APPENDIX: THE LOCAL FLATNESS CRITERION

Theorem A.1: Let A be a ring, B a Noetherian A-algebra, M a finitely generated B-module, and $\mathfrak{J} \subseteq \mathrm{Jrad}(B)$ an ideal. For $n \geq 0$ set $M_n = M/\mathfrak{J}^{n+1}M$. If M_n is flat over A for all $n \geq 0$, then M is flat over A.

Proof: We have to show that for every finitely generated ideal I of A the natural map $u: I \otimes_A M \longrightarrow M$ is injective. Then $I \otimes_A M = M'$ is a finitely generated B-module and hence separated in the $\mathfrak{J}$-adic topology. Let $x \in \ker(u)$. We want to show that $x \in \bigcap \mathfrak{J}^n M' = 0$. For all $n \geq 0$: $M'_n = M'/\mathfrak{J}^{n+1}M' = (I \otimes_A M) \otimes_B B/\mathfrak{J}^{n+1} = I \otimes_A M_n$ and the map $u_n: M'_n \longrightarrow M_n$ is injective, since M_n is A-flat. From the commutative diagram:

$$\begin{array}{ccc} M' & \xrightarrow{u} & M \\ \downarrow & & \downarrow \\ M'_n & \xrightarrow{u_n} & M_n \end{array}$$

we obtain that $x \in \mathfrak{J}^{n+1}M$.

Theorem A.2: Let A be a ring, B a Noetherian A-algebra, and M a finitely generated B-module. Let $b \in \mathrm{Jrad}(B)$ be an M-regular element. If M/bM is flat over A, so is M.

Proof: For all $i > 0$, the sequence $0 \longrightarrow M/b^iM \xrightarrow{b} M/b^{i+1}M \longrightarrow M/bM \longrightarrow 0$ is exact. Using Theorem (7.102) it follows by induction on i that M/b^iM is A-flat for all $i > 0$. Apply Theorem A.1.

Definition: Let A be a ring and $I \subseteq A$ an ideal. An A-module M is called I-*adically ideal-separated* if for every finitely generated ideal $K \subseteq A$ the A-module $K \otimes_A M$ is separated in the I-adic topology.

Example: If B is a Noetherian A-algebra and $IB \subseteq \mathrm{Jrad}(B)$, then a finitely generated B-module M is I-adically ideal-separated as an A-module. (Since $K\otimes_A M$ is a finitely generated B-module for every finitely generated ideal $K\subseteq A$).

Let A be a ring, $I\subseteq A$ an ideal, and M an A-module. For all $n\geq 0$ set $A_n = A/I^{n+1}$ and $M_n = M/I^{n+1}M$. Consider $gr_I(A) = gr(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and $gr_I(M) = gr(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$. For all $n\geq 0$ there are natural maps

$$\gamma_n : (I^n/I^{n+1})\otimes_{A_0} M_0 \longrightarrow I^nM/I^{n+1}M$$

which induce a morphism of the $gr(A)$-modules:

$$\gamma : gr(A)\otimes_{A_0} M_0 \longrightarrow gr(M).$$

<u>Theorem</u> A.3: With the above notation suppose that A is a Noetherian ring and that M is I-adically ideal-separated. Then the following are equivalent:

(a) M is flat over A.

(b) $\mathrm{Tor}_1^A(N,M)=0$ for every A_0-module N.

(c) M_0 is flat over A_0 and $I\otimes_A M\cong IM$.

(c') M_0 is flat over A_0 and $\mathrm{Tor}_1^A(A_0,M)=0$.

(d) M_0 is flat over A_0 and γ_n is an isomorphism for all $n\geq 0$.

(d') M_0 is flat over A_0 and γ is an isomorphism.

(e) M_n is flat over A_n for all $n\geq 0$.

<u>Proof</u>: (a)$\Rightarrow$(b): clear

(b)$\Rightarrow$(c): Let N be an A_0-module. Then $N\otimes_A M\cong (N\otimes_{A_0}A_0)\otimes_A M\cong N\otimes_{A_0}M_0$. For every exact sequence $0\to N_1\to N_2\to N_3\to 0$ of A_0-modules, the sequence $0=\mathrm{Tor}_1^A(N_3,M)\to N_1\otimes_{A_0}M_0\to N_2\otimes_{A_0}M_0\to N_3\otimes_{A_0}M_0\to 0$ is exact and M_0 is flat over A_0. From the exact sequence $0\to I\to A\to A_0\to 0$ we get the exact sequence $0=\mathrm{Tor}_1^A(A_0,M)\to I\otimes M\to M\to M_0\to 0$ and $I\otimes_A M\cong IM$.

(c)$\Leftrightarrow$(c') Consider the exact sequence $0\to I\to A\to A_0\to 0$ and the induced longexact sequence: $0=\mathrm{Tor}_1^A(A,M)\to \mathrm{Tor}_1^A(A_0,M)\to I\otimes_A M\to M\to M_0\to 0$.

