Solutions to Homework 4 (due: 12-10-07).

(1) Prove the *Five Lemma:*
Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

and prove:

(a) If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
(b) If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.
(c) If t_1, t_2, t_4 and t_5 are isomorphisms, then t_3 is an isomorphism.

Proof. (a) Let $b_3 \in B_3$.

$\Rightarrow \exists a_4 \in A_4$ with $t_4(a_4) = g_3(b_3)$ (t_4 surjective)
$\Rightarrow g_4 g_3(b_3) = 0 = g_4 t_4(a_4) = t_5 f_4(a_4)$
$\Rightarrow f_4(a_4) = 0$ (t_5 injective)
$\Rightarrow \exists a_3 \in A_3$ with $f_3(a_3) = a_4$
$\Rightarrow g_3(b_3 - t_3(a_3)) = g_3(b_3) - g_3 t_3(a_3) = t_4(a_4) - t_4 f_3(a_3) = t_4(a_4) - t_4(a_4) = 0$
$\Rightarrow \exists b_2 \in B_2$ with $g_2(b_2) = b_3 - t_3(a_3)$
$\Rightarrow \exists a_2 \in A_2$ with $t_2(a_2) = b_2$ (t_2 surjective)

Then

$$\begin{aligned}
t_3(f_2(a_2) + a_3) &= t_3 f_2(a_2) + t_3(a_3) \\
&= g_2 t_2(a_2) + t_3(a_3) \\
&= g_2(b_2) + t_3(a_3) \\
&= b_3 - t_3(a_3) + t_3(a_3) \\
&= b_3
\end{aligned}$$

(b) Let $a_3 \in A_3$ with $t_3(a_3) = 0$.

$\Rightarrow t_4 f_3(a_3) = g_3 t_3(a_3) = 0$
$\Rightarrow f_3(a_3) = 0$ (t_4 injective)
$\Rightarrow \exists a_2 \in A_2$ with $f_2(a_2) = a_3$
$\Rightarrow g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(a_3) = 0$
$\Rightarrow \exists b_1 \in B_1$ with $g_1(b_1) = t_2(a_2)$
$\Rightarrow \exists a_1 \in A_1$ with $t_1(a_1) = b_1$ (t_1 surjective)
$\Rightarrow g_1 t_1(a_1) = g_1(b_1) = t_2 f_1(a_1) = t_2(a_2)$
$\Rightarrow f_1(a_1) = a_2$ (t_2 injective)
$\Rightarrow f_2(a_2) = f_2 f_1(a_1) = 0 = a_3$

Thus t_3 is injective.
(c) trivial

(2) Let A be a commutative ring with $1 \neq 0$ and let P and Q be projective A-modules. Show that $Q \otimes_A P$ is a projective A-module.

Proof. Since P and Q are projective, there are projective A-modules P' and Q' such that $P \oplus P' = F_1$ and $Q \oplus Q' = F_2$ where F_1 and F_2 are free A-modules. Since the tensor product commutes with direct sums, the module $F_1 \otimes_A F_2$ is free and

$$\begin{aligned} F_1 \otimes_A F_2 &\cong (P \oplus P') \otimes_A (Q \oplus Q') \\ &\cong (P \otimes (Q \oplus Q')) \oplus (P' \otimes_A (Q \oplus Q')) \\ &\cong (P \otimes_A Q) \oplus [(P \otimes_A Q') \oplus (P' \otimes_A (Q \oplus Q'))] \end{aligned}$$

$P \otimes_A Q$ is projective.

(3) Let A be a commutative local ring,

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow 0$$

an exact sequence of finitely generated free A-modules. Prove that

$$\sum_{i=0}^{n} (-1)^i \mathrm{rk}(F_i) = 0$$

where for a finitely generated free A-module F $\mathrm{rk}(F)$ denotes the number of elements in a basis of F.

Proof. We use the following facts:
(i) If k is a field and

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

an exact sequence of finite dimensional k-vector spaces, then

$$\dim(V) = \dim(V') + \dim(V'').$$

(ii) Let A be a commutative ring and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a split exact sequence of A-modules. Then for every A-module N the sequence

$$0 \longrightarrow M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is split exact.

Assume now that $(A, \mathfrak{m}, k)$ is a commutative local ring and let

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \to 0$$

be an exact sequence of finitely generated free A-modules. We proceed by induction on n. If $n = 1$ the statement is trivial.

If $n = 2$ the short exact sequence:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is split exact. Thus, after tensoring with $\otimes_A k$ the sequence

$$0 \longrightarrow F_2/\mathfrak{m}F_2 \longrightarrow F_1/\mathfrak{m}F_1 \longrightarrow F_1/\mathfrak{m}F_0 \longrightarrow 0$$

is exact. By (i)

$$\dim(F_1/\mathfrak{m}F_1) = \mathrm{rk}(F_1) = \dim(F_2/\mathfrak{m}F_2) + \dim(F_0/\mathfrak{m}F_0) = \mathrm{rk}(F_2) + \mathrm{rk}(F_0).$$

For the induction step, set $P = \mathrm{im}(f_2) = \ker(f_1)$ and consider the induced exact sequences:

$$0 \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_3} F_2 \longrightarrow P \longrightarrow 0$$

and

$$0 \longrightarrow P \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow 0.$$

Since F_0 is a free A-module, the last sequence is split exact and P is a finitely generated projective A-module. By (6.69) P is a free A-module. By induction hypothesis

$$\mathrm{rk}(P) + \sum_{i=1}^{n-1}(-1)^i \mathrm{rk}(F_{i+1}) = 0$$

and

$$\mathrm{rk}(P) = \mathrm{rk}(F_1) - \mathrm{rk}(F_0).$$

This yields the formula:

$$\sum_{i=0}^{n}(-1)^i \mathrm{rk}(F_i) = 0.$$