Solutions to Homework 8.

(1) Let A be a ring, $I \subseteq A$ an ideal, and M an A-module with $I^n M = (0)$ for some $n \in \mathbb{N}$. Show that M is I-adically complete.

Proof. Let $n \in \mathbb{N}$ with $I^n M = (0)$ and $\Gamma = \{i \in \mathbb{N} \mid i \geq n\}$. By (7.85)

$$\varprojlim_{\mathbb{N}} M/I^r M = \varprojlim_{\Gamma} M/I^r M \equiv M.$$

(2) Let A be a Noetherian ring, $I, J \subseteq A$ ideals of A.

(a) If $J \subseteq I$ and if A is I-adically complete, then A is J-adically complete.

(b) If A is complete in the I-adic and the J-adic topology, then A is complete in the $I + J$-adic topology.

Proof. (a) Let $J = (a_1, \ldots, a_s)$ and $J = (a_1, \ldots, a_s, b_1, \ldots, b_t)$. Since A is I-adically complete, by (9.25):

$$A \cong A[[x_1, \ldots, x_s, y_1, \ldots, y_t]]/(x_1 - a_1, \ldots, x_s - a_s, y_1 - b_1, \ldots, y_t - b_t) = T.$$

Since

$$A \subseteq A[[x_1, \ldots, x_s]]/(x_1 - a_1, \ldots, x_s - a_s) \subseteq T \cong A,$$

A is also I-adically complete.

(b) Let $I = (a_1, \ldots, a_s)$ and $J = (b_1, \ldots, b_t)$. Since A is I-adically complete

$$A \cong A[[x_1, \ldots, x_s]]/(x_1 - a_1, \ldots, x_s - a_s) = B.$$

Then B is complete with respect to the JB-adic topology and

$$\begin{aligned} B &\cong B[[y_1, \ldots, y_t]]/(y_1 - b_1, \ldots, y_t - b_t) \\ &= A[[x_1, \ldots, x_s, y_1, \ldots, y_t]]/(x_1 - a_1, \ldots, x_s - a_s, y_1 - b_1, \ldots, y_t - a_t) \\ &\cong A. \end{aligned}$$

A is complete with respect to the $I + J$-adic topology.

(3) Let k be a field of characteristic $\neq 2$ and let

$$f = \sum_{i=0}^{\infty} a_i x^i \in k[[x]]$$

be a power series with $a_0 \neq 0$ and $a_0 = b_0^2$ for some $b_0 \in k$. Use Hensel's Lemma to show that there is a power series

$$g = \sum_{i=0}^{\infty} b_i x^i \in k[[x]]$$

with $f = g^2$. Note that $1 + x$ is not a square in $k[x]_{(x)}$, thus the ring $k[x]_{(x)}$ does not satisfy Hensel's Lemma!

Proof. The power series ring $A = k[[x]]$ is a complete local Noetherian ring with maximal ideal $\mathfrak{m} = (x)$ and residue class field k. By (9.31) Hensel's Lemma holds over A. Consider the monic polynomial in y: $F(y) = y^2 - f \in k[[x]][y] = A[y]$. Modulo $\mathfrak{m}$ we obtain $\bar{F}(y) = y^2 - a_0 = (y - b_0)(y + b_0) \in k[y]$. Since $\operatorname{char} k \neq 2$, the monic polynomials $y - b_0$ and $y + b_0$ are relatively prime in $k[y]$. By Hensel's Lemma there are power series $g_1, g_2 \in A = k[[x]]$ so that $F(y) = (y - g_1)(y - g_2)$. Thus $y^2 - f = y^2 - (g_1 + g_2)y + g_1 g_2$ and $g_1 = -g_2$. This shows that $f = g_1^2$ in $k[[x]]$.