$(c') \Rightarrow (b)$: Let N be an A_0-module and $0 \to R \xrightarrow{\varphi} F_0 \to N \to 0$ an exact sequence with F_0 a free A_0-module. Then
$0 = \mathrm{Tor}_1^A(F_0, M) \to \mathrm{Tor}_1^A(N,M) \to R \otimes_{A_0} M_0 \xrightarrow{\varphi \otimes M_0} F_0 \otimes_{A_0} M_0 \to N \otimes_{A_0} M_0 \to 0$
is exact. Since M_0 is A_0-flat, $\varphi \otimes M_0$ is injective and $\mathrm{Tor}_1^A(N,M) = 0$.

$(c) \Rightarrow (d)$: By (b) $\mathrm{Tor}_1^A(I/I^2, M) = 0$. Thus the exact sequence $0 \to I^2 \to I \to I/I^2 \to 0$ induces an exact sequence $0 \to I^2 \otimes M \to I \otimes M \to (I/I^2) \otimes M \to 0$. Since $I \otimes M \cong IM$ we get that $I^2 \otimes M \cong I^2 M$. Proceeding by induction on n we obtain from the exact sequence $0 \to I^{n+1} \to I^n \to I^n/I^{n+1} \to 0$ that $I^{n+1} \otimes M \cong I^{n+1} M$. Thus $(I^n/I^{n+1}) \otimes M \cong I^n M / I^{n+1} M$.

$(d) \Leftrightarrow (d')$ clear.

$(d) \Rightarrow (e)$: Fix $n > 0$. For $i \leq n$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & (I^{i+1}/I^{n+1}) \otimes M & \longrightarrow & (I^i/I^{n+1}) \otimes M & \longrightarrow & (I^i/I^{i+1}) \otimes M & \longrightarrow & 0 \\ & & \downarrow \alpha_{i+1} & & \downarrow \alpha_i & & \downarrow \gamma_i & & \\ 0 & \longrightarrow & I^{i+1}M/I^{n+1}M \cong I^{i+1}M_n & \longrightarrow & I^i M/I^{n+1}M \cong I^i M_n & \longrightarrow & I^i M/I^{i+1}M & \longrightarrow & 0 \end{array}$$

By assumption γ_i is an isomorphism and α_{n+1} is an isomorphism. Thus, by decending induction, α_i is an isomorphism for all $1 \leq i \leq n+1$. In particular, $\alpha_1 : (I/I^{n+1}) \otimes_A M = IA_n \otimes_{A_n} M_n \cong IM_n$. Thus A_n, M_n, and $I/I^{n+1} \subseteq A_n$ satisfy the conditions of (c). By $(c) \Rightarrow (b)$ $\mathrm{Tor}_1^{A_n}(N, M_n) = 0$ for all A_0-modules N. If N is an A_1-module, then $0 \to IN \to N \to N/IN \to 0$ is exact with A_0-modules IN and N/IN. Thus $\mathrm{Tor}_1^{A_n}(N, M_n) = 0$. Proceeding by induction on i we see that if N is an A_i-module, then IN and N/IN are A_{i-1}-modules. Assuming that $\mathrm{Tor}_1^{A_n}(K, M_n) = 0$ for all A_{i-1}-modules, the exact sequence $0 \to IN \to N \to N/IN \to 0$ yields that $\mathrm{Tor}_1^{A_n}(N, M_n) = 0$. Thus M_n is A_n-flat.

$(e) \Rightarrow (a)$: We have to show that the natural map $\psi : J \otimes_A M \to M$ is injective for every ideal $J \subseteq A$. Since M is I-adically ideal-separated, $\bigcap_n I^n(J \otimes M) = 0$ and it suffices to show that $\ker(\psi) \subseteq I^n(J \otimes M)$ for all $n > 0$. Fix $n > 0$. By Artin-Rees $I^k \cap J \subseteq I^n J$ for sufficiently large $k > n$. Consider the natural map: $J \otimes M \xrightarrow{f} (J/J \cap I^k) \otimes M \xrightarrow{g} (J/I^n J) \otimes_A M \cong J \otimes M / I^n (J \otimes M)$.

Since M_{k-1} is flat over $A_{k-1} = A/I^k$ the map

$$(J/J\cap I^k)\otimes_A M \cong (J/J\cap I^k)\otimes_{A_{k-1}} M_{k-1} \longrightarrow M_{k-1}$$

is injective. From the commutative diagram

$$\begin{array}{ccc} J\otimes M & \xrightarrow{f} & (J/J\cap I^k)\otimes M \\ \psi\downarrow & & \downarrow \text{inj} \\ M & \longrightarrow & M_{k-1} \end{array}$$

we get that $\ker(\psi) \subseteq \ker(f) \subseteq \ker(gf) = I^k(J\otimes M)$.

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