(A) = 0$ or $\mathrm{char}(A) = p > 0$, where p is a prime number.

(b) Let (A, m, k) be a local ring. If $\mathrm{char}(k) = 0$, then $\mathrm{char}(A) = 0$. In this case A contains a subfield which is isomorphic to $\mathbb{Q}$.

(c) Let (A, m, k) be a local ring. If $\mathrm{char}(k) = p > 0$, then $\mathrm{char}(A) = 0$ or $\mathrm{char}(A) = p^n$ for some $n \ge 1$.

(d) For a local ring (A, m, k) the following four cases can occur:

(i) $\mathrm{char}(A) = \mathrm{char}(k) = 0$ $\quad (\Longleftrightarrow \mathbb{Q} \subseteq A)$

(ii) $\mathrm{char}(A) = \mathrm{char}(k) = p$, prime $\quad (\Longleftrightarrow \mathbb{Z}/(p) \subseteq A)$

(iii) $\mathrm{char}(A) = 0$ and $\mathrm{char}(k) = p$, prime

(iv) $\mathrm{char}(A) = p^n$, p prime and $1 < n < \infty$, and $\mathrm{char}(k) = p$ (cannot occur if A is reduced).

Proof: (b) The composition of natural maps $\varphi = \pi\varepsilon : \mathbb{Z} \xrightarrow{\varepsilon} A \xrightarrow{\pi} k$ is injective. Thus $\varphi(\mathbb{Z}-(0)) \subseteq A - m = A^*$. φ extends to a map $\tilde{\varphi} : \mathbb{Q} \longrightarrow A$.
(c) If $\mathrm{char}(A) = r = p^n t$ with $n>0$ and $p \nmid t$. Then $\varepsilon(t) \notin m$ and $\varepsilon(t)$ is a unit in A. Thus $t=1$.

(9.35) Definition: A local ring (A, m, k) is called of equal characteristic if $\mathrm{char}(A) = \mathrm{char}(k)$ and of unequal characteristic if $\mathrm{char}(A) \neq \mathrm{char}(k)$.

(9.36) Remark: (a) A local ring (A, m, k) is of equal characteristic if and only if A contains a field.
(b) Let A be a ring which contains a field $k_0 \subseteq A$. Then by Zorn's Lemma A contains a maximal subfield k_1 with $k_0 \subseteq k_1$.

(9.37) Definition: Let (A, m, k) be a local ring with $p = \mathrm{char}(k) \geq 0$. A coefficient ring of A is a subring $A_0 \subseteq A$ so that (A_0, m_0) is a complete Noetherian local ring with $m_0 = pA_0$ and $A_0/m_0 = k$ (i.e. $A = A_0 + m$).

Notice that a coefficient ring is a field if and only if $\mathrm{char}(A) = \mathrm{char}(k)$, in which case it is called a coefficient field.

(9.38) Theorem (Cohen) Let A be a complete local (Noetherian) ring. Then A has a coefficient ring. If $\mathrm{char}(A) = p^n$, p prime and $1 \leq n \leq \infty$, then $A_0 \cong V/(p^n)$ with $(V, (p))$ a complete discrete valuation ring.

We only prove a special case:

(9.39) Theorem: Let (A, m, k) be a complete local (Noetherian) ring. If $\mathrm{char}(k) = 0$, then A has a coefficient field.

Proof: Since A contains a field, by (9.36) A contains a maximal subfield. Let $\pi: A \longrightarrow k$ be the natural map. We claim that $\pi(K) = k$. Suppose not and let $\alpha \in k - \pi(K)$.

Case 1: α is transcendental over $\pi(k)$. Let $a \in A$ with $\pi(a) = \alpha$. For every nonzero polynomial $F \in K[x]$, $\pi(F) \neq 0$, and $\pi(F)(\alpha) = \pi(F(a)) \neq 0$. Thus a is transcendental over K and for all $F \in K[x] - (0)$, $F(a)$ is a unit in A. Therefore $K(a) \subseteq A$ contradicting that K is maximal

Case 2: α is algebraic over $\pi(K)$.

Since $\mathrm{char}(k) = 0$, the element α is separable over $\pi(K)$. Thus there is a monic polynomial $F \in K[x]$ so that $\pi(F)$ is the minimal polynomial of α over $\pi(K)$. Note that F is irreducible in $K[x]$. Thus $\pi(F) = (x - \alpha) \cdot h$ in $k[x]$ with $\gcd(x - \alpha, h) = 1$. By Hensel's Lemma (9.31) there are monic polynomials $G, H \in A[x]$ with $F = GH$ and $\pi(G) = x - \alpha$ and $\pi(H) = h$. Then $G = x - a$ with $\pi(a) = \alpha$, in particular, $F(a) = 0$. The morphism $\partial: K[x] \longrightarrow A$ with $\partial|_K = id_K$ and $\partial(x) = a$ factors through the field $K[x]/(F)$ which is isomorphic to a subfield of A that strictly contains K.

(9.40) Theorem: (Cohen's Structure Theorem) Let (A, m, k) be a complete local Noetherian ring.
(a) If $\mathrm{char}(A) = \mathrm{char}(k)$, then $A \cong k[[x_1, -, x_n]]/I$ for some n and some ideal I.
(b) If $\mathrm{char}(A) \neq \mathrm{char}(k)$, then $A \cong V[[x_1, -, x_n]]/I$ for some n and some ideal I, where $(V, (p))$ is a complete discrete valuation ring.

