Homework 3 (due: 11-14-07).

(1) For a polynomial $P(t) \in \mathbb{Q}$ show that the following conditions are equivalent:

(a) $P(n) \in \mathbb{Z}$ for all integers $n \in \mathbb{Z}$.
(b) $P(n) \in \mathbb{Z}$ for all but finitely many integers $n \in \mathbb{Z}$.
(c) $P(t) = \sum_{i=0}^{n} a_i \binom{t}{i}$ with $a_i \in \mathbb{Z}$ and $n \in \mathbb{N}$ suitable.

(2) Show that $S = \{P(t) \in \mathbb{Q}[t] \,|\, P(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\}$ is a non-Noetherian subring of $\mathbb{Q}[t]$.

(3) Let A be a ring and $n \in \mathbb{N}$ an integer. Suppose that every ideal of A is generated by st most n elements. Show that $\dim(A) \leq 1$.

(4) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial and let $Y = Z(f)$ be the algebraic variety defined by f. Y is called *non-singular* or *smooth* at a point $P \in Y$ if not all of the partial derivatives $\partial f/\partial x_i$ vanish at P. Let $A(Y)$ be the coordinate ring of Y and let $\mathfrak{m}_P \subseteq A(Y)$ be the maximal ideal of $A(Y)$ corresponding to P (that is, if $P = (a_1, \ldots, a_n)$, then $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)/(f)$). Show that Y is smooth at P if and only if the ring $A(Y)_{\mathfrak{m}_P}$ is regular.

(5) Let K be a field, $R = K[x_1, \ldots, x_n]$ the polynomial ring over K, and $I \subseteq R$ an ideal. Show that;

$$\mathrm{ht} I + \dim(R/I) = \dim(R).$$

(6) Let $A \subseteq B$ be an extension of rings such that the set $B - A$ is closed under multiplication. Show that A is integrally closed in B.

(7) Let A be a normal domain, $K = Q(A)$ its field of quotients, and $f(x) \in A[x]$ a monic polynomial. Show that $f(x)$ is irreducible in $K[x]$ if and only if $f(x)$ is irreducible in $A[x]$.

(8) Let $K \subseteq L$ be an extension of fields, $Q \subseteq l[x_1, \ldots, x_n]$ a prime ideal in the polynomial ring in n variables over L, and $P = Q \cup K[x_1, \ldots, x_n]$ its contraction to the polynomial ring over K. Show that $\mathrm{ht} Q \geq \mathrm{ht} P$ and that equality holds if L is algebraic over K, Use this to show that if two polynomials $f, g \in K[x_1, \ldots, x_n]$ have no common divisor in $K[x_1, \ldots, x_n]$, then f and g have no common divisor in $L[x_1, \ldots, x_n]$.

(9) Let $A \subseteq B$ be an integral extension of domains with A a normal domain, and $K = Q(A)$ the field of quotients of A. Let $I \subseteq A$ be an ideal, $b \in B$ an element, and $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ the minimal polynomial of b over K. Show that $b \in \mathrm{rad}(IB)$ if and only if $a_i \in I$ for all $0 \leq i \leq n-1$.