Solutions to Homework 7.

(1) Let A be a local Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A/I-module. Show:

$$\operatorname{projdim}_{(A/I)}(M) + \operatorname{projdim}_A(A/I) = \operatorname{projdim}_A(M)$$

provided that each of them is finite.

Proof. By the Auslander-Buchsbaum Theorem:

$$\begin{aligned} \operatorname{projdim}_A(M) + \operatorname{depth}(M) &= \operatorname{depth}(A) \\ \operatorname{projdim}_A(A/I) + \operatorname{depth}(A/I) &= \operatorname{depth}(A) \\ \operatorname{projdim}_{(A/I)}(M) + \operatorname{depth}(M) &= \operatorname{depth}(A/I) \end{aligned}$$

Thus

$$\begin{aligned} & \operatorname{projdim}_A(M) + \operatorname{depth}(M) = \operatorname{projdim}_A(A/I) + \operatorname{depth}(A/I) \\ \Rightarrow \quad & \operatorname{projdim}_A(M) + \operatorname{depth}(M) = \operatorname{projdim}_A(A/I) + \operatorname{projdim}_{(A/I)}(M) + \operatorname{depth}(M) \\ & \qquad \Rightarrow \quad \operatorname{projdim}_A(M) = \operatorname{projdim}_A(A/I) + \operatorname{projdim}_{(A/I)}(M). \end{aligned}$$

(2) Let A be a regular local ring of dimension n, and $M \neq 0$ a finitely generated A-module. Show that the following are equivalent:

(a) M is free.
(b) M is a CM-module of dimension n.

Proof. (a) $\Rightarrow$ (b) trivial
(b) $\Rightarrow$ (a) Let M be a CM A-module with $\dim(M) = \dim(A) = n$. Then $\operatorname{depth}(M) = n$. Since A is a RLR, $\operatorname{projdim}_A(M) < \infty$ and by Auslander-Buchsbaum:

$$\operatorname{projdim}_A(M) = 0.$$

Thus M is free.

(3) Let A be a Noetherian domain of dimension 1, and let M be a nonzero finitely generated A-module of dimension 1. Show that M is a CM-module if and only if M is torsionfree.

Proof. $\Leftarrow$: Suppose that M is torsionfree and let $\mathfrak{m} \subseteq A$ be a maximal ideal of A. Then $M_\mathfrak{m}$ is a torsionfree $A_\mathfrak{m}$-module. Thus every $a \in A_\mathfrak{m} - (0)$ is a NZD of $M_\mathfrak{m}$ and $\operatorname{depth}(M_\mathfrak{m}) \geq 1$. Since $\dim(A_\mathfrak{m}) = 1$, the module $M_\mathfrak{m}$ is CM.
$\Rightarrow$: Suppose that M is a CM-module of dimension 1. Let $a \in A$ and $m \in M - (0)$ with $am = 0$. Then $a \in P$ for some $P \in \operatorname{Ass}_A(M)$. If $P \neq (0)$, then $M_P \neq 0$ is a CM-module over A_P. Since $\dim(M) = 1$ and A a domain, $\operatorname{ann}_A(M) = (0)$ and since M is finitely generated, $\operatorname{ann}_A(M_P) = \operatorname{ann}(M)_P = (0)$. Thus $\dim(M_P) = 1$ with $PA_P \in \operatorname{Ass}_{A_P}(M_P)$, a contradiction to M_P a CM-module.

(4) Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension n, and let $a, b \in \mathfrak{m} - (0)$ be elements with $a | b$ and $b \nmid a$. Let $S = A/(a)$ and $T = A/(b)$. Show:

(a) S and T are CM-rings.
(b) S is a CM-module over T with $\dim(S) = \dim(T)$.
(c) S is not a free T-module.

Proof. By assumption $b = ar$ for some $r \in A$. Let $\varphi : A/(b) = T \longrightarrow A/(a) = S$ denote the natural map.
(a) A is a RLR and a, b are regular elements of A. Thus S and T are CM-rings.
(b) Obviously, $\dim(S) = \dim(T) = 1$. Let $b, x_1, \dots, x_{n-1}$ be a maximal regular sequence of A. Then $x_1, \dots, x_{n-1}, b$ is also a regular sequence of A and for all $Q \in \mathrm{Ass}_A(A/(x_1, \dots, x_{n-1}))$, $b \notin Q$. Thus for all $Q \in \mathrm{Ass}_A(A/(x_1, \dots, x_{n-1}))$, $a \notin Q$ and $a, x_1, \dots, x_{n-1}$ is a regular sequence of A. In particular, $x_1 + (b), \dots, x_{n-1} + (b) \in T$ is a regular sequence of the T-module S. S is a CM-module over T.
(c) Since $b \not| \, a$, $\mathrm{ann}_T(S) = (a)/(b) \neq (0)$ and S is not a free T-module.