Proof: By (9.38) there is a homomorphism $k \longrightarrow A$ or $V \longrightarrow A$, respectively, so that the composition with $\pi: A \longrightarrow A/m$ is surjective. If $m = (a_1, -, a_n)$ by (9.32) there is a surjective homomorphism $k[[x_1, -, x_n]] \longrightarrow A$ or $V[[x_1, -, x_n]] \longrightarrow A$, respectively.

(9.41) Proposition: Let (A, m, k) be a complete local Noetherian ring with $\mathrm{char}(A) = \mathrm{char}(k)$ If A is regular with $\dim(A) = d$ then $A \cong k[[x_1, -, x_d]]$.

Proof: Let $a_1, -, a_d$ be a regular system of parameters of A. By the proof of (9.40) there is a surjective homomorphism $\varphi : k[\![x_1, \ldots, x_d]\!] \longrightarrow A$ with $\varphi(x_i) = a_i$. Since $\dim(k[\![x_1, \ldots, x_d]\!]) = \dim(A) = d$ (and since $k[\![x_1, \ldots, x_d]\!]$ is a domain), φ is injective.

(9.41) <u>Remark</u>: Assumptions as in (9.41) with $\mathrm{char}(A) \neq \mathrm{char}(k) = p$. If $p \cdot 1_A \notin m^2$ then $A \simeq V[\![x_1, \ldots, x_{d-1}]\!]$, where $(V, (p))$ is a complete discrete valuation ring.

§4: ASCENT AND DESCENT

(9.42) Proposition: Let $\varphi: A \longrightarrow B$ be a homomorphism of Noetherian rings, $Q \subseteq B$ a prime ideal and $P = \varphi^{-1}(Q) = Q \cap A \in \mathrm{Spec}(A)$. Then:

(a) $\mathrm{ht}\, Q \leq \mathrm{ht}\, P + \dim B_Q/PB_Q$

(b) If φ is flat, equality holds in (a).

Proof: We may replace A by A_P, B by B_Q, and assume that $\varphi: (A, m) \longrightarrow (B, n)$ is a local homomorphism of Noetherian rings.

(a) We have to show that $\dim B \leq \dim A + \dim B/mB$. Let $x_1, \dots, x_r$ be a SoP of A and $y_1, \dots, y_s \in B$ so that $y_1 + mB, \dots, y_s + mB$ is a SoP of B/mB. Then there are $k, \ell \in \mathbb{N}$ so that $n^\ell \subseteq mB + \sum y_i B$ and $m^k \subseteq \sum x_i A$. Thus $n^{k\ell} \subseteq \sum x_i B + \sum y_i B$ and $\dim B \leq r+s$.

(b) Let $\dim B/mB = s$ and let $n = Q_0 \supsetneqq Q_1 \supsetneqq \cdots \supsetneqq Q_s (\supseteq mB)$ be a strictly decreasing chain of prime ideals between n and mB. Then $Q_i \cap A = \varphi^{-1}(Q_i) = m$ for all $0 \leq i \leq s$. Let $\dim A = r$ and let $m = P_0 \supsetneqq P_1 \supsetneqq \cdots \supsetneqq P_r$ be a strictly decreasing chain of primes in A. Since $\varphi: A \longrightarrow B$ is flat, by (7.99) going down holds and there is a descending chain of prime ideals $Q_s \supsetneqq Q_{s+1} \supsetneqq \cdots \supsetneqq Q_{s+r}$ in B with $Q_{s+i} \cap A = P_i$. Thus $\dim B \geq r+s$.

(9.43) Corollary: If $\varphi: (A, m) \longrightarrow (B, n)$ is a faithfully flat homomorphism of Noetherian rings then $\dim B = \dim A + \dim B/mB$.

For the remainder of this section let (A, m) be a local Noetherian ring and $(\widehat{A}, \widehat{m})$ its m-adic completion. The natural map $\varphi: A \longrightarrow \widehat{A}$ is faithfully flat with $\widehat{m} = m\widehat{A}$. This yields (see also (9.26))

(9.44) Corollary: $\dim A = \dim \widehat{A}$

Corollary (9.27) shows that A is regular if and only if $\hat{A}$ is regular. The goal is to investigate if other properties of the local Noetherian ring A pass to the completion $\hat{A}$ and vice versa.

(9.45) <u>Theorem</u>: Let $\varphi: A \longrightarrow B$ be a homomorphism of Noetherian rings, M a finitely generated A-module, and N a B-module which is flat over A.

(a) If $P \in \operatorname{Spec}(A)$ with $N/PN \neq 0$, then ${}^a\varphi(\operatorname{Ass}_B(N/PN)) = \operatorname{Ass}_A(N/PN) = \{P\}$.

(b) If $Q \in \operatorname{Ass}_B(M \otimes_A N)$, then $P = Q \cap A \in \operatorname{Ass}_A(M)$. In particular, if $Q \in \operatorname{Ass}_B(N)$ then $P = Q \cap A \in \operatorname{Ass}_A(A)$.

(c) $\operatorname{Ass}_B(M \otimes_A N) = \bigcup_{P \in \operatorname{Ass}_A(M)} \operatorname{Ass}_B(N/PN)$

<u>Proof</u>: (a) Since A/P is an integral domain and $N/PN \cong N \otimes_A A/P$ is flat over A/P, every non zero element of A/P is regular on N/PN. Thus $\operatorname{Ass}_A(N/PN) = \{P\}$. If $Q \in \operatorname{Ass}_B(N/PN)$ then there is an element $y \in N/PN$ with $\operatorname{ann}_B(y) = Q$. Then $\operatorname{ann}_A(y) = Q \cap A \in \operatorname{Ass}_A(N/PN) = \{P\}$.

