Solutions to Homework 5.

(1) Let A be a Noetherian ring, M be a finitely generated A-module, and $\{N_i\}_{i\in I}$ a set of A-modules. Show:

$$\mathrm{Hom}_A(M, \oplus_{i\in I} N_i) \cong \oplus_{i\in I}\mathrm{Hom}_A(M.N_i).$$

Proof. Obviously,

$$\begin{aligned}\mathrm{Hom}_A(A, \oplus_{i\in I} N_i) &\cong \oplus_{i\in I} N_i \\ &\cong \oplus_{i\in I}\mathrm{Hom}_A(A, N_i).\end{aligned}$$

Thus for every finitely generated free A-module $F = A^r$:

$$\mathrm{Hom}_A(F, \oplus_{i\in I} N_i) \cong \oplus_{i\in I}\mathrm{Hom}_A(F, N_i).$$

If M is a finitely generated A-module, consider an exact sequence:

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i are finitely generated A-modules. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(M, \oplus_{i\in I} N_i) & \longrightarrow & \mathrm{Hom}_A(F_0, \oplus_{i\in I} N_i) & \longrightarrow & \mathrm{Hom}_A(F_1, \oplus_{i\in I} N_i) \\
\big\uparrow & & f\big\uparrow & & g\big\uparrow & & h\big\uparrow \\
0 & \longrightarrow & \oplus_{i\in I}\mathrm{Hom}_A(M, N_i) & \longrightarrow & \oplus_{i\in I}\mathrm{Hom}_A(F_0, N_i) & \longrightarrow & \oplus_{i\in I}\mathrm{Hom}_A(F_1, N_i)
\end{array}$$

Since g and h are isomorphisms, the five Lemma yields that f is an isomorphism.

(2) Let A be a commutative ring and M an A-module. Suppose that

$$0 \to K_1 \to P_1 \to M \to 0 \quad \text{and} \quad 0 \to K_2 \to P_2 \to M \to 0$$

are exact sequences with projective modules P_1 and P_2. Show that $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proof. Consider the diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \xrightarrow{f_1} & P_1 & \xrightarrow{g_1} & M & \longrightarrow & 0 \\
& & \alpha\big\downarrow & & \beta\big\downarrow & & \mathrm{id}_M\big\downarrow & & \\
0 & \longrightarrow & K_2 & \xrightarrow{f_2} & P_2 & \xrightarrow{g_2} & M & \longrightarrow & 0
\end{array}$$

where the maps α and β have to be constructed so that the diagram commutes. Since P_1 is projective and g_2 surjective there is a map $\beta : P_1 \longrightarrow P_2$ so that $g_2\beta = g_1$ and the right square commutes. This implies that $\mathrm{im}(\beta f_1) \subseteq \ker(g_2) = \mathrm{im}(f_2) \cong K_2$. Thus there is a linear map $\alpha : K_1 \longrightarrow K_2$ so that $\beta f_1 = f_2\alpha$ and the left square commutes. Consider the sequence:

$$(*) \quad 0 \to K_1 \xrightarrow{\theta} P_1 \oplus K_2 \xrightarrow{\tau} P_2 \to 0$$

where θ and τ are defined as follows: $\theta(k_1) = (f_1(k_1), \alpha(k_1))$ and $\tau(p_1, k_2) = \beta(p_1) - f_2(k_2)$. This definition makes the sequence (*) exact. Since P_2 is projective, the assertion follows.

(3) Let A be a commutative domain, K its field of quotients. Prove:

(a) A torsion-free A-module M is injective if and only if M is divisible.
(b) K is the injective hull of A.

Proof. (a) By (6.79) every injective module is divisible. It remains to show that a torsion-free divisible A-module M is injective. Let $I \subseteq A$ be an ideal and $f : I \longrightarrow M$ an A-linear map. We need to show that f extends to a A-linear map $g : A \longrightarrow M$. If $f = 0$ there is nothing to show. If $f \neq 0$ let $a \in I$ with $f(a) = m \neq 0$. Since A is a domain, there is an element $n \in M$ so that $m = an$. We claim that the map $g : A \longrightarrow M$ defined by $g(1) = n$ extends f. Let $b \in I$, then $f(ab) = af(b) = bm = abn = g(ab) = ag(b)$. Since M is torsion-free, $f(b) = g(b)$. By (6.27) M is injective.
(b) By (a) K is an injective A-module. Since K is an essential extension of A, the quotient field K is the injective hull of A.

(4) Let A be a Noetherian ring. Show that a direct sum of injective A-modules is an injective A-module.

