Homework 2: Sample solutions.

(1) Consider the polynomial ring $A = K[x, y, z]$ over a field K and the prime ideals $P_1 = (x, y)$ and $P_2 = (x, z)$ of A. Find two distinct shortest primary decompositions of $I = P_1P_2$.

Proof.

$$\begin{aligned} I &= (x^2, xy, xz, yz) \\ &= (x, y) \cap (x, z) \cap (x^2, y, z) \\ &= (x, y) \cap (x, z) \cap (x, y, z)^2 \end{aligned}$$

(2) Let A be a Noetherian ring, $P \subseteq A$ a prime ideal, and $i_{A,P} : A \longrightarrow A_P$ the canonical map into the localization. Define $P^{(n)} = i_{A,P}^{-1}(P^nA_P)$ and show:

(a) $P^{(n)}$ is a P-primary ideal.
(b) $P^{(n)}$ is the P-primary component of P^n.
(c) $P^{(n)} = P^n$ if and only if P^n is a primary ideal.

Proof.

Lemma. *Let $\varphi : A \longrightarrow B$ be a homomorphism of rings and let $Q' \subseteq B$ be a P'-primary ideal. Then $Q = \varphi^{-1}(Q')$ is a $P = \varphi^{-1}(P')$-primary ideal of A.*

Proof of Lemma. Let $a, b \in A$ with $ab \in Q$ and $a \notin Q$. Then $\varphi(a)\varphi(b) \in Q'$ and $\varphi(a) \notin Q'$. Since Q' is primary there is an $m \in \mathbb{N}$ so that $\varphi(b)^m \in Q'$. Thus $b^m \in Q$ and Q is a primary ideal. Since $\mathrm{rad}(Q) = P$ it follows that Q is P-primary.

(a) Since PA_P is the maximal ideal of A_P the ideal P^nA_P is PA_P-primary. The lemma shows that $P^{(n)} = i_{A,P}^{-1}(P^nA_P)$ is P-primary.
(b) Let $P^n = Q_1 \cap \ldots \cap Q_r$ be a shortest primary decomposition of P^n with Q_i a P_i-primary ideal. Since $\mathrm{rad}(P^n) = P = P_1 \cap \ldots \cap P_r$ the prime ideal $P = P_1$ is minimal in $\mathrm{Ass}(A/P^n)$. By Theorem (2.40) $Q_1 = i_{A,P}^{-1}(P^nA_P) = P^{(n)}$.
(c) follows from (a) and (b).

(3) Let A be a Noetherian ring and $P \subseteq A$ a prime ideal. Let $S_P(0)$ denote the kernel of the canonical map $i_{A,P} : A \longrightarrow A_P$. Show:

(a) $S_P(0) \subseteq P$
(b) $\mathrm{rad}(S_P(0)) = P$ if and only if P is a minimal prime of A.
(c) If P is a minimal ideal of A then $S_P(0)$ is the smallest P-primary ideal.

Proof. (a) Let $a \in A$ with $i_{A,P}(a) = a/1 = 0$. Then there is a $t \in A - P$ with $ta = 0$. Since $t \notin P$ we have that $a \in P$.
(b) $\Rightarrow$: Suppose that $\mathrm{rad}(S_P(0)) = P$ and let $x \in PA_P$. Then $x = p/s$ for some $p \in P$ and $s \notin P$. By assumption $p^n \in S_P(0)$ for some $n \in \mathbb{N}$ and therefore $x^n = p^n/s^n = 0$. This shows that $PA_P \subseteq \mathrm{rad}(A_P)$. Since $\mathrm{rad}(A_P)$ is the intersection of all minimal prime ideals of A_P, it follows that $PA_P = \mathrm{nil}(A_P)$ and that PA_P is the only prime ideal of A_P. Thus P is a minimal prime ideal of A.