(b) Let $Q \in \operatorname{Ass}_B(M \otimes_A N)$. Since M is a finitely generated A-module, $(0) \subseteq M$ has a shortest primary decomposition $(0) = L_1 \cap \dots \cap L_r$. The quotient modules $M_i = M/L_i$ are primary, that is, $\operatorname{Ass}_A(M/L_i) = \{P_i\}$ and $\operatorname{Ass}_A(M) = \{P_1, \text{—}, P_r\}$. Moreover, there is an injective A-linear map $M \longrightarrow \oplus_{i=1}^r M_i$. Thus $M \otimes_A N \longrightarrow \oplus_{i=1}^r M_i \otimes_A N$ is injective and $\operatorname{Ass}_B(M \otimes_A N) \subseteq \bigcup_{i=1}^r \operatorname{Ass}_B(M_i \otimes N)$. This implies that $Q \in \operatorname{Ass}_B(M_i \otimes_A N)$ for some $1 \le i \le r$. Since $\operatorname{Ass}_A(M_i) = \{P_i\} \subseteq \operatorname{Ass}_A(M)$, every element of $A - P_i$ is regular on M_i and thus, by flatness, regular on $M_i \otimes_A N$. This implies that $Q \cap A \subseteq P_i$. Conversely, since $\operatorname{Ass}_A(M_i) = P_i$ and M_i finitely generated, $P_i^t M_i = 0$ for some $t \in \mathbb{N}$. Hence $P_i^t(M_i \otimes_A N) = 0$ and $P_i \subseteq Q \cap A$.

(c) If $P \in \operatorname{Ass}_A(M)$, then there is an exact sequence $0 \longrightarrow A/P \longrightarrow M$. Since N is flat over A, the sequence $0 \longrightarrow N/PN \longrightarrow M \otimes_A N$ is exact and $\operatorname{Ass}_B(N/PN) \subseteq \operatorname{Ass}_B(M \otimes_A N)$.

Conversely, let $Q \in \operatorname{Ass}_B(M \otimes_A N)$. As in the proof of (b), let $(0) = L_1 \cap \dots \cap L_r$ be a shortest primary decomposition and set $M_i = M/L_i$ for $1 \leq i \leq r$. As shown in (b), $Q \in \mathrm{Ass}_B(M_i \otimes_A N)$ for some $1 \leq i \leq r$ and $Q \cap A = P_i$ where $\mathrm{Ass}_A(M_i) = \{P_i\}$. Consider a chain of submodules $M_i = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_s = (0)$ with $K_j/K_{j+1} \cong A/W_j$ for some $W_j \in \mathrm{Spec}(A)$. Then, by flatness of N, $M_i \otimes_A N \supseteq K_1 \otimes_A N \supseteq \cdots \supseteq K_s \otimes_A N = (0)$ is a chain of submodules of $M_i \otimes_A N$ with $K_j \otimes_A N / K_{j+1} \otimes_A N \cong (A/W_j) \otimes_A N = N/W_j N$. Moreover, $\mathrm{Ass}_B(M_i \otimes_A N) \subseteq \bigcup_{i=1}^{s} \mathrm{Ass}_B(N/W_j N)$ and $Q \in \mathrm{Ass}_B(N/W_j N)$ for some $1 \leq j \leq s$. Thus, by (a), $Q \cap A = W_j = P_i$ and $Q \in \mathrm{Ass}_B(N/P_i N)$.

Recall from Chapter $\underline{\overline{\mathrm{VII}}}$:

(9.46) <u>Proposition</u>: Let (A, m, k) be a local Noetherian ring and M a finitely generated A-module. M is flat over A if and only if $\mathrm{Tor}_1^A(k, M) = 0$.

<u>Proof</u>: The forward direction is known. Conversely, if $\mathrm{Tor}_1^A(k, M) = 0$, the first Betti number of M is 0 and M is free. (see (7.53))

(9.47) <u>Remark</u>: Proposition (9.46) is true for a larger class of A-modules M. Following Matsumura's notion, if M is m-adically ideal-separated then M is flat over A if and only if $\mathrm{Tor}_1^A(k, M) = 0$. For example, if $(A, m, k) \longrightarrow (B, n, k')$ is a local morphism of local Noetherian rings and M is a finitely generated B-module, then M is m-adically ideal-separated. Thus M is flat over A if and only if $\mathrm{Tor}_1^A(k, M) = 0$. We will use this result without proof.

(9.48) <u>Theorem</u>: Let $\varphi : (A, m, k) \longrightarrow (B, n, k')$ be a local homomorphism of local Noetherian rings, and $u : M \longrightarrow N$ a linear map of finitely generated B-modules. If N is flat over A, the following conditions are equivalent:

(a) u is injective and $N/u(M)$ is flat over A.

(b) $\bar{u} : M \otimes_A k \longrightarrow N \otimes_A k$ is injective.

Proof: (a) $\Rightarrow$(b): Since $\mathrm{Tor}_1^A(k, N/u(M))=0$, $\bar{u}$ is injective.