(4) Let k be a field of characteristic $\neq 2$, and let $f = x^2(1+x) - y^2 \in k[x, y]$. Show that f is irreducible in $k[x, y]$, while f is a product of two irreducible power series (non units) in $k[[x, y]]$. This implies that the ring $A = k[x, y]_{(x,y)}/(f)$ is a domain while its completion $\widehat{A} = k[[x, y]]/(f)$ is not a domain.

Proof. Apply Eisenstein to see that $f = -y^2 + (x+1)x^2$ is irreducible in $k[x][y] = k[x, y]$. By Problem 3 there is a power series $h \in k[[x]]$ so that $x + 1 = h^2$. Thus over $k[[x]]$: $f = x^2h^2 - y^2 = (xh - y)(xh + y)$, and $k[[x, y]]/(f)$ is not a domain.

(5) Prove Chevalley's Theorem: Let (A, m) be a local Noetherian ring, which is m-adically complete. Let $I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ be a decreasing chain of ideals in A for which $\bigcap_{n \in \mathbb{N}} I_n = (0)$. Show that for all $n \in \mathbb{N}$ there is an integer $\nu(n) \in \mathbb{N}$ so that $I_{v(n)} \subseteq m^n$.

Proof. For all $k \in \mathbb{N}$ the ring $A/\mathfrak{m}^k$ is Artinian. In particular, for any fixed $k \in \mathbb{N}$ the decreasing sequence of ideals $(I_n + \mathfrak{m}^k)_{n \in \mathbb{N}}$ is stationary. Thus for all $k \in \mathbb{N}$ there is an integer $\mu(k) \in \mathbb{N}$ so that

$$I_n + \mathfrak{m}^k = I_{\mu(k)} + \mathfrak{m}^k$$

for all $n \geq \mu(k)$. Choose the integers $\mu(k)$ so that $\mu(k+1) \geq \mu(k) \geq k$ for all $k \in \mathbb{N}$.

Suppose that the statement of Chevalley's Theorem is false. Then there is an $N \in \mathbb{N}$ so that $I_n \not\subseteq \mathfrak{m}^N$ for all $n \in \mathbb{N}$. In order to obtain a contradiction we want to construct an nonzero element $a \in \cap_{n \in \mathbb{N}} I_n$. Since $I_{\mu(N)} \not\subseteq \mathfrak{m}^N$ let $a_1 \in I_{\mu(N)} - \mathfrak{m}^N$. Since

$$I_{\mu(N)} + \mathfrak{m}^N = I_{\mu(N+1)} + \mathfrak{m}^N,$$

let $a_2 \in I_{\mu(N+1)}$ so that $a_1 \equiv a_2 \bmod \mathfrak{m}^N$. Suppose that we have constructed $a_1, \ldots, a_t$ so that $a_i \in I_{\mu(N+i-1)}$ for all $1 \leq i \leq t$ and $a_i \equiv a_{i+1} \bmod \mathfrak{m}^{N+i}$ for all $1 \leq i \leq t - 1$. Since

$$I_{\mu(N+t-1)} + \mathfrak{m}^{N+t-1} = I_{\mu(N+t)} + \mathfrak{m}^{N+t-1},$$

there is an $a_{t+1} \in I_{\mu(N+t)}$ with $a_t \equiv a_{t+1} \bmod \mathfrak{m}^{N+t-1}$. This yields a sequence

$$a = (a_t + \mathfrak{m}^{N+t-1}) \in \varprojlim_{t \geq N} A/\mathfrak{m}^t$$

with $a \neq 0$. Since $a_i \in I_{\mu(N+t)}$ for all $i \geq t + 1$ we have that $a \in I_{\mu(k)}$ for all $k \in \mathbb{N}$ and thus $a \in \cap_{n \in \mathbb{N}} I_n$, a contradiction.

(6) Let A be a PID with field of quotients K. Prove that $0 \longrightarrow K \longrightarrow K/A \longrightarrow 0$ is an injective resolution of A.

Proof. K and K/A are divisible A-modules. Since A is a PID, by (6.80) K and K/A are injective A-modules.

