Homework 4 (due: 12-10-07).

(1) Prove the *Five Lemma:*
Consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

and prove:

(a) If t_2 and t_4 are surjective and t_5 is injective, then t_3 is surjective.
(b) If t_2 and t_4 are injective and t_1 is surjective, then t_3 is injective.
(c) If t_1, t_2, t_4 and t_5 are isomorphisms, then t_3 is an isomorphism.

(2) Let A be a commutative ring with $1 \neq 0$ and let P and Q be projective A-modules. Show that $Q \otimes_A P$ is a projective A-module.

(3) Let A be a commutative local ring, and

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow 0$$

an exact sequence of finitely generated free A-modules. Prove that

$$\sum_{i=0}^{n} (-1)^i \mathrm{rk}(F_i) = 0$$

where for a finitely generated free A-module F $\mathrm{rk}(F)$ denotes the number of elements in a basis of F.

(4) Let A be a ring, $S = A[x_1, \ldots, x_n]$ the polynomial ring over A in n variables, and let

$$f = \sum_{(i) \in \mathbf{N}^n} a_{(i)} x_1^{i_1} \ldots x_n^{i_n}$$

be a polynomial in S. Put $T = S/(f)$. Show:

(a) If $(a_{(i)})_{(i) \in \mathbf{N}^n} = A$ then T is flat over A.
(b) If $(a_{(i)})_{(i) \in \mathbf{N}^n - (0)} = A$ then T is faithfully flat over A.