(b)$\Rightarrow$(a): Let $x\in M$ with $u(x)=0$. Then $\bar{u}(\bar{x})=0$ and $\bar{x}=0$, i.e. $x\in mM$. We will show by induction on n that $x\in m^nM$ for all $n\in\mathbb{N}$. Assume that $x\in m^nM$ and let $a_1,\ldots,a_r$ be a minimal system of generators of m^n. Let $y_i\in M$ with $x=\sum_{i=1}^r a_iy_i$. Then $0=\sum_{i=1}^r a_i u(y_i)$. Since N is flat over A, by (7.103) there are $c_{ij}\in A$ and $z_j\in N$ so that $\sum_{i=1}^r a_i c_{ij}=0$ for all j and $u(y_i)=\sum_j c_{ij}z_j$ for all i. Since $a_1,\ldots,a_r$ is a minimal system of generators of m^n, for all i,j: $c_{ij}\in m$ and $u(y_i)\in mN$. Thus $\bar{u}(\bar{y_i})=0$ and $y_i\in mM$ implying that $x\in m^{n+1}M$. This shows that $x\in\bigcap_{n\in\mathbb{N}} m^nM\subseteq\bigcap_{n\in\mathbb{N}} n^nM=0$ and u is injective. From the exact sequence $0\longrightarrow M\xrightarrow{u} N\longrightarrow N/u(M)\longrightarrow 0$ we obtain that $\mathrm{Tor}_1^A(k, N/u(M))=0$ (since N is A-flat). By (9.47) $N/u(M)$ is flat over A.

(9.49) Corollary: Let A and B be as in (9.48) and let M be a finitely generated B-module. Set $\bar{B}=B\otimes_A k=B/mB$ and for $x\in n$ set $\bar{x}=x+mB$. Let $x_1,\ldots,x_r\in n$. If M is a flat A-module and $\bar{x}_1,\ldots,\bar{x}_r$ is an $M\otimes_A k$-sequence, then $x_1,\ldots,x_r$ is an M-sequence and $M/(x_1,\ldots,x_n)M$ is flat over A.

Proof: Set $M_0=M$, $M_i=M/(x_1,\ldots,x_i)M$ for $1\le i\le r$ and $\bar{M}_i=M_i\otimes_A k$. Let $u_i: M_{i-1}\longrightarrow M_{i-1}$ denote multiplication by x_1. By assumption $\bar{u}_1$ is injective and $M=M_0$ is A-flat. By (9.48) u_1 is injective and M_1 is A-flat. Continue.

(9.50) Remark: The converse of (9.49) is also true, that is, if $x_1,\ldots,x_r$ is an M-sequence and $M_r=M/(x_1,\ldots,x_r)M$ is A-flat, then $\bar{x}_1,\ldots,\bar{x}_r$ is an $\bar{M}$-sequence and M is A-flat. (without proof)

(9.51) Theorem: Let $\varphi: (A,m,k)\longrightarrow(B,n,k')$ be a local homomorphism of local Noetherian rings, M a finitely generated A-module, and N a finitely generated B-module. Assume that N is flat over A. Then

$$\mathrm{depth}_B(M\otimes_A N)=\mathrm{depth}_A M+\mathrm{depth}_B(N/mN)$$

<u>Proof</u>: Let $x_1, \ldots, x_s \in m$ be a maximal M-sequence and $y_1, \ldots, y_t \in n$ a maximal N/mN-sequence. Setting $x_i' = \varphi(x_i)$ for $1 \le i \le s$, we want to show that $x_1', \ldots, x_s', y_1, \ldots, y_t$ is a maximal $M \otimes_A N$-sequence. Since N is A-flat, $x_1', \ldots, x_s'$ is an $M \otimes_A N$-sequence. Set $M_s = M/(x_1, \ldots, x_s)M$. Then $m \in \mathrm{Ass}_A(M_s)$ and $M \otimes N/(x_1', \ldots, x_s')(M \otimes N) \cong M_s \otimes_A N$. By (9.49) y_1 is N-regular and $N_1 = N/y_1 N$ is A-flat. Consider the exact sequence: $0 \longrightarrow N \xrightarrow{y_1} N \longrightarrow N_1 \longrightarrow 0$. Since $\mathrm{Tor}_1^A(M_s, N_1) = 0$, the sequence $0 \longrightarrow M_s \otimes_A N \xrightarrow{y_1} M_s \otimes_A N \longrightarrow M_s \otimes_A N_1 \longrightarrow 0$ is exact. By (9.49) y_2 is N_1-regular and $N_2 = N/(y_1, y_2)N$ is A-flat, etc. Continue.
Thus $y_1, \ldots, y_t$ is an $M_r \otimes_A N$-sequence and $N_t = N/(y_1, \ldots, y_t)N$ is A-flat. Moreover, $M_s \otimes_A N_t \cong M \otimes_A N/(x_1', \ldots, x_s', y_1, \ldots, y_t) M \otimes_A N$. It remains to show that $\mathrm{depth}_B(M_s \otimes N_t) = 0$.
By (9.45) $\mathrm{Ass}_B(M_s \otimes_A N_t) = \bigcup_{P \in \mathrm{Ass}(M_r)} \mathrm{Ass}_B(N_t/PN_t) \supseteq \mathrm{Ass}_B(N_t/mN_t)$.
Since $y_1, \ldots, y_t$ is a maximal N/mN-sequence, $n \in \mathrm{Ass}_B(N_t/mN_t)$ and therefore $n \in \mathrm{Ass}_B(M_s \otimes_A N_t)$. This implies that $\mathrm{depth}_B(M_s \otimes_A N_t) = 0$.

(9.52) <u>Corollary</u>: Let $\varphi: (A, m, k) \longrightarrow (B, n, k')$ be a local homomorphism of local Noetherian rings and assume that B is flat over A. Then:

(a) $\mathrm{depth}\, B = \mathrm{depth}\, A + \mathrm{depth}\, B/mB$

(b) B is CM if and only if A and B/mB are CM.

<u>Proof</u>: (a) follows from (9.51).