Proof. Let $\{E_i\}_{i \in J}$ be a set of injective A-modules and $E = \oplus_{i \in J} E_i$. By (6.27) we have to show that every A-linear map $g : I \longrightarrow E$ extends to an A-linear map $f : A \longrightarrow E$ for every ideal I of A. Let $I \subseteq A$ be an ideal and $g : I \longrightarrow E$ an A-linear map. Since A is Noetherian, I is finitely generated, say: $I = (a_1, \ldots, a_m)$. For all $1 \leq k \leq m$ the ith component $g(a_k)_i$ is zero for all but finitely many $i \in J$. Let $i_1, \ldots, i_r \in J$ so that $g(a_k)_i = 0$ for all $i \in J - \{i_1, \ldots, i_r\}$ and all $1 \leq k \leq m$. Then

$$\operatorname{im}(g) \subseteq E' = \oplus_{j=1}^{r} E_{i_j} \subseteq E.$$

Let $g' : I \longrightarrow E'$ be the map defined by $g(a) = g'(a)$ for all $a \in I$. Since the direct product of injective modules is injective, E' is an injective A-module and g' extends to an A-linear map $f' : A \longrightarrow E'$. f' combined with the embedding $\epsilon : E' \longrightarrow E$ extends g.

(5) Let A be a Noetherian ring and $P \subseteq A$ a prime ideal.

(a) If E is an injective A-module show that E_P is both A_P-injective and A-injective.
(b) Let M be an A-module and $E(M)$ the injective hull of M. Then $E(M)_P$ is the injective hull of the A_P-module M_P.
(c) Let $E^\bullet$ be a minimal injective resolution of an A-module M. Show that $E^\bullet_P$ is a minimal injective resolution of the A_P-module M_P.

Proof. (a) Let $P, Q \subseteq A$ be prime ideals and let $E(A/Q)$ be the injective hull of A/Q. If $Q \subseteq P$, by (7.59) and (7.60):

$$E(A/Q) = E(A/Q)_P = E_{A_P}(A_P/QA_P)$$

is an injective A_P-module and an injective A-module. If $Q \not\subseteq P$, let $a \in Q - P$ and $\zeta \in E(A/Q)$. By (7.59) there is an $n \in \mathbb{N}$ so that $Q^n \zeta = 0$. Thus $a^n \zeta = 0$ and $E(A/Q)_P = 0$.

If E be any injective A-module, then by (7.53)

$$E \cong \oplus_{\alpha\in\Lambda} E(A/Q_\alpha)$$

where $Q_\alpha \in \operatorname{Spec}(A)$ for all $\alpha \in \Lambda$. Thus

$$E_P = \oplus_{\alpha\in\Lambda} E(A/Q_\alpha)_P$$

and E_P is an injective A_P-module and an injective A-module (Problem 3).
(b) By (7.63):

$$E(M) = \oplus_{\alpha\in\Lambda} E(A/Q_\alpha).$$

Let $\Lambda' = \{\alpha \in \Lambda \,|\, Q_\alpha \subseteq P\}$. Then

$$E(M)_P = \oplus_{\alpha\in\Lambda'} E(A/Q_\alpha).$$

Moreover, we can consider $E(M)_P$ as an A-submodule of $E(M)$. By (a) $E(M)_P$ is an injective A_P-module which contains M_P. It remains to show that $E(M)_P$ is an essential extension of M_P. Let $N \subseteq E(M)_P$ be a nonzero A_P-submodule of $E(M)_P$. Considering N as an A-submodule of $E(M)$ we obtain that $U = N \cap M \neq 0$ since $E(M)$ is an essential extension of M. Then $U_P \subseteq N \cap M_P$. By (7.59) every $a \in A - P$ induces an automorphism on $E(M)_P$. This implies that $U_P \neq 0$ and that $E(M)_P$ is an essential extension of M_P (as A_P-modules).
(c) Let $(E^\bullet, \partial^\bullet)$ be a minimal injective resolution of M. Then $E^i = E(Z^i(E^\bullet)) = E(\ker(\partial^i))$ for all i. By (a) $(E^\bullet_P, \partial^\bullet_P)$ is an injective resolution of M_P. Since localization is flat, for all i:

$$\ker(\partial^i)_P = \ker(\partial^i_P)$$

and $(E^\bullet_P, \partial^\bullet_P)$ is a minimal resolution of M_P.

(6) Let M be an A-module. Show that the following are equivalent:

(a) M is a flat A-module.

(b) For every ideal $I \subseteq A$ the canonical morphism:

$$I \otimes_A M \longrightarrow IM$$

is injective.

