**Homework 2 (due: 10-26-07).**

(1) Consider the polynomial ring $A = K[x, y, z]$ over a field $K$ and the prime ideals $P_1 = (x, y)$ and $P_2 = (x, z)$ of $A$. Find two distinct shortest primary decompositions of $I = P_1 P_2$.

(2) Let $A$ be a Noetherian ring, $P \subseteq A$ a prime ideal, and $i_{A,P} : A \longrightarrow A_P$ the canonical map into the localization. Define $P^{(n)} = i_{A,P}^{-1}(P^n A_P)$ and show:

(a) $P^{(n)}$ is a $P$-primary ideal.
(b) $P^{(n)}$ is the $P$-primary component of $P^n$.
(c) $P^{(n)} = P^n$ if and only if $P^n$ is a primary ideal.

(3) Let $A$ be a Noetherian ring and $P \subseteq A$ a prime ideal. Let $S_P(0)$ denote the kernel of the canonical map $i_{A,P} : A \longrightarrow A_P$. Show:

(a) $S_P(0) \subseteq P$
(b) $\mathrm{rad}(S_P(0)) = P$ if and only if $P$ is a minimal prime of $A$.
(c) If $P$ is a minimal ideal of $A$ then $S_P(0)$ is the smallest $P$-primary ideal.

(4) Let $A$ be a Noetherian ring and $I, J \subseteq A$ ideals with $IA_P \subseteq JA_P$ for all $P \subseteq \mathrm{Ass}(A/J)$. Show that $I \subseteq J$.

(5) Let $A$ be a Noetherian ring and $a \in A$ a NZD of $A$. Show that $\mathrm{Ass}(A/(a)) = \mathrm{Ass}(A/(a^n))$ for all $n \in \mathbb{N}$.

(6) Let $A$ be a ring so that for every maximal ideal $\mathfrak{m} \subseteq A$ the localization $A_{\mathfrak{m}}$ is Noetherian. Suppose that for every element $a \in A - (0)$ there are at most finitely many maximal ideals $\mathfrak{m} \subseteq A$ so that $a \in \mathfrak{m}$. Show that $A$ is a Noetherian ring. Is the converse true?

(7) Let $K$ be a field and $T = K[\{x_i | i \in \mathbb{N}\}]$ the polynomial ring in infinitely many (countably) many variables over $K$. Let $\{n_i\}$ be a strictly increasing sequence of positive integers which satisfies the condition: $0 < n_i - n_{i-1} < n_{i+1} - n_i$ for all $i \in \mathbb{N}$. Consider the prime ideals $P_i = (x_j | n_i \leq j < n_{i+1})$ in $T$ and set $S = T - \cup_{i \in \mathbb{N}} P_i$ and $A = S^{-1}T$. Show

(a) The maximal ideals of $A$ are exactly the ideals $S^{-1}P_i$ for all $i \in \mathbb{N}$.
(b) The ring $A_{S^{-1}P_i}$ is Noetherian of dimension $n_{i+1} - n_i$.
(c) $A$ is a Noetherian ring of infinite dimension.

(This example is due to M. Nagata.)

(8) Let $K$ be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an irreducible algebraic variety of dimension $r$. Let $H$ be a hypersurface of $\mathbb{A}^n_K$ with $Y \not\subseteq H$. Show that every irreducible component of $Y \cap H$ has dimension $\leq r - 1$.

(9) Show:

- (a) A Noetherian topological space is quasi-compact, that is, every open cover has a finite subcover.
- (b) Any subset of a Noetherian topological space is Noetherian.
- (c) A Hausdorff Noetherian space is a finite set with the discrete topology.