Homework 1: Sample solutions.

(1) Let $\mathfrak{m}$ denote the maximal ideal of A. Then:

$$\begin{aligned} e^2 = e &\Rightarrow e(e-1) = 0 \\ &\Rightarrow e \in \mathfrak{m} \text{ or } 1-e \in \mathfrak{m} \\ &\Rightarrow 1-e \in A^* \text{ or } e \in A^* \\ &\Rightarrow e = 0 \text{ or } 1-e = 0 \end{aligned}$$

(2) (b) $\Rightarrow$ (a) trivial

(a) $\Rightarrow$ (b) Let $\mathfrak{m} \subseteq A$ be a maximal ideal with $I \subseteq A$. Then $IA_{\mathfrak{m}} = I^2 A_{\mathfrak{m}}$. Since I is finitely generated by Nakayama $IA_{\mathfrak{m}} = 0$. Again, since I is finitely generated there is an element $t \in A - \mathfrak{m}$ so that $tI = 0$. This shows that the annihilator of I, $J = \operatorname{ann}(I)$ is not the zero ideal.

Claim: $I + J = A$

Proof of Claim. Suppose not. Then there is a maximal ideal $\mathfrak{n} \subseteq A$ with $I + J \subseteq \mathfrak{n}$. As before we see that $IA_{\mathfrak{n}} = 0$ and there is an element $s \in A - \mathfrak{n}$ with $sI = 0$. Thus $s \in J$, a contradiction.

Take elements $e \in I$ and $t \in J$ so that $e + t = 1$. Then

$$I = Ie + It = Ie$$

and

$$Ae \subseteq I \subseteq Ae.$$

Thus $I = Ae$ and $e = e^2 + et = e^2$.

(3) (a) Let $a \in \operatorname{Jrad}(A)$. Then for all $b \in A$ the element $1 - ab$ is a unit in A. Thus $\varphi(1 - ab) = 1 - \varphi(a)\varphi(b)$ is a unit in B for all $b \in A$. Since φ is surjective $\varphi(a)$ is in the Jacobson radical of B.

In general, we do not have that $\varphi(\operatorname{Jrad}(A)) = \operatorname{Jrad}(B)$. For example, the canonical map $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}/(4)$ is a surjective homomorphism. Then $\operatorname{Jrad}(\mathbb{Z}) = (0)$ but $\operatorname{Jrad}(\mathbb{Z}/(4)) = (2)/(4) \neq (0)$.

(b) We may assume that $B = A/I$ for some ideal $I \subseteq A$ and that $\varphi : A \longrightarrow A/I$ is the canonical map. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of A and suppose that:

$$I \subseteq \mathfrak{m}_i \text{ for } 1 \leq i \leq r \text{ and } I \not\subseteq \mathfrak{m}_i \text{ for } r+1 \leq i \leq n.$$

Then

$$\operatorname{Jrad}(A/I) = \bigcap_{i=1}^{r} (\mathfrak{m}_i/I) = (\cap_{i=1}^{r} \mathfrak{m}_i)/I.$$

Since $I + (\mathfrak{m}_{r+1} \cap \ldots \cap \mathfrak{m}_n) = A$ there are elements $a \in I$ and $b \in \mathfrak{m}_{r+1} \cap \ldots \cap \mathfrak{m}_n$ so that $1 = a + b$. Let $t \in \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$. Then $t = at + bt$ and $\varphi(t) = \varphi(bt)$ with $bt \in \operatorname{Jrad}(A)$. This shows $\varphi(\operatorname{Jrad}(A)) = \operatorname{Jrad}(B)$.

(4) (a) $\Rightarrow$ (b): Suppose that

$$A - S = \bigcup_{P \text{ prime and } P\cap S=\emptyset} P.$$

Assume that $ab \in S$ and $a \notin S$. Then there is a prime ideal P with $P \cap S = \emptyset$ and $a \in P$. But then $ab \in P$, a contradiction.
(b) $\Rightarrow$ (a): Obviously,

$$\bigcup_{P \text{ prime and } P\cap S=\emptyset} P \subseteq A - S.$$

Suppose there is an element

$$a \in (A - S) - (\bigcup_{P \text{ prime and } P\cap S=\emptyset} P).$$

By assumption (b) $ad \notin S$ for all $d \in A$. This shows that $S \cap (a) = \emptyset$. By Theorem (1.10) there is a prime ideal $Q \subseteq A$ with $a \in Q$ and $S \cap Q = \emptyset$, a contradiction. (Note that the conditions of (b) imply that S is a multiplicative subset of A.)