(c) For every finitely generated ideal $I \subseteq A$ the canonical morphism:

$$I \otimes_A M \longrightarrow IM$$

is injective.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c): trivial
(c) $\Rightarrow$ (b): Suppose that $I \subseteq A$ is an ideal and that the canonical map $\varphi : I \otimes_A M \longrightarrow IM$ is not injective. Then there are elements $a_i \in I$ and $m_i \in M$ for $1 \leq i \leq n$ with $\sum_{i=1}^n a_i \otimes m_i \neq 0$ in $I \otimes_A M$ and $\sum_{i=1}^n a_i m_i = 0$ in M. Let $I_0 = (a_1, \ldots, a_n) \subseteq A$ and consider the commutative diagram:

$$\begin{array}{ccc} I_0 \otimes_A M & \xrightarrow{\delta} & I_0 M \\ \psi \big\downarrow & & \epsilon \big\downarrow \\ I \otimes_A M & \xrightarrow{\varphi} & IM \end{array}$$

where ϵ is the embedding. By assumption (c) δ is injective. Then $\sum_{i=1}^{n} a_i \otimes m_i \neq 0$ in $I_0 \otimes_A M$ and $\epsilon\delta(\sum_{i=1}^{n} a_i \otimes m_i) = 0$, a contradiction.

(b) $\Rightarrow$ (a): Let $0 \longrightarrow N' \longrightarrow N$ be an exact sequence of A-modules. We have to show that the sequence $0 \longrightarrow N' \otimes_A M \longrightarrow N \otimes_A M$ is exact. We may consider $N' \subseteq N$ as a submodule of N. Consider the following set of submodules of N: $\Gamma = \{T \subseteq N \,|\, \text{T a submodule of N with } N' \subseteq T \text{ and } 0 \longrightarrow N' \otimes_A M \longrightarrow T \otimes_A M \text{ exact}\}$. Since $N' \in \Gamma$, $\Gamma \neq \emptyset$. Γ is partially ordered by inclusion. We want to show that Γ is inductively ordered. Let $\{T_i\}_{i \in I}$ be a chain in Γ and let $T_0 = \cup_{i \in I} T_i$. Let $\rho : N' \otimes_A M \longrightarrow T_0 \otimes_A M$ be the induced map and let $m_i \in M$ and $t_i \in N'$ with $\rho(\sum_{i=0}^{n} n_i \otimes m_i) = 0$. Write $T_0 \otimes_A M$ as $A^{(T_0 \times M)}/U$ where $A^{(T_0 \times M)}$ is the free module with basis $T_0 \times M$ and U is the submodule generated by elements $(x+x', y) - (x, y) - (x', y), (x, y+y') - (x, y) - (x, y'), (ax, y) - a(x, y), (x, ay) - a(x, y)$ for $x, x' \in T_0, y, y' \in M$ and $a \in A$. Since $\sum_{i=0}^{n} n_i \otimes m_i = 0$ in $T_0 \otimes M$, $\sum_{i=0}^{n} n_i \otimes m_i$ is a finite linear combination of the generators of U. Thus there is an $j \in I$ so that $\sum_{i=0}^{n} n_i \otimes m_i = 0$ in $T_j \otimes_A M$. This implies that $\sum_{i=0}^{n} n_i \otimes m_i = 0$ in $N' \otimes_A M$ and ρ is injective.

By Zorn's Lemma Γ contains a maximal element T. If $T \neq N$ take an element $n \in N - T$ and consider the submodule $K = T + An$. Then $K/T \cong A/I$ for some ideal $I \subseteq A$. Consider the exact sequence

$$0 \to I \to A \to A/I \to 0.$$

Tensoring with M yields a long exact sequence:

$$\operatorname{Tor}_1^A(A, M) = 0 \to \operatorname{Tor}_1^A(A/I, M) \to I \otimes_A M \xrightarrow{h} M \to M/IM \to 0.$$

By assumption h is injective and therefore $\operatorname{Tor}_1^A(A/I, M) = 0$. The exact sequence

$$0 \to T \to K \to K/T \to 0$$

yields a long exact sequence:

$$\to \operatorname{Tor}_1^A(K/T, M) \to T \otimes_A M \xrightarrow{\tau} K \otimes_A M \to K/T \otimes_A M \to 0$$

Since $\operatorname{Tor}_1^A(K/T, M) \cong \operatorname{Tor}_1^A(A/I, M) = 0$, τ is injective and so is the composition of maps:

$$N' \otimes_A M \to T \otimes_A M \xrightarrow{\tau} K \otimes_A M.$$

Thus $K \in \Gamma$ and the assertion that $T = N$ follows.