(b) By (9.43) $\dim B = \dim A + \dim B/mB$. Thus if A and B/mB are CM so is B. Since $\mathrm{depth}\, A \le \dim A$ and $\mathrm{depth}\, B/mB \le \dim B/mB$, the converse follows immediately.

(9.53) <u>Corollary</u>: Let (A, m) be a local Noetherian ring and $\hat{A}$ the m-adic completion of A. Then:

(a) $\mathrm{depth}\, A = \mathrm{depth}\, \hat{A}$

(b) A is CM if and only if $\hat{A}$ is CM.

(9.54) Definition: Consider the following conditions (R_i) and (S_i) (for $i\in\mathbb{N}$) on a Noetherian ring A:

(R_i) A_P is regular for all $P\in\operatorname{Spec}A$ with $\operatorname{ht}P\leq i$.

(S_i) $\operatorname{depth}A_P\geq\min(\operatorname{ht}P,i)$ for all $P\in\operatorname{Spec}A$.

Conditions (R_i) and (S_i) are refered to as Serre conditions.

(9.55) Remark: (S_0) always holds. (S_1) says that all associated prime ideals of A are minimal, that is, A has no embedded associated prime ideals.

Recall that a Noetherian ring A is called reduced if A has no nonzero nilpotent elements.

(9.56) Proposition: Let A be a Noetherian ring. A is reduced if and only if (R_0) and (S_1) hold.

Proof: "$\Rightarrow$": Suppose that A is reduced. Thus for all $P\in\operatorname{Spec}A$ with $\operatorname{ht}P=0$ the ring A_P is reduced and 0-dimensional. Hence A_P is a field and (R_0) holds. Let $P\in\operatorname{Spec}A$ be any prime ideal. Since A is reduced, the zero ideal of A is the intersection of the minimal prime ideals of A and all prime ideals of $\operatorname{Ass}(A)$ are minimal. Thus $\operatorname{depth}A_P\geq\min(\operatorname{ht}P,1)$ and (S_1) holds.

"$\Leftarrow$": Suppose that (R_0) and (S_1) hold. All associated prime ideal of A are minimal, that is, $\operatorname{Ass}(A)=\{P_1,\dots,P_r\}$ is the set of minimal prime ideals of A. By (R_0) A_{P_i} is a field for all $1\leq i\leq r$. Let $S=A-(P_1\cup\dots\cup P_r)$. Then $S^{-1}A$ is a semilocal Noetherian ring of dimension 0. Thus $S^{-1}A$ is Artinian and by (1.88) $S^{-1}A\cong A_{P_1}\times\dots\times A_{P_r}$. By (R_0), $S^{-1}A$ is reduced. Let $a\in A$ with $a^n=0$ for some $n\in\mathbb{N}$. Suppose that $a\neq0$. Then $\frac{a}{1}=0$ in $S^{-1}A$ and there is a $t\in S$ with $ta=0$. This implies that $\operatorname{ann}(a)\not\subseteq P_i$ for all $1\leq i\leq r$. On the other hand, $\operatorname{ann}(a)\subseteq\bigcup_{P\in\operatorname{Ass}A}P=P_1\cup\dots\cup P_r$, contradiction.

(9.57) Definition: A Noetherian ring A is called normal if A_P is a normal domain for all $P \in \mathrm{Spec}(A)$, that is, A_P is a domain and it is integrally closed in its field of quotients.

(9.58) Remarks: Normal rings need not to be integral domains. For example, if K is a field, then $A = K \times K$ is a normal ring but not a domain.

(9.59) Theorem: A Noetherian ring A is normal if and only if A satisfies (R_1) and (S_2).

Proof: "$\Rightarrow$": Suppose that A is normal and let $P \in \mathrm{Spec}(A)$ be a prime ideal of height one. Then A_P is a normal Noetherian local domain of dimension one. By (5.38) A_P is a DVR and (R_1) holds. Moreover, $\operatorname{depth} A_P \geq \min(\operatorname{ht} P, 2)$ for all prime ideals of height ≤ 1. Let $P \in \mathrm{Spec}(A)$ be a prime ideal of height ≥ 2. Since A_P is normal, by (5.44) every prime ideal of $\mathrm{Ass}_{A_P}(A_P/aA_P)$ is minimal for all $a \in P$ and $\operatorname{depth} A_P \geq 2$. Thus (S_2) holds.
"$\Leftarrow$": We may assume that (A, m) is local. Since A satisfies (R_0) and (S_1), A is reduced and $(0) = P_1 \cap \dots \cap P_r$ where P_i are the minimal primes of A. In particular, $\mathrm{Ass}(A) = \{P_1, \dots, P_r\}$ and $S = A - (P_1 \cup \dots \cup P_r)$ is the set of NZD of A. Then $A \subseteq S^{-1}A = Q(A) \cong K_1 \times \dots \times K_r$ where $K_i = Q(A/P_i) = k(P_i)$ is the quotient field of A/P_i. ($Q(A)$ is called the total ring of fractions of A). We first show that A is integrally closed in $Q(A)$. Suppose that $a/b \in Q(A)$ with $a \in A$, $b \in S$, is integral over A. For all $P \in \mathrm{Spec}(A)$ with $\operatorname{ht} P = 1$, A_P is a RLR and $\frac{a}{1} \in \frac{b}{1} A_P$. b is A-regular and by (S_2) all associated primes of $A/(b)$ have height one. Thus if $bA = Q_1 \cap \dots \cap Q_m$ is a shortest primary decomposition of bA with Q_i W_i-primary, then $\operatorname{ht} W_i = 1$ for all $1 \leq i \leq m$. If $J = (b) :_A a$, then $J \subseteq W_i$ for some i or $J = A$. Since A_{W_i} is a DVR, $\frac{a}{1} \in \frac{b}{1} A_{W_i}$ and $J \nsubseteq W_i$. Thus $J = A$ and $a \in bA$. Hence A is integrally closed in $Q(A)$. Thus the idempotents $e_i = (0, \dots, 1, \dots, 0) \in Q(A)$ are in A since $e_i^2 - e_i = 0$. Since $1 = \sum e_i$ and $e_i e_j = 0$ for $i \neq j$, $A \cong Ae_1 \times \dots \times Ae_r$. Since A is local, $r = 1$, and A is an integrally closed domain.