(5) Consider the A-module $\bar{M} = M/IM$ and let $\mathfrak{m} \subseteq A$ be a maximal ideal of A. If $I \nsubseteq \mathfrak{m}$ then $I_{\mathfrak{m}} = A_{\mathfrak{m}}$ and $\bar{M}_{\mathfrak{m}} = 0$. If $I \subseteq \mathfrak{m}$ then by assumption $M_{\mathfrak{m}} = 0$ and $\bar{M}_{\mathfrak{m}} = 0$. By the local-global principle $\bar{M} = 0$.

(6) Suppose that $M_{\mathfrak{m}}$ is torsion free for all maximal ideals $\mathfrak{m} \subseteq A$. Let $n \in M$ and $a \in A - (0)$ with $an = 0$. If $n \neq 0$ the annihilator $\text{ann}(n)$ is a proper ideal of A and there is a maximal ideal $\mathfrak{m}$ of A with $\text{ann}(n) \subseteq \mathfrak{m}$. This implies that $n/1 \neq 0$ in $M_{\mathfrak{m}}$. Since A is an integral domain the canonical map $A \longrightarrow A_{\mathfrak{m}}$ is injective. Let $t \in \text{ann}(n) - (0)$, then $t/1 \neq 0$ in $A_{\mathfrak{m}}$ and $(t/1)(n/1) = 0$ in $M_{\mathfrak{m}}$, a contradiction.

The converse also holds true: Suppose that M is torsion free. Let $\mathfrak{m} \subseteq A$ be a maximal ideal of A. Suppose that $r, t \in A - \mathfrak{m}$, $r \in A$ and $n \in M$ with $(s/t)(n/r) = 0$ in $M_{\mathfrak{m}}$. Then there is an element $u \in A - \mathfrak{m}$ so that $(us)n = 0$ in M. Since M is torsion free either $n = 0$ or $us = 0$. Thus $n/r = 0$ or $s/t = 0$.

(7) (a) $\Leftarrow$: Let $f = \sum_{i=0}^n a_i x^i \in A[x]$ with $a_0 \in A^*$ and a_i nilpotent for all $1 \leq i \leq n$. Then there is an $N \in \mathbb{N}$ so that

$$(\sum_{i=1}^n a_i x^i)^N = 0.$$

With $g = (1/a_0) \sum_{i=1}^n a_i x^i$ we have that

$$(1 + g)(1 - g + g^2 - \ldots \pm g^{N-1}) = 0$$

and f is invertible.

$\Rightarrow$ Let $g = \sum_{i=0}^m b_i x^i \in A[x]$ with $fg = 1$. Then $a_0 b_0 = 1$ and $a_0 \in A^*$. The case where $n = 0$ or $m = 0$ is trivial. Assume $n, m \geq 1$.

Claim: For all $0 \le r \le m$ it holds that $a_n^{r+1} b_{m-r} = 0$.

Proof. By induction on r. If $r = 0$ then $a_n b_m = 0$. Suppose that $a_n^{k+1} b_{m-k} = 0$ for all $0 \le k < r \le m$. Then

$$a_n^r = a_n^r fg = \sum_{\ell=0}^{n+m} a_n^r \Big(\sum_{i+j=\ell} a_i b_j \Big) x^\ell.$$

Consider the coefficient of x^{n+m-r}:

$$a_n^r (a_n b_{m-r} + a_{n-1} b_{m-r+1} + \ldots + a_{n-r} b_m).$$

Since $a_n^r b_\ell = 0$ for all $\ell > m - r$ it follows that $a_n^{r+1} b_{m-r} = 0$.

Thus

$$a_n^{m+1} b_0 = 0.$$

Since b_0 is invertible we obtain that $a_n^{m+1} = 0$. Since a_n is nilpotent the element $a_n x^n$ is contained in every maximal ideal of $A[x]$. Thus $h = f - a_n x^n$ is invertible and we can apply the same argument in order to obtain that a_{n-1} is nilpotent, etc.
(b) The backward direction is trivial. In order to prove the forward direction let $f = \sum_{i=0}^n a_i x^i, g = \sum_{i=0}^m b_i x^i \in A[x]$ with $fg = 0$. We may assume that $f \ne 0$ and that g is a polynomial of minimal degree with $fg = 0$. Since $a_n b_m = 0$ the polynomial $a_n g$ is either zero or has a smaller degree than g. Since $f(a_n g_m) = 0$ it follows that $a_n g = 0$.

Claim: $a_i g = 0$ for all $0 \le i \le n$.