$\Leftarrow$ If P is a minimal prime ideal of A then $\mathrm{nil}(A_P) = PA_P$. Thus for all $p \in P$ there is an $n \in \mathbb{N}$ with $(p/1)^n = 0$. This implies that $p^n \in S_P(0)$ and $P \subseteq \mathrm{rad}(S_P(0)$.

(c) Suppose that P is a minimal prime ideal of A and that $Q \subseteq A$ is a P-primary ideal. We claim that $Q = i_{A,P}^{-1}(QA_P)$. Obviously, $Q \subseteq i_{A,P}^{-1}(QA_P)$. In order to show the other inclusion let $a \in i_{A,P}^{-1}(QA_P)$. Then $a/1 \in QA_P$ and there is a $t \in A - P$ with $ta \in Q$. Since Q is primary with $\mathrm{rad}(Q) = P$, we have that $t^n \notin Q$ for all $n \in \mathbb{N}$. Thus $a \in Q$. This shows that $Q = i_{A,P}^{-1}(QA_P)$ and therefore $S_P(0) = i_{A,P}^{-1}(0) \subseteq i_{A,P}^{-1}(QA_P) = Q$. It remains to show that $S_P(0)$ is P-primary. Since A_P is a Noetherian local ring with exactly one prime ideal PA_P the zero ideal of A_P is PA_P-primary. By the Lemma of Problem 2, $S_P(0)$ is P-primary.

(4) Let A be a Noetherian ring and $I, J \subseteq A$ ideals with $IA_P \subseteq JA_P$ for all $P \subseteq \mathrm{Ass}(A/J)$. Show that $I \subseteq J$.

Proof. Suppose that $\mathrm{Ass}(A/J) = \{P_1, \ldots, P_n\}$ with $P_1, \ldots, P_r$ the maximal elements in $\mathrm{Ass}(A/J)$. Then $S = A - \cup_{i=1}^n P_i = A - \cup_{i=1}^r P_i$ is a multiplicative subset of A. The localization $S^{-1}A$ is a semilocal ring with r maximal ideals $P_1S^{-1}A, \ldots, P_rS^{-1}A$. Moreover, for all $1 \leq i \leq r$:

$$(S^{-1}A)_{P_iS^{-1}A} \cong A_{P_i}.$$

For all $1 \leq i \leq r$:

$$((I+J)S^{-1}A/JS^{-1}A)_{P_iS^{-1}A} \cong (I+J)A_{P_i}/JA_{P_i} = (0)$$

and by the local-global principle $IS^{-1}A \subseteq JS^{-1}A$. Since I is a finitely generated ideal there is a $t \in S$ so that $tI \subseteq J$ or equivalently $t(I+J)/J = 0$ in A/J. But t is a NZD in A/J and hence $I = J$.

(5) Let A be a Noetherian ring and $a \in A$ a NZD of A. Show that $\mathrm{Ass}(A/(a)) = \mathrm{Ass}(A/(a^n))$ for all $n \in \mathbb{N}$.

Proof. Since $a \in A$ is a NZD, the A-linear map:

$$\varphi : A/(a^{n-1}) \longrightarrow aA/(a^n)$$

defined by $\varphi(x + (a^{n-1})) = ax + (a^n)$ is an isomorphism of A-modules. From the short exact sequence;

$$0 \longrightarrow A/(a^{n-1}) \xrightarrow{\varphi} A/(a^n) \to A/(a) \to 0$$

we obtain that

$$\mathrm{Ass}(A/(a^n)) \subseteq \mathrm{Ass}(A/(a^{n-1})) \cup \mathrm{Ass}(A/(a)).$$

The proof is by induction on n. Since

$$\mathrm{Ass}(A/(a^n)) \subseteq \mathrm{Ass}(A/(a^{n-1})) \cup \mathrm{Ass}(A/(a))$$

and

$$\mathrm{Ass}(A/(a^{n-1})) = \mathrm{Ass}(aA/(a^n)) \subseteq \mathrm{Ass}(A/(a^n))$$

the induction hypothesis

$$\mathrm{Ass}(A/(a^{n-1})) = \mathrm{Ass}(A/(a))$$

yields that

$$\text{Ass}(A/(a^n)) = \text{Ass}(A/(a)).$$

(6) Let A be a ring so that for every maximal ideal $\mathfrak{m} \subseteq A$ the localization $A_{\mathfrak{m}}$ is Noetherian. Suppose that for every element $a \in A - (0)$ there are at most finitely many maximal ideals $\mathfrak{m} \subseteq A$ so that $a \in \mathfrak{m}$. Show that A is a Noetherian ring. Is the converse true?