(9.60) Theorem: Let $\varphi: (A, m) \longrightarrow (B, n)$ be a local homomorphism of local Noetherian rings and suppose that B is flat over A. Then for all $i \in \mathbb{N}$:

(a) If B satisfies (R_i) (or (S_i), respectively) then A satisfies (R_i) (or (S_i), respectively).
(b) If A and the fiber rings $B \otimes_A k(P)$ for all $P \in \operatorname{Spec} A$ satisfy (R_i) (or (S_i), respectively), then B satisfies (R_i) (or (S_i), respectively).

Proof: (a) Let $P \in \operatorname{Spec} A$. Since B is flat over A, there is a prime ideal $Q \subseteq B$ lying over P. Choose Q minimal, that is, $Q \in \operatorname{Spec} B$ is such that $Q \cap A = P$ and $\operatorname{ht}(Q/PB) = 0$. Then by (9.42) $\operatorname{ht} P = \operatorname{ht} Q$. Moreover, the induced map $\varphi_P : A_P \longrightarrow B_Q$ is faithfully flat. By (8.63) if B_Q is regular so is A_P. This implies that if B satisfies (R_i) so does A. If B satisfies (S_i) then by (9.52) depth B_Q = depth A_P and A satisfies (S_i).
(b) Let $Q \in \operatorname{Spec} B$ with $Q \cap A = P$. Suppose that A and the fiber rings $B \otimes_A k(P)$ satisfy (R_i). If $\operatorname{ht} Q \leq i$, then $\operatorname{ht} P \leq i$ and A_P is regular. Moreover, $\operatorname{ht}(Q/PB) \leq i$ and by assumption B_Q/PB_Q is regular. Apply (8.63) to the flat morphism $\varphi_P : A_P \longrightarrow B_Q$ to obtain that B_Q is regular. Thus B satisfies (R_i).
Suppose now that A and the fiber rings satisfy (S_i). By (9.52) depth B_Q = depth A_P + depth $(B_Q/PB_Q) \geq \min(\operatorname{ht} P, i) + \min(\operatorname{ht} Q/PB, i) \geq \min(\operatorname{ht} P + \operatorname{ht} Q/PB, i) =$
$= \min(\operatorname{ht} Q, i)$ by (9.42). B satisfies (S_i).

(9.61) Corollary: Let (A, m) be a local Noetherian ring and $\hat{A}$ its m-adic completion. Then:
(a) If $\hat{A}$ is normal, A is normal.
(b) If $\hat{A}$ is reduced, A is reduced.

(9.62) Remark: (a) In general, the converse of (9.61) is false. There are examples of local Noetherian normal rings A whose completion $\hat{A}$ is not reduced. On the other hand, the converse of (9.61) is true for a large class of local Noetherian rings. For example, if A is the localization of a finitely generated algebra over a field (or over $\mathbb{Z}$) the converse of (9.61) holds.
(b) So far we only discussed properties of rings which descent from the completion to the ring. There are, however, few properties which ascent, but do not descent. For example, if A is a local CM ring which has a canonical module, so does $\hat{A}$. There are local CM-rings A which fail to have a canonical module but $\hat{A}$ admits a canonical module.

§5: NORMALIZATION THEOREMS

(9.63) Theorem (Noether normalization theorem) Let k be a field, $A = k[x_1, \dots, x_n]/\mathfrak{J}$ a finitely generated k-algebra, and $I \subseteq A$ an ideal with $I \neq A$. Then there are integers $\delta \leq d$ and elements $y_1, \dots, y_d \in A$ such that:

(a) $y_1, \dots, y_d$ are algebraically independent over k, that is, the natural map from the polynomial ring into A $\varphi: k[t_1, \dots, t_d] \longrightarrow A$ defined by $\varphi(t_i) = y_i$ and $\varphi|_k = id_k$ is injective.

(b) A is a finitely generated $k[y_1, \dots, y_d]$-module.

(c) $I \cap k[y_1, \dots, y_d] = (y_{\delta+1}, \dots, y_d)$

If k is an infinite field then in addition:

(d) For all $1 \leq i \leq \delta$ the elements $y_i \in A$ can be chosen so that $y_i = \sum_{j=1}^{n} a_{ij} (x_j + \mathfrak{J})$ for some $a_{ij} \in k$.

The proof of the theorem requires a Lemma:

(9.64) Lemma: Let k be a field, $x_1, \dots, x_n$ variables over k, and $F \in k[x_1, \dots, x_n] - k$.

(a) By a substitution of the form $x_i = y_i + x_n^{r_i}$ for $1 \leq i \leq n-1$ and suitable $r_i \in \mathbb{N}$, F can be written as $F = a\, x_n^m + \rho_1 x_n^{m-1} + \dots + \rho_m$ where $m > 0$, $a \in k^*$ and $\rho_i \in k[y_1, \dots, y_{n-1}]$ for all $1 \leq i \leq m$.