(5) Let A be a Noetherian ring, M a finitely generated A-module and $I \subseteq A$ an ideal of A. Show that $\mathrm{depth}_I(M) \geq 2$ if and only if the natural homomorphism $M \longrightarrow \mathrm{Hom}_A(I, M)$ is an isomorphism.

Proof. $\Gamma : M \longrightarrow \mathrm{Hom}_A(I, M)$ is defined by: $\Gamma(m)(a) = am$ for all $m \in M, a \in I$.
$\Rightarrow$: Let $x, y \in I$ be a regular sequence in M. If $m, n \in M$ with $\Gamma(m) = \Gamma(n)$, then $xm = xn$ and $m = n$, since x is regular. Thus Γ is injective.

In order to show that Γ is surjective, let $\varphi : I \longrightarrow M$ be an A-linear map and set $\varphi(x) = m$. Then $\varphi(xy) = ym = x\varphi(y) \in xM$. Thus there is an element $n \in M$ so that $m = xn$. We claim that $\varphi = \Gamma(n)$. Let $t \in I$, then $\varphi(tx) = txn = x\varphi(t)$. Since x is regular, $\varphi(t) = tn$ and thus $\varphi = \Gamma(n)$.
$\Leftarrow$: If $\mathrm{depth}_I(M) = 0$ then there is a prime ideal $P \in \mathrm{Ass}(M)$ with $I \subseteq P$. Let $m \in M$ with $\mathrm{ann}(m) = P$. Then $\Gamma(m) = 0$ and Γ fails to be injective. Thus $\mathrm{depth}_I(M) \geq 1$. The exact sequence: $0 \to I \to A \to A/I \to 0$ induces a long exact sequence:

$$0 \to \mathrm{Hom}_A(A/I, M) \to \mathrm{Hom}_A(A, M) \xrightarrow{*} \mathrm{Hom}_A(I, M) \to \mathrm{Ext}^1_A(A/I, M) \to 0.$$

Since Γ is an isomorphism, the map $*$ is an isomorphism and $\mathrm{Ext}^1_A(A/I, M) = 0$. By (8.14) $\mathrm{depth}_I(M) \geq 2$.

(6) Let $\varphi : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a local homomorphism of local Noetherian rings, and M an B-module which is finitely generated as an A-module.

(a) Suppose that $P \in \mathrm{Ass}_B(M)$ and let $x \in M$ with $\mathrm{ann}_B(x) = P$. Prove that φ induces an embedding $A/(P \cap A) \longrightarrow B/P \cong Bx$ which makes B/P into a finitely generated $A/(P \cap A)$-module. Conclude that $P \cap A \neq \mathfrak{m}$ if $P \neq \mathfrak{n}$.
(b) Show that $\mathrm{depth}_A(M) = \mathrm{depth}_B(M)$.

Proof. Since M is finitely generated as an A-module, M is also finitely generated as a B-module.
(a) If $P \in \mathrm{Ass}_B(M)$ and $x \in M$ with $\mathrm{ann}(x) = P$, then $Bx \cong B/P$. The induced map

$$A/(P \cap A) \longrightarrow B/P \cong Bx$$

is injective. Since Bx is an A-submodule of M, $Bx \cong B/P$ is a finitely generated A-module. Thus B/P is finite (integral) over $A/(P \cap A)$ and $A/(P \cap A)$ is a field if and only if B/P is a field.
(b) By induction on $\mathrm{depth}_A(M)$. If $\mathrm{depth}_A(M) = 0$, then $\mathfrak{m} \in \mathrm{Ass}_A(M)$. Then there is a $P \in \mathrm{Ass}_B(M)$ with $\varphi(\mathfrak{m}) \subseteq P$. Then $P \cap A = \mathfrak{m}$ and by (a) we have that $P = \mathfrak{n}$. Thus $\mathrm{depth}_B(M) = 0$.

If $\operatorname{depth}_A(M) \geq 1$, let $x \in \mathfrak{m}$ be an M-regular element. Then $x = \varphi(x)$ is also a regular element of the B-module M. Thus $\operatorname{depth}_A(M/xM) = \operatorname{depth}_A(M) - 1$ and $\operatorname{depth}_B(M/xM) = \operatorname{depth}_B(M) - 1$. Obviously, M/xM is a B-module and finitely generated as A-module. By induction hypothesis $\operatorname{depth}_A(M/xM) = \operatorname{depth}_B(M/xM)$ and the statement follows.