Proof. Let $I \subseteq A$ be a nonzero ideal. Since every $a \in I - (0)$ is contained in at most finitely many maximal ideals, the ideal I is contained in at most finitely many maximal ideals. Suppose that $I \neq A$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be the maximal ideals containing I. Since $A_{\mathfrak{m}_i}$ is Noetherian for all $1 \leq i \leq s$ there are elements $a_1, \ldots, a_n \in I$ so that

$$IA_{\mathfrak{m}_i} = (a_1/1, \ldots, a_n/1)A_{\mathfrak{m}_i} \quad \text{for all} \quad 1 \leq i \leq s.$$

Let $J = (a_1, \ldots, a_n)$ be the ideal of A which is generated by the $a_i's$. Obviously, $J \subseteq I$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s, \mathfrak{m}_{s+1}, \ldots, \mathfrak{m}_t$ be the maximal ideals containing J. If $s = t$ then $I = J$ since $I_{\mathfrak{m}} = J_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subseteq A$. Suppose that $s < t$. Then for all $s+1 \leq i \leq t$ we have $I \not\subseteq \mathfrak{m}_i$. For all $s+1 \leq i \leq t$ take an element $b_i \in I - \mathfrak{m}_i$. We claim that

$$I = (a_1, \ldots, a_n, b_{s+1}, \ldots, b_t).$$

Let $K = (a_1, \ldots, a_n, b_{s+1}, \ldots, b_t)$ and let $\mathfrak{m} \subseteq A$ be a maximal ideal of A. If $\mathfrak{m} \neq \mathfrak{m}_i$ for all $1 \leq i \leq t$ then $I_{\mathfrak{m}} = K_{\mathfrak{m}} = A_{\mathfrak{m}}$, since $J \subseteq K$. If $\mathfrak{m} = \mathfrak{m}_i$ for some $s+1 \leq i \leq t$ then $I_{\mathfrak{m}} = A_{\mathfrak{m}} = K_{\mathfrak{m}}$ since $b_i \notin \mathfrak{m} = \mathfrak{m}_i$. If $\mathfrak{m} = \mathfrak{m}_i$ for some $1 \leq i \leq s$, then $I_{\mathfrak{m}} = J_{\mathfrak{m}} = K_{\mathfrak{m}}$ since $J \subseteq K \subseteq I$. Thus for all maximal ideals $\mathfrak{m} \subseteq A$ we have that $I_{\mathfrak{m}} = K_{\mathfrak{m}}$. By the local-global principle $I = K$.

The converse is false. Let $A = K[x, y]$ where K is an infinite field. Then $x \in (x, y + a)$ for all $a \in K$.

(7) Let K be a field and $T = K[\{x_i | i \in \mathbb{N}\}]$ the polynomial ring in infinitely many (countably) many variables over K. Let $\{n_i\}$ be a strictly increasing sequence of positive integers which satisfies the condition: $0 < n_i - n_{i-1} < n_{i+1} - n_i$ for all $i \in \mathbb{N}$. Consider the prime ideals $P_i = (x_j | n_i \leq j < n_{i+1})$ in T and set $S = T - \cup_{i \in \mathbb{N}} P_i$ and $A = S^{-1}T$. Show

(a) The maximal ideals of A are exactly the ideals $S^{-1}P_i$ for all $i \in \mathbb{N}$.
(b) The ring $A_{S^{-1}P_i}$ is Noetherian of dimension $n_{i+1} - n_i$.
(c) A is a Noetherian ring of infinite dimension.

