Homework 8 (due: 4-23-08).

(1) Let A be a ring, $I \subseteq A$ an ideal, and M an A-module with $I^n M = (0)$ for some $n \in \mathbb{N}$. Show that M is I-adically complete.

(2) Let A be a Noetherian ring, $I, J \subseteq A$ ideals of A.

(a) If $J \subseteq I$ and if A is I-adically complete, then A is J-adically complete.
(b) If A is complete in the I-adic and the J-adic topology, then A is complete in the $I + J$-adic topology.

(3) Let k be a field of characteristic $\neq 2$ and let

$$f = \sum_{i=0}^{\infty} a_i x^i \in k[[x]]$$

be a power series with $a_0 \neq 0$ and $a_0 = b_0^2$ for some $b_0 \in k$. Use Hensel's Lemma to show that there is a power series

$$g = \sum_{i=0}^{\infty} b_i x^i \in k[[x]]$$

with $f = g^2$. Note that $1 + x$ is not a square in $k[x]_{(x)}$, thus the ring $k[x]_{(x)}$ does not satisfy Hensel's Lemma!

(4) Let k be a field of characteristic $\neq 2$, and let $f = x^2(1+x) - y^2 \in k[x,y]$. Show that f is irreducible in $k[x,y]$, while f is a product of two irreducible power series (non units) in $k[[x,y]]$. This implies that the ring $A = k[x,y]_{(x,y)}/(f)$ is a domain while its completion $\widehat{A} = k[[x,y]]/(f)$ is not a domain.

(5) Prove Chevalley's Theorem: Let (A, m) be a local Noetherian ring, which is m-adically complete. Let $I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$ be a decreasing chain of ideals in A for which $\bigcap_{n \in \mathbb{N}} I_n = (0)$. Show that for all $n \in \mathbb{N}$ there is an integer $\nu(n) \in \mathbb{N}$ so that $I_{v(n)} \subseteq m^n$.

(6) Let A be a PID with field of quotients K. Prove that $0 \to K \to K/A \to 0$ is an injective resolution of A.

(7) Let A be a local Noetherian ring. Show that if there is a nonzero finitely generated injective A-module then A is Artinian.

(8) Let A be a local Gorenstein ring and M a finitely generated A-module. Show that

$$\mathrm{projdim}_A(M) < \infty \quad \Leftrightarrow \quad \mathrm{injdim}_A(M) < \infty.$$