(7) Let A be a local Noetherian ring. Show that if there is a nonzero finitely generated injective A-module then A is Artinian.

Proof. Let M be a finitely generated injective A-module. By (7.63) M is a direct sum of copies of $E_A(A/P)$ for some $P \in \operatorname{Spec}(A)$. We may assume that $M = E_A(A/P)$ is a finitely generated A-module for some $P \in \operatorname{Spec}(A)$. By (7.60) $k(P) = (A/P)_P \subseteq E_A(A/P)$ and $k(P)$ is a finitely generated A/P-module. Thus $P = \mathfrak{m}$, the maximal ideal of A and $M = E = E_A(k)$, where k is the residue field of A. By (7.57) every element of E is annihilated by some power of $\mathfrak{m}$. Since E is finitely generated, there is an $n \in \mathbb{N}$ with $\mathfrak{m}^n E = 0$. By (10.22) the A-linear map $\theta : A \longrightarrow \operatorname{Hom}_A(\operatorname{Hom}_A(A, E), E)$ is injective. This implies that A is annihilated by $\mathfrak{m}^n$.

(8) Let A be a local Gorenstein ring and M a finitely generated A-module. Show that

$$\operatorname{projdim}_A(M) < \infty \quad \Leftrightarrow \quad \operatorname{injdim}_A(M) < \infty.$$

Proof. Let $\dim A = d$.

$\Rightarrow$: Let $\operatorname{projdim}(M) = n < \infty$. The proof is by induction on n. If $n = 0$ then $M \cong A^r$ and $\operatorname{injdim}_A M < \infty$ since A is Gorenstein. If $\operatorname{projdim}_A M = n$ let $F_\bullet$ be a minimal free resolution of M. From the long exact sequence

$$0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

we obtain exact sequences:

$$(1) \quad 0 \to F_n \to \ldots \to F_1 \to K \to 0$$

and

$$(2) \qquad 0 \to K \to F_0 \to M \to 0.$$

From the first sequence we get that $\operatorname{projdim}_A K = n - 1$ and thus, by induction hypothesis, $\operatorname{injdim}_A K = m < \infty$. From the second exact sequence we obtain for every ideal $I \subseteq A$ a long exact sequence:

$$\ldots \to \operatorname{Ext}^i_A(A/I, F_0) \to \operatorname{Ext}^i_A(A/I, M) \to \operatorname{Ext}^{i+1}_A(A/I, K) \to \ldots$$

For all $i > \max(d, m-1)$ we have that $\operatorname{Ext}^i_A(A/I, F_0) = 0$ and $\operatorname{Ext}^{i+1}_A(A/I, K) = 0$. Thus $\operatorname{Ext}^i_A(A/I.M) = 0$ for all $i > \max(d, m-1)$ and $\operatorname{injdim}_A M < \infty$ by (7.42).

$\Leftarrow$: Let K be the dth syzygy module of M. A similar argument as in " $\Rightarrow$ " shows that $\operatorname{injdim}_A K < \infty$ if $\operatorname{injdim}_A M < \infty$. By (8.22) K is a MCM A-module. Let $\underline{x} = x_1, \ldots, x_d$ be an A-regular sequence. Set $\bar{A} = A/(\underline{x})$ and $\bar{K} = K/(\underline{x})K$. By (10.15) $\operatorname{injdim}_{\bar{A}} \bar{K} < \infty$. Since A is Gorenstein, so is $\bar{A}$ and $\bar{A} = E_{\bar{A}}(k)$. Since every exact sequence $0 \to L \to \bar{A}^r \to \bar{A}^s \to 0$ splits, $\bar{K}$ is an injective $\bar{A}$-module. Thus there is an isomorphism $\bar{\varphi} : \bar{A}^r \to \bar{K}$. By (10.32) the surjective A-linear map $\varphi : A^r \to K$ with $\bar{\varphi} = \varphi \otimes_A \bar{A}$ is an isomorphism and K is free.