**Homework 5 (due: 2-6-08).**

(1) Let $A$ be a Noetherian ring, $M$ be a finitely generated $A$-module, and $\{N_i\}_{i\in I}$ a set of $A$-modules. Show:

$$\mathrm{Hom}_A(M, \oplus_{i\in I} N_i) \cong \oplus_{i\in I}\mathrm{Hom}_A(M.N_i).$$

(2) Let $A$ be a commutative ring and $M$ an $A$-module. Suppose that

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow K_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

are exact sequences with projective modules $P_1$ and $P_2$. Show that $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

(3) Let $A$ be a commutative domain, $K$ its field of quotients. Prove:

(a) A torsion-free $A$-module $M$ is injective if and only if $M$ is divisible.
(b) $K$ is the injective hull of $A$.

(4) Let $A$ be a Noetherian ring. Show that a direct sum of injective $A$-modules is an injective $A$-module.

(5) Let $A$ be a Noetherian ring and $P \subseteq A$ a prime ideal.

(a) If $E$ is an injective $A$-module show that $E_P$ is both $A_P$-injective and $A$-injective.
(b) Let $M$ be an $A$-module and $E(M)$ the injective hull of $M$. Then $E(M)_P$ is the injective hull of the $A_P$-module $M_P$.
(c) Let $E^\bullet$ be a minimal injective resolution of an $A$-module $M$. Show that $E^\bullet_P$ is a minimal injective resolution of the $A_P$-module $M_P$.

(6) Let $M$ be an $A$-module. Show that the following are equivalent:

(a) $M$ is a flat $A$-module.
(b) For every ideal $I \subseteq A$ the canonical morphism:

$$I \otimes_A M \longrightarrow IM$$

is injective.

(c) For every finitely generated ideal $I \subseteq A$ the canonical morphism:

$$I \otimes_A M \longrightarrow IM$$

is injective.

(7) Let $A$ be a ring, $S = A[x_1, \dots, x_n]$ the polynomial ring over $A$ in $n$ variables, and let

$$f = \sum_{(i) \in \mathbf{N}^n} a_{(i)} x_1^{i_1} \dots x_n^{i_n}$$

be a polynomial in $S$. Put $T = S/(f)$. Show:

(a) If $(a_{(i)})_{(i) \in \mathbf{N}^n} = A$ then $T$ is flat over $A$.
(b) If $(a_{(i)})_{(i) \in \mathbf{N}^n - (0)} = A$ then $T$ is faithfully flat over $A$.

(8) Let $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a local homomorphism of local Noetherian rings, and let $N$ be an $R$-flat $S$-module such that $N/\mathfrak{m}N$ has finite length (as an $S$-module). Show that for every finite length $R$-module $M$:

$$l_S(M \otimes_R N) = l_R(M) l_S(N/\mathfrak{m}N).$$