(7) Let A be a local Noetherian ring, M a finitely generated A-module and N an nth syzygy of M in a finite free resolution of M. Show that $\operatorname{depth}(N) \geq \min(n, \operatorname{depth}(A))$.

Proof. Let

$$F_\bullet : \quad \ldots F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 (\xrightarrow{\varphi_0} M) \longrightarrow 0$$

be a finite free resolution of M and let $L_m = \ker(\varphi_{m-1})$ be an mth syzygy module of M where $m \geq 1$. The proof is by induction on m. If $m = 1$, then L_1 is a submodule of a free A-module, thus $\operatorname{depth}(L_1) \geq \min(1, \operatorname{depth}(A)$. If $n > 1$ consider the induced exact sequence:

$$0 \longrightarrow L_m \longrightarrow F_{n-1} \longrightarrow L_{m-1} \longrightarrow 0.$$

By induction hypothesis

$$\operatorname{depth}(L_{m-1}) \geq \min(m-1, \operatorname{depth}(A))$$

and by (8.22);

$$\operatorname{depth}(L_m) \geq \min(\operatorname{depth}(F_{m-1}), \operatorname{depth}(L_{m-1}) + 1).$$

Thus

$$\operatorname{depth}(L_m) \geq \min(m, \operatorname{depth}(A)).$$

(8) Let A be a local Noetherian ring, and

$$0 \longrightarrow L_s \longrightarrow L_{s-1} \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0$$

a complex of finitely generated A-modules. Suppose that the following hold for $i > 0$:

(i) $\operatorname{depth}(L_i) \geq i$
(ii) $\operatorname{depth}(H_i(L_\bullet)) = 0$ or $H_i(L_\bullet) = 0$.

Show that $L_\bullet$ is acyclic.

(This is Peskine and Szpiro's 'lemme d'acyclité'.)

Hint: Set $C_i = \operatorname{coker}(L_{i+1} \longrightarrow L_i)$ and show by descending induction that $\operatorname{depth}(C_i) \geq i$ and $H_i(L_\bullet) = 0$ for $i > 0$.

Proof. Consider

$$(*_s) \qquad 0 \longrightarrow L_s \xrightarrow{\varphi_s} L_{s-1} \longrightarrow C_{s-1} \longrightarrow 0.$$

We claim that φ_s is injective. Let $K_s = H_s(L_\bullet) = \ker(\varphi_s)$. Since K_s is a submodule of L_s and $\operatorname{depth}(L_s) \geq s$ it follows that $\operatorname{depth}(K_s) > 0$ or $K_s = 0$. Thus $K_s = 0$. In particular, $(*_s)$ is an exact sequence and by (8.22): $\operatorname{depth}(C_s) \geq$

$\min(\operatorname{depth}(L_s), \operatorname{depth}(L_{s-1})) \geq s-1$. The proof is by decreasing induction. For the induction step assume that $\operatorname{depth}(C_j) \geq j$ for all $j \geq i+1$ and that

$$0 \longrightarrow L_s \longrightarrow L_{s-1} \longrightarrow \ldots \longrightarrow L_{i+1}$$

is exact. We need to show that $\operatorname{depth}(C_i) \geq i$ and that

$$(**) \quad 0 \longrightarrow L_s \longrightarrow L_{s-1} \longrightarrow \ldots \longrightarrow L_{i+2} \xrightarrow{\varphi_{i+2}} L_{i+1} \xrightarrow{\varphi_{i+1}} L_i$$

is exact. Note that φ_{i+1} factors:

$$\begin{array}{ccc} L_{i+1} & \xrightarrow{\varphi_{i+1}} & L_i \\ \downarrow & \nearrow & \\ C_{i+1} & & \end{array}$$

Then

$$H_{i+1}(L_\bullet) = \ker(\varphi_{i+1})/\operatorname{im}(\varphi_{i+2}) \subseteq L_{i+1}/\operatorname{im}(\varphi_{i+2}).$$

Since $\operatorname{depth}(C_{i+1}) \geq i+1 > 0$, either $H_{i+1}(L_\bullet) = 0$ or $\operatorname{depth}(H_{i+1}(L_\bullet)) > 0$. Thus by assumption $H_{i+1}(L_\bullet) = 0$ and the sequence $(**)$ is exact. This yields a short exact sequence:

$$(*_i) \qquad 0 \longrightarrow C_{i+1} \longrightarrow L_i \longrightarrow C_i \longrightarrow 0.$$

Thus by (8.22):

$$\operatorname{depth}(C_i) \geq \min(\operatorname{depth}(C_{i+1}) - 1, \operatorname{depth}(L_i)) \geq i.$$