(This example is due to M. Nagata.)

Proof. (a) Let $i_{T,S} : T \longrightarrow A$ be the canonical map into the localization and $\mathfrak{m} \subseteq A$ be a maximal ideal of A. The preimage $i_{T,S}^{-1}(\mathfrak{m}) = P$ is a prime ideal of T with $P \cap S = \emptyset$. This implies that

$$P \subseteq \cup_{i \in \mathbb{N}} P_i.$$

Note that for every nonzero element $f \in T$ there is an integer $t \in \mathbb{N}$ so that $f \in K[x_1, \ldots, x_t]$. If $f \in P$ is a nonzero element with $f \in K[x_1, \ldots, x_t]$, then there is a maximal j so that $n_j \leq t$. We claim that for this integer j:

$$(*) \quad P \subseteq \cup_{i=1}^{j} P_i.$$

Proof of (*). Suppose that there is an element $g \in P - \cup_{i=1}^{j} P_i$. Since $P \subseteq \cup_{i\in\mathbb{N}} P_i$, there is an $\ell > j$ so that $g \in P_\ell$. Then we can write

$$g = \sum a_\alpha m_\alpha$$

where $a_\alpha \in K - (0)$ and m_α monomials with the following property:

(i) For all α there is an i with $n_\ell \leq i < n_{\ell+1}$ so that x_i divides m_α.
(ii) For all $1 \leq k \leq j$ there is an $m_{\alpha(k)}$ such that x_i does not divide $m_{\alpha(k)}$ for all $n_k \leq i < n_{k+1}$.

If $f = \sum b_\beta n_\beta$ with $b_\beta \in K - (0)$ and n_β monomials, consider the sum:

$$f + g = \sum b_\beta n_\beta + \sum a_\alpha m_\alpha.$$

Since $f \in K[x_1, \ldots, x_t]$ with $t < n_\ell$ the monomials n_β and m_α do not cancel each other. Furthermore $n_\beta \notin P_r$ for all $r > j$ and $g \notin P_i$ for $i \leq j$ and $f + g \notin P_i$ for all $i \in \mathbb{N}$, a contradiction. This proves the claim.

(*) implies that $P \subseteq P_i$ for some $i \in \mathbb{N}$ and hence $\mathfrak{m} = PA \subseteq P_iA$. Since $\mathfrak{m}$ is maximal $\mathfrak{m} = P_iA$ (and $P = P_i$).
(b)

$$A_{S^{-1}P_i} \cong T_{P_i} = K(x_j | j \in \mathbb{N} \text{ with } j < n_i \text{ or } j \geq n_{i+1})[x_{n_i}, \ldots, x_{n_{i+1}-1}]_{\tilde{P}_i}$$

where $\tilde{P}_i$ is the prime ideal generated by $x_{n_i}, \ldots, x_{n_{i+1}-1}$. This shows that $A_{S^{-1}P_i}$ is a Noetherian ring of dimension $n_{i+1} - n_i$.
(c) Let $d \in A - (0)$ be a nonunit of A. Then $d = f/g$ where $f, g \in T$ and $g \in S$. Since f is contained in only finitely many P_i the element d is contained in only finitely many maximal ideals of A. By Problem (6) the ring A is Noetherian. Since $\dim(A_{S^{-1}P_i}) = n_{i+1} - n_i$ and $n_{i+1} - n_i \to \infty$ if $i \to \infty$, the dimension of A is infinite.

(8) Let K be an algebraically closed field and $Y \subseteq \mathbb{A}_K^n$ an irreducible algebraic variety of dimension r. Let H be a hypersurface of $\mathbb{A}_K^n$ with $Y \not\subseteq H$. Show that every irreducible component of $Y \cap H$ has dimension $\leq r - 1$.