(7) Let A be a ring, $S = A[x_1, \ldots, x_n]$ the polynomial ring over A in n variables, and let

$$f = \sum_{(i) \in \mathbb{N}^n} a_{(i)} x_1^{i_1} \ldots x_n^{i_n}$$

be a polynomial in S. Put $T = S/(f)$. Show:

(a) If $(a_{(i)})_{(i) \in \mathbb{N}^n} = A$ then T is flat over A.
(b) If $(a_{(i)})_{(i) \in \mathbb{N}^n - (0)} = A$ then T is faithfully flat over A.

Proof. (a) Let $S = A[x_1, \dots, x_n]$ and

$$f = \sum_{(i)\in\mathbb{N}^n} a_(i)x_1^{i_1}\dots x_n^{i_n} \in S$$

with $(a_{(i)}) = A$. Problem 7 of Homework 1 implies that f is a regular element of A. The same argument shows that for all ideals $I \subseteq A$ the element $f + IS$ is a regular element of $(A/I)[x_1, \dots, x_n] \cong S/IS$. Fix an ideal $I \neq (0)$ of A. We claim that $\mathrm{Tor}_1^A(A/I, S) = 0$. Let $T = S/(f)$. Then there is an exact sequence:

$$0 \to S \xrightarrow{f} S \to T \to 0$$

where the first map is multiplication by f. Tensoring with A/I yields a long exact sequence:

$$\mathrm{Tor}_1^A(A/I, S) \to \mathrm{Tor}_1^A(A/I, S/(f)) \to S/IS \xrightarrow{f} S/IS \to T/IT \to 0.$$

Since S is a free A-module, $\mathrm{Tor}_1^A(A/I, S) = 0$ and since f is S/IS-regular, it follows that $\mathrm{Tor}_1^A(A/I, S/(f)) = 0$.

The exact sequence $0 \to I \to A \to A/I \to 0$ yields the long exact sequence:

$$\mathrm{Tor}_1^A(A/I, S/(f)) = 0 \to I \otimes_A S/(f) \to S/(f) \to A/I \otimes_A (S/(f)) \to 0.$$

Thus the canonical map $I\otimes_A(S/(f)) \to IS/(f)$ is bijective and the assertion follows with Problem 6.

(b) Assume that $(a_{(i)})_{(i)\in\mathbb{N}^n-(0)} = A$. By (a) $S/(f)$ is a flat A-module. We have to show that for every maximal ideal $\mathfrak{m}$ of A: $\mathfrak{m}S/(f) \neq S/(f)$ or equivalently that $(\mathfrak{m}S, f) \neq S$. Since $(a_{(i)})_{(i)\in\mathbb{N}^n-(0)} = A$ there is a $(j) \in \mathbb{N}^n$ so that $a_{(j)} \notin \mathfrak{m}$. Thus $f + \mathfrak{m}S \in S/\mathfrak{m}S \cong (A/\mathfrak{m})[x_1, \dots, x_n]$ is a non constant polynomial. In particular, $f + \mathfrak{m}S$ is not a unit and thus $(\mathfrak{m}S, f) \neq S$.

(8) Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local homomorphism of local Noetherian rings, and let N be an R-flat S-module such that $N/\mathfrak{m}N$ has finite length (as an S-module). Show that for every finite length R-module M:

$$l_S(M \otimes_R N) = l_R(M)l_S(N/\mathfrak{m}N).$$

Proof. We show by induction on $n = l_R(M)$ that $l_S(M \otimes_R N) < \infty$ and that $l_S(M \otimes_R N) = l_R(M)l_S(N/\mathfrak{m}N)$. If $n = 1$, then $M \cong R/\mathfrak{m}$ and $M \otimes_R N \cong R/\mathfrak{m} \otimes_R N \cong N/\mathfrak{m}N$.

Suppose that $l_R(M) = n+1$. Since M is an R-module of finite length, we know that $\mathfrak{m} \in \mathrm{Ass}(M)$. Thus there is an element $v \in M$ so that $U = Av \cong A/\mathfrak{m}$. Consider the exact sequence $0 \to U \to M \to M/U \to 0$. Then $l_R(U) = 1$ and $l_R(M/U) = n$. Since N is R-flat, the sequence

$$0 \to U \otimes_R N \to M \otimes_R N \to M/U \otimes_R N \to 0$$

is exact. By induction hypothesis $l_S(U \otimes_R N) < \infty$ and $l_S((M/U) \otimes_R N) < \infty$, thus $l_S(M \otimes_R N) < \infty$. Moreover,

$$\begin{aligned} l_S(M \otimes_R N) &= l_S(U \otimes_R N) + l_S((M/U) \otimes_R N) \\ &= l_S(N/\mathfrak{m}N) + (l_R(M) - 1)l_S(N/\mathfrak{m}N) \\ &= l_R(M)l_S(N/\mathfrak{m}N). \end{aligned}$$