(b) If k is an infinite field, the same result as in (a) can be achieved by substituting $x_i = y_i + a_i x_n$ with suitable $a_i \in k$.

Proof: Let $F = \sum a_{(\nu_1, \dots, \nu_n)} x_1^{\nu_1} \cdots x_n^{\nu_n}$ with $a_{(\nu_1, \dots, \nu_n)} \in k$.

(a) Replacing x_i by $x_i = y_i + x_n^{r_i}$, F can be written as a polynomial in $y_1, \dots, y_{n-1}, x_n$:

$$F = \sum a_{(\nu_1, \dots, \nu_n)} (x_n^{r_1} + y_1)^{\nu_1} \cdots (x_n^{r_{n-1}} + y_{n-1})^{\nu_{n-1}} x_n^{\nu_n}$$

$$= \sum a_{(\nu_1, \dots, \nu_n)} [x_n^{\nu_n + \nu_1 r_1 + \dots + \nu_{n-1} r_{n-1}} + \text{terms of lower degree in } x_n]$$

Let $q - 1 = \max\{\nu_i \mid F \text{ contains a coefficient } a_{(\nu_1, \dots, \nu_i, \dots, \nu_n)} \neq 0\}$ and set $r_i = q^i$.

For all $(\nu)=(\nu_1,\dots,\nu_n)$, $(\mu)=(\mu_1,\dots,\mu_n)\in\mathbb{N}^n$ with $a_{(\nu)}\neq 0$, $a_{(\mu)}\neq 0$, and $(\nu)\neq(\mu)$ we have that $\nu_n+\nu_1 q+\dots+\nu_{n-1}q^{n-1}\neq\mu_n+\mu_1 q+\dots+\mu_{n-1}q^{n-1}$. Thus with $m=\max\{\nu_n+\nu_1 q+\dots+\nu_{n-1}q^{n-1}\mid a_{(\nu_1,\dots,\nu_n)}\neq 0\}$ F can be written as a polynomial in $y_1,\dots,y_{n-1},x_n$ with $F=a x_n^m+$ terms of lower degree in x_n, where $a\in k^*$.

(b) Write $F=F_0+\dots+F_m$ where F_i are homogeneous polynomials of degree i and $F_m\neq 0$. Substitute $x_i=y_i+a_i x_n$ for $1\le i\le n-1$, then

$$\begin{aligned}F_m &= \sum_{\nu_1+\dots+\nu_n=m} a_{(\nu_1,\dots,\nu_n)}(y_1+a_1x_n)^{\nu_1}\cdots(y_{n-1}+a_{n-1}x_n)^{\nu_{n-1}}x_n^{\nu_n}\\ &= F_m(a_1,\dots,a_{n-1},1)\,x_n^m+\text{terms of lower degree in } x_n.\end{aligned}$$

If $F_m\neq 0$, then $F_m(x_1,\dots,x_{n-1},1)\neq 0$ and, since k is infinite, there are $a_1,\dots,a_{n-1}\in k$ with $F_m(a_1,\dots,a_{n-1},1)\neq 0$. Then $F(y_1,\dots,y_{n-1},x_n)=F_m(a_1,\dots,a_{n-1},1)x_n^m+$ terms of lower degree in x_n.

<u>Proof</u> of <u>Theorem</u> (9.63): We first prove the case where I is a principal ideal and $A=k[x_1,\dots,x_n]$ is the polynomial ring.

<u>Case 1</u>: $A=k[x_1,\dots,x_n]$ and $I=(F)$, F a nonconstant polynomial.

By Lemma (9.64) there are integers $r_i\in\mathbb{N}$ (or elements $a_i\in k$, if k is infinite) such that with $x_i=y_i+x_n^{r_i}$ (or $x_i=y_i+a_ix_n$, respectively) $F=a x_n^m+\rho_1x_n^{m-1}+\dots+\rho_m$, where $a\in k^*$ and $\rho_i\in k[y_1,\dots,y_{n-1}]$. Consider the homomorphism of rings

$$\varphi: k[y_1,\dots,y_n]\longrightarrow k[x_1,\dots,x_n]=A$$

defined by $\varphi|_k=\mathrm{id}_k$, $\varphi(y_i)=x_i-x_n^{r_i}$ (or $\varphi(y_i)=x_i-a_ix_n$, respectively) for $1\le i\le n-1$, and $\varphi(y_n)=F$. Note that $A=k[y_1,\dots,y_{n-1},x_n]$ and that x_n is integral over $k[y_1,\dots,y_n]$ since $a x_n^m+\rho_1x_n^{m-1}+\dots+\rho_m-y_n=0$, where $\rho_i\in k[y_1,\dots,y_{n-1}]$. This implies that $\dim(k[y_1,\dots,y_n]/\ker(\varphi))=\dim A=n$ and thus $\ker(\varphi)=0$ and $y_1,\dots,y_n$ are algebraically independent over k. Let $\overline{\varphi}$ denote the composition map $\pi\varphi$ where $\pi: A\longrightarrow A/(F)$ is the natural map. Then $A/(F)$ is integral over $k[y_1,\dots,y_n]/\ker\overline{\varphi}$ and thus $\dim k[y_1,\dots,y_n]/\ker\overline{\varphi}=\dim A/(F)=n-1$. Since (y_n) is a prime ideal of height one and $(y_n)\subseteq\ker\overline{\varphi}$, by dimension reasons

$\ker \bar{\varphi} = (y_n)$ and thus $(F) \cap k[y_1, \ldots, y_n] = (y_n)$.

