Homework 7 (due: 3-26-08).

(1) Let A be a local Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A/I-module. Show:

$$\mathrm{pd}_{(A/I)}(M) + \mathrm{pd}_A(A/I) = \mathrm{pd}_A(M)$$

provided that each of them is finite.

(2) Let A be a regular local ring of dimension n, and $M \neq 0$ a finitely generated A-module. Show that the following are equivalent:

(a) M is free.
(b) M is a CM-module of dimension n.

(3) Let A be a Noetherian domain of dimension 1, and let M be a nonzero finitely generated A-module of dimension 1. Show that M is a CM-module if and only if M is torsionfree.

(4) Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension n, and let $a, b \in \mathfrak{m} - (0)$ be elements with $a|b$ and $b \not| a$. Let $S = A/(a)$ and $T = A/(b)$. Show:

(a) S and T are CM-rings.
(b) S is a CM-module over T with $\dim(S) = \dim(T)$.
(c) S is not a free T-module.

(5) Let A be a Noetherian ring, M a finitely generated A-module and $I \subseteq A$, and ideal of A. Show that $\mathrm{depth}_I(M) \geq 2$ if and only if the natural homomorphism $M \longrightarrow \mathrm{Hom}_A(I, M)$ is an isomorphism.

(6) Let $\varphi : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a local homomorphism of local Noetherian rings, and M an B-module which is finitely generated as an A-module.

(a) Suppose that $P \in \mathrm{Ass}_B(M)$ and let $x \in M$ with $\mathrm{ann}_B(x) = P$. Prove that φ induces an embedding $A/(P \cap A) \longrightarrow B/P \cong Bx$ which makes B/P into a finitely generated $A/(P \cap A)$-module. Conclude that $P \cap A \neq \mathfrak{m}$ if $P \neq \mathfrak{n}$.
(b) Show that $\mathrm{depth}_A(M) = \mathrm{depth}_B(M)$.

(7) Let A be a local Noetherian ring, M a finitely generated A-module and N an nth syzygy of M in a finite free resolution of M. Show that $\mathrm{depth}(N) \geq \min(n, \mathrm{depth}(A))$.

(8) Let A be a local Noetherian ring, and

$$0 \to L_s \to L_{s-1} \to \ldots \to L_1 \to L_0 \to 0$$

a complex of finitely generated A-modules. Suppose that the following hold for $i > 0$:

(i) $\mathrm{depth}(L_i) \geq i$

(ii) $\mathrm{depth}(H_i(L_\bullet)) = 0$ or $H_i(L_\bullet) = 0$.

Show that $L_\bullet$ is acyclic.

(This is Peskine and Szpiro's 'lemme d'acyclité'.)

Hint: Set $C_i = \mathrm{coker}(L_{i+1} \to L_i)$ and show by descending induction that $\mathrm{depth}(C_i) \geq i$ and $H_i(L_\bullet) = 0$ for $i > 0$.