Proof. We know that $Y = Z(P)$ where $P \subseteq K[x_1, \ldots, x_n]$ is a prime ideal. Since H is a hypersurface, $H = Z(f)$ for some $f \in K[x_1, \ldots, x_n]$ and $Y \not\subseteq H$ implies that $f \notin P$. Then

$$Y \cap H = Z(P) \cap Z(f) = Z(P + (f)).$$

If $P + (f) = k[x_1, \ldots, x_n]$, then $Y \cap H = \emptyset$ and $\dim(Y \cap H) \leq r - 1$. If $P + (f) \neq K[x_1, \ldots, x_n]$ then f is a nonzero nonunit in the domain $A(Y) = k[x_1, \ldots, x_n]/P$. Thus

$$\dim(K[x_1, \ldots, x_n]/(P + (f))) < \dim(k[x_1, \ldots, x_n]/P) = \dim(A(Y)).$$

In particular, $\dim(A(Y \cap H)) = \dim(K[x_1, \ldots, x_n]/(\text{rad}(P + (f))) \leq r - 1$.

(9) Show:

(a) A Noetherian topological space is quasi-compact, that is, every open cover has a finite subcover.
(b) Any subset of a Noetherian topological space is Noetherian.
(c) A Hausdorff Noetherian space is a finite set with the discrete topology.

Proof. (a) Let X be a Noetherian topological space and $X = \cup_{i \in I} U_i$ an open cover of X. Construct an ascending chain of open subsets as follows: If $i_1 \in I$ with $U_{i_1} \neq X$ then there is an $i_2 \in I$ so that $U_{i_2} \not\subseteq U_{i_1}$. Then $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$. Suppose $i_1, \ldots, i_m$ have been chosen so that for all $1 < k \leq m$

$$U_{i_1} \cup \ldots \cup U_{i_{k-1}} \subsetneq U_{i_1} \cup \ldots U_{i_k}.$$

If $X = U_{i_1} \cup \ldots U_{i_m}$ we are done. Otherwise there is an $i_{m+1} \in I$ so that $U_{i_{m+1}} \not\subseteq U_{i_1} \cup \ldots U_{i_m}$, etc. Since every ascending chain of open sets in X is stationary, this process stops after finitely many steps with $X = U_{i_1} \cup \ldots U_{i_\ell}$.
(b) Let X be a Noetherian topological space and $Y \subseteq X$ a nonempty subset. Suppose that for all $i \in \mathbb{N}$ there are given open subsets $\tilde{U}_i \subseteq Y$ of Y so that

$$\tilde{U}_1 \subseteq \tilde{U}_2 \subseteq \ldots \subseteq \tilde{U}_n \subseteq \ldots$$

is an increasing chain of open subsets of Y. Then there are open subsets $U_i \subseteq X$ of X so that $\tilde{U}_i = U_i \cap Y$. Set $V_n = U_1 \cup U_2 \cup \ldots \cup U_n$ and note that $V_n \cap Y = \tilde{U}_n$.

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots$$

is an ascending chain of open subsets of X. Since X is Noetherian there is an $n \in \mathbb{N}$ so that $V_n = V_{n+k}$ for all $k \in \mathbb{N}$. This implies that $\tilde{U}_n = V_n \cap Y = V_{n+k} \cap Y = \tilde{U}_{n+k}$ for all $k \in \mathbb{N}$ and Y is Noetherian.
(c) Since X is Noetherian, $X = X_1 \cup \ldots \cup X_n$ where X_i are the irreducible components of X. If X is a Noetherian Hausdorff space so is every X_i and we may assume that X is irreducible. We claim that $X = \{P\}$ is a one point space. Suppose that $P, Q \in X$ with $P \neq Q$. Then there are open subsets $U, V \subseteq X$ with $P \in U$, $Q \in V$ and $U \cap V = \emptyset$. But then $X = (X - U) \cup (X - V)$ with $X - U \neq X$ and $X - V \neq X$, contradicting that X is irreducible.