<u>Case</u> 2: Assume that $A = k[x_1, \ldots, x_n]$ and that $I \subseteq A$ is any ideal with $I \neq A$. The proof is by induction on n. We may assume that $I \neq (0)$ and take $F \in I - (0)$. By the first case there are $y_1, \ldots, y_n \in A$ with $y_n = F$ so that the ring extension $k[y_1, \ldots, y_n] \subseteq A$ is finite integral. If k is infinite, $y_1, \ldots, y_{n-1}$ can be chosen as linear combinations of $x_1, \ldots, x_n$ (over k). Apply the induction hypothesis to $k[y_1, \ldots, y_{n-1}]$ and the ideal $I \cap k[y_1, \ldots, y_{n-1}]$. Then there are elements $t_1, \ldots, t_{n-1} \in k[y_1, \ldots, y_{n-1}]$ such that:

(i) $k[y_1, \ldots, y_{n-1}]$ is a finitely generated $k[t_1, \ldots, t_{n-1}]$-module.

(ii) $I \cap k[t_1, \ldots, t_{n-1}] = (t_{\delta+1}, \ldots, t_{n-1})$ for some $\delta \leq n-1$.

(iii) If k is infinite, then for $1 \leq i \leq \delta$: $t_i = \sum_{j=1}^{n-1} b_{ij} y_j$ where $b_{ij} \in k$.

This implies that $A = k[x_1, \ldots, x_n]$ is a finitely generated $k[t_1, \ldots, t_{n-1}, y_n]$-module. In particular, $t_1, \ldots, t_{n-1}, y_n$ are algebraically independent over k. Let $f \in I \cap k[t_1, \ldots, t_{n-1}, y_n]$, then $f = g + h y_n$ where $g \in k[t_1, \ldots, t_{n-1}]$ and $h \in k[t_1, \ldots, t_{n-1}, y_n]$. Since $y_n \in I$, it follows that $g \in I \cap k[t_1, \ldots, t_{n-1}] = (t_{\delta+1}, \ldots, t_{n-1})$ and hence $f \in (t_{\delta+1}, \ldots, t_{n-1}, y_n)$. If k is infinite, $y_1, \ldots, y_{n-1}$ are linear combinations of $x_1, \ldots, x_n$. Thus for $1 \leq i \leq \delta$, t_i are linear combinations of $x_1, \ldots, x_n$.

<u>Case</u> 3: $A = k[x_1, \ldots, x_n]/J$ and $I \subseteq A$ any ideal.

First apply case 2 to $k[x_1, \ldots, x_n]$ and the ideal J. Hence there is a subalgebra $k[y_1, \ldots, y_n] \subseteq k[x_1, \ldots, x_n]$ such that $k[x_1, \ldots, x_n]$ is a finitely generated $k[y_1, \ldots, y_n]$-module and $J \cap k[y_1, \ldots, y_n] = (y_{d+1}, \ldots, y_n)$. If k is infinite, for $1 \leq i \leq d$, the y_i are linear combinations of $x_1, \ldots, x_n$. This implies that A is a finitely generated $k[y_1, \ldots, y_d]$-module. Let $I' = I \cap k[y_1, \ldots, y_d]$. By case 2 there is a subalgebra $k[t_1, \ldots, t_d] \subseteq k[y_1, \ldots, y_d]$ so that $k[y_1, \ldots, y_d]$ is a finitely generated $k[t_1, \ldots, t_d]$-module and $I' \cap k[t_1, \ldots, t_d] = (t_{\delta+1}, \ldots, t_d)$. If k is infinite, for $1 \leq i \leq \delta$, t_i is a linear combination of $y_1, \ldots, y_d$. Then A is a finitely generated $k[t_1, \ldots, t_d]$-module and $I \cap k[t_1, \ldots, t_d] = (t_{\delta+1}, \ldots, t_d)$. Moreover, if k is infinite, for $1 \leq i \leq \delta$, t_i is a linear combination of $x_1, \ldots, x_n$.

(9.65) <u>Theorem</u>: Let (A, m, k) be a complete local Noetherian ring of dimension d. If

$\mathrm{char}(A) = \mathrm{char}(k)$, then there is a subring B of A such that A is a finitely generated B-module and $B \cong k[\![y_1, -, y_d]\!]$, the power series ring in d variables over k.

Proof: Let $t_1, -, t_d$ be a SOP of A, that is, $\dim A = d$, $(t_1, -, t_d) \subseteq m$ and $m^r \subseteq (t_1, -, t_d)$ for some $r \in \mathbb{N}$. By (9.38) A contains a coefficient field K and there is an isomorphism $\psi: k \xrightarrow{\cong} K$. Then, by (9.32), there is a homomorphism of rings $\varphi: k[\![y_1, -, y_d]\!] \longrightarrow A$ with $\varphi(y_i) = t_i$ and $\varphi|_k = \psi$. $A/(\varphi(y_1), -, \varphi(y_d)) = A/(t_1, -, t_d)$ is an Artinian ring and thus a finitely generated k-vector space. By (9.29) A is a finitely generated module over $k[\![y_1, -, y_d]\!]$. By dimension reasons, φ is injective and A is a finitely generated module over the subring $B = \mathrm{im}\,\varphi \cong k[\![y_1, -, y_d]\!]$.

(9.66) Remark: Let (A, m, k) be a complete local Noetherian ring of dimension d with $\mathrm{char}(A) \neq \mathrm{char}(k) = p > 0$. If $p \cdot 1_A$ is a non zero divisor of A, then there is a complete DVR V with maximal ideal $m_V = (p)$ so that (i) A contains a subring $B \cong V[\![y_1, -, y_{d-1}]\!]$ and (ii) A is a finitely generated B-module.