MTH910 Homework 1 (due: 9/28/07).

All rings are commutative with identity!

(1) Let A be a ring with exactly one maximal ideal and $e \in A$ an element. e is called an *idempotent element* if $e^2 = e$. Show that the only idempotent elements of A are 0 and 1.

(2) Let A be a ring and $I \subseteq A$ a finitely generated ideal. Show that the following are equivalent:

(a) $I^2 = I$

(b) $I = Ae$ for some idempotent element $e \in A$.

(3) Let $\varphi : A \longrightarrow B$ be a surjective homomorphism of rings. Show:

(a) $\varphi(\mathrm{Jrad}(A)) \subseteq \mathrm{Jrad}(B)$. Give an example to show that equality fails in general.

(b) If A is a semilocal ring then $\varphi(\mathrm{Jrad}(A)) = \mathrm{Jrad}(B)$.

(4) Let A be a ring and $S \subseteq A$ a subset. Show that the following are equivalent:

(a) $A - S$ is the union of prime ideals.

(b) $1 \in S$ and ($ab \in S \Leftrightarrow a \in S$ and $b \in S$).

(5) Let M be an A-module and $I \subseteq A$ an ideal. Suppose that $M_{\mathfrak{m}} = (0)$ for every maximal ideal $\mathfrak{m} \subseteq A$ with $I \subseteq \mathfrak{m}$. Show that $M = IM$.

(6) Suppose that A is an integral domain. An A-module M is called *torsionfree* if for all $a \in A - (0)$ and all $m \in M - (0)$: $am \neq 0$. Show that M is a torsionfree A-module if $M_{\mathfrak{m}}$ is a torsionfree $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \in A$.

(7) Let $f = \sum_{i=0}^{n} \in A[x]$ be an element in the polynomial ring over A. Show:

(a) f is invertible in $A[x]$ if and only if $a_0 \in A^*$ is invertible and a_i are nilpotent for all $i \geq 1$.

(b) f is a zerodivisor in $A[x]$ if and only if there is an element $b \in A - (0)$ so that $bf = 0$.

(8) Show that $\mathrm{Jrad}(A[x]) = \mathrm{nil}(A[x])$ for any ring A.

(9) Let A be a ring and M a finitely generated A-module. Consider the set of submodules of M: $\Lambda = \{IM \mid I \subseteq A \text{ an ideal}\}$ and suppose that Λ satisfies the a.c.c. and the d.c.c. Show that M is an Artinian A-module.