Proof. We show by induction on r that $a_{n-r} g = 0$ for all $0 \le r \le n$. We already know that $a_n g = 0$. Suppose the statement is shown for all $0 \le k < r$. If $a_{n-r} g \ne 0$ then $\deg(g) = \deg(a_{n-r} g)$ since $f(a_{n-r} g) = 0$ and g of minimal degree. This implies that $a_{n-r} b_m$, the leading coefficient of $a_{n-r} g$, is nonzero. The coefficient of x^{n+m-r} of $fg (= 0)$ is:

$$\sum_{i+j=m+n-r} a_i b_j = a_{n-r} b_m + \sum_{i+j=m+n-r; i>n-r} a_i b_j = 0.$$

By induction hypothesis the right hand sum is 0. Thus $a_{n-r} b_m = 0$.

The claim implies that $b_m f = 0$.

(8) Obviously, $\mathrm{nil}(A[x]) \subseteq \mathrm{Jrad}(A[x])$. Let $f = \sum_{i=0}^n a_i x^i \in \mathrm{Jrad}(A[x])$. Then for all $g \in A[x]$:

$$1 - fg \in A[x]^*.$$

Thus, for $g = x$ the polynomial $1 - xf = 1 - \sum_{i=0}^n a_i x^{i+1} \in A[x]^*$. By Problem 7: $a_i \in \mathrm{nil}(A)$ for all $0 \le i \le n$ and $f \in \mathrm{nil}(A[x])$.

(9) We consider M as an $(A/\mathrm{ann})(M)$-module and assume that $\mathrm{ann}(M) = (0)$.

Claim (a) : A is a semilocal ring.

Proof (a). Consider the following subset of Λ:

$$\Gamma_1 = \{JM \mid J \text{ is a finite product of maximal ideals of } A\}.$$

By assumption Γ_1 has a minimal element JM where $J = \mathfrak{m}_1 \dots \mathfrak{m}_n$. If $\mathfrak{w} \in$ m-Spec(A) is a maximal ideal different from the $\mathfrak{m}_i$ then by the minimality of JM: $JM = \mathfrak{w}JM$. Then, since $JA_{\mathfrak{w}} = A_{\mathfrak{w}}$ we have that $M_{\mathfrak{w}} = \mathfrak{w}M_{\mathfrak{w}}$. By Nakayama's Lemma $M_{\mathfrak{w}} = 0$. Using again that M is finitely generated we see that there is an element $t \in A - \mathfrak{w}$ with $tM = 0$, a contradiction to $\operatorname{ann}(M) = 0$.

Claim (b) : For every ideal $I \subseteq A$ the A-module IM is finitely generated.

Proof (b). Let $I \subseteq A$ be an ideal. Consider the set

$$\Gamma_2 = \{JM \mid J \subseteq I \text{ and } JM \text{ is a finitely generated } A\text{-module}\}.$$

By assumption Γ_2 has a maximal element J_0M. If $x \in IM$ then there are finite many elements $a_1, \dots, a_n \in I$ and $m_1, \dots, m_n \in M$ with $x = \sum_{i=1}^n a_i m_i \in (J_0 + (a_1, \dots, a_n))M$ and by the maximality of J_0M we have that $x \in J_0M$. Thus $J_0M = IM$ and IM is a finitely generated A-module.

(c) Suppose that m-Spec$(A) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$. By the a.c.c. there is an integer $k \in \mathbb{N}$ so that $(\mathfrak{m}_1, \dots, \mathfrak{m}_n)^k M = (\mathfrak{m}_1, \dots, \mathfrak{m}_n)^{k+1} M$. By (b) the A-module $(\mathfrak{m}_1, \dots, \mathfrak{m}_n)^k M$ is finitely generated and thus by Nakayama $(\mathfrak{m}_1, \dots, \mathfrak{m}_n)^k M = 0$.

(d) Set $I_{(k_1,\dots,k_n)} = \mathfrak{m}_1^{k_1}\mathfrak{m}_2^{k_2}\dots\mathfrak{m}_n^{k_n}$ and consider the chain of submodules:

$$M \supseteq I_{(1,0,\dots,0)}M \supseteq \dots \supseteq I_{(k,0,\dots,0)}M \supseteq I_{(k,0,\dots,0)}M \supseteq \dots \supseteq I_{(k,k,\dots,k)}M.$$

Each factor module

$$N_{(k,\dots,k,t,0,\dots,0)} = \mathfrak{m}_1^k \dots \mathfrak{m}_i^k \mathfrak{m}_{i+1}^t M / \mathfrak{m}_1^k \dots \mathfrak{m}_i^k \mathfrak{m}_{i+1}^{t+1} M$$

is a finitely generated vector space over $K_{i+1} = A/\mathfrak{m}_{i+1}$ and every $N_{(k,\dots,k,t,0,\dots,0)}$ is an A-module of finite length. Thus M is an A-module of finite length.