SUPPLEMENT TO CHAPTER I

Vector spaces

Please read and memorize the definition of a vector space on pp.33 and 34. We will consider vector spaces over arbitrary fields F. In many instances F will be the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$.

Properties of vector spaces:.
Let V be a vector space over F and $u \in V$, $a \in F$. Then:

(a) The additive inverse $-u$ of u is unique.
(b) $0u = 0$ (Note that the first 0 denotes the zero element of F while the second 0 is the zero vector in V.)
(c) a0 $= 0$ (Here 0 is the zero vector in V.)
(d) $(-1)u = -u$.

Read and memorize the definition of a linear combination of vectors. Also read Examples 1.1 (1), (2), (3).

Subspaces

The definition of a subspace states that $S \subseteq V$ is a subspace of V if the following conditions are satisfied:

(1) For all $u, v \in S$: $u + v \in S$ (closed under addition)
(2) For all $u \in S$ and $a \in F$: $au \in S$ (closed under scalar multiplication)
(3) S is a vector space under these operations.

One can easily show that if S is a nonempty set which satisfies conditions (1) and (2) then condition (3) is satisfied. In addition, the zero vector of S is the same as the zero vector of V. This is summarized in Theorem 1.1 of the book. (Read it!)

Notation. Let V be a vector space. Then $\mathcal{S}(V)$ denotes the set of all subspaces of V.

In particular:

(a) $\{0\} \in \mathcal{S}(V)$ and $V \in \mathcal{S}(V)$.
(b) If $S, T \in \mathcal{S}(V)$ then $S \cap T \in \mathcal{S}(V)$, this means that $S \cap T$ is a subspace of V.
(c) More generally, if $\{S_i \mid i \in K\} \subseteq \mathcal{S}(V)$ is a collection of subspaces of V then

$$\bigcap_{i \in K} S_i \in \mathcal{S}(V).$$

Note that the union of subspaces is in general NOT a subspace as the following proposition shows:

Proposition 1A. *Let $S, T \in \mathcal{S}(V)$ then $S \cup T \in \mathcal{S}(V)$ if and only if $S \subseteq T$ or $T \subseteq S$.*

Proof. The backward direction is easy. For the forward direction we show the contrapositive: Suppose that $S \not\subseteq T$ and $T \not\subseteq S$ and take elements in the complements $u \in S - T$ and $v \in T - S$. If $S \cup T$ is a subspace then $u + v \in S \cup T$ and therefore $u + v \in S$ or $u + v \in T$. If $u + v \in S$ then $v = (u + v) - u \in S$, a contradiction. Similarly, if $u + v \in T$ then $u = (u + v) - v \in T$, contradiction. Thus $S \cup T$ is not a subspace of V.

From the section on "The lattice of subspaces" we will use the definition of the sum of subspaces on p.37.

Proposition 1B. *Let V be a vector space and $S, T \subseteq V$ subspaces of V. Then $S + T$ is a subspace of V. More generally, the sum of a family of subspaces of V is a subspace of V.*

Proof. The proof is routine. Verify that the condition of Theorem 1.1 is satisfied.

Direct sums

Read the definition on external direct sums and Example 1.4 on page p. 38.

For internal direct sums we will use the following definition:

Definition. Let V be a vector space over a field F and $S_1, \ldots, S_n$ subspaces of V. We say that V is an *(internal) direct sum* of $S_1, \ldots, S_n$ if every vector $v \in V$ can be written in a **unique way** as a sum of vectors in S_i:

$$v = u_1 + \ldots + u_n \quad \text{where} \quad u_i \in S_i.$$

Remark. Note that the definition of internal direct sums contains two conditions:

(1) $V = S_1 + \ldots + S_n$, that is, for every element $v \in V$ there are vectors $u_i \in S_i$ so that $v = u_1 + \ldots + u_n$.
(2) The representation of a vector $v \in V$ as a sum of vectors in S_i is unique. This means that if $u_i, w_i \in S_i$ with $v = u_1 + \ldots + v_n = w_1 + \ldots + w_n$ then $u_i = w_i$ for all $i = 1, 2, \ldots, n$.

Proposition 1C. *Suppose that $S_1, \ldots, S_n$ are subspaces of V. Then $V = S_1 \oplus \ldots \oplus S_n$ if and only if:*

(a) $V = S_1 + \ldots + S_n$
(b) *The only way to write 0 as a sum $0 = u_1 + \ldots + u_n$ with $u_j \in S_j$ is by taking $u_j = 0$ for all $j = 1, \ldots, n$.*

Proof. The forward direction follows immediately from the definition.

$\Leftarrow$: Suppose that (a) and (b) hold and let $v \in V$ and $u_i, w_i \in S_i$ with

$$v = u_1 + \ldots + u_n = w_1 + \ldots + w_n.$$

Then

$$0 = (v_1 - w_1) + (v_2 - w_2) + \ldots + (v_n - w_n).$$

By condition (b) $v_i - w_i = 0$ for all $i = 1, 2, \ldots, n$.

Proposition 1D. *Suppose that S and T are subspaces of V. Then $V = S \oplus T$ if and only if $V = S + T$ and $S \cap T = \{0\}$.*

Proof. $\Rightarrow$: If $V = S \oplus T$ then $V = S + T$. Suppose that $v \in S \cap T$. Then $0 = v + (-v)$ with $v \in S$ and $-v \in T$. Thus $v = 0$.

$\Leftarrow$: Since $V = S + T$ we only need to verify condition (b) in Proposition $1C$. Suppose that $0 = u + w$ with $u \in S$ and $w \in T$. Then $u = -w \in S \cap T$ and therefore $u = w = 0$.

Read Example 1.5 on page 41.

Spanning sets and linear independence

In the following V is a vector space over the field F.

Definition. A *subspace spanned* by a set S of vectors in V is the set of all linear combinations of vectors from S:

$$\langle S \rangle = \operatorname{span}(S) = \{ r_1 v_1 + \ldots + r_n v_n \mid r_i \in F, v_i \in S \}.$$

When $S = \{v_1, \ldots, v_n\}$ is a finite set we use the notation $\langle v_1, \ldots, v_n \rangle$ or $\operatorname{span}(v_1, \ldots, v_n)$. A set S of vectors in V is set to *span*, or *generate* V, if $V = \operatorname{span}(S)$, that is, if every vector $v \in V$ can be written in the form

$$v = r_1 v_1 + \ldots + r_n v_n$$

for some scalars $r_1, \ldots, r_n \in F$ and vectors $v_1, \ldots, v_n \in S$.

Definition. (a) A nonempty set S of vectors in V is *linearly independent* if for any $v_1, \ldots, v_n \in S$ with $v_i \neq v_j$ if $i \neq j$, we have:

$$r_1 v_1 + \ldots + r_n v_n = 0 \quad \text{implies that} \quad r_1 = r_2 = \ldots = r_n = 0.$$

If a set of vectors is not linearly independent, it is said to be *linearly dependent*.

(b) A list of vectors $v_1, \ldots, v_n \in V$ is called *linearly independent* if whenever $r_1 v_1 + \ldots + r_n v_n = 0$ with $r_i \in F$ then $r_1 = r_2 = \ldots = r_n = 0$.

Remark. Let $S = \{v_1, \ldots, v_n\}$ be a finite set of vectors. The difference between definitions (a) and (b) is that in a list of vectors repetitions are allowed. For example the list of vectors in F^2: $v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, 0)$ is linearly dependent while the set of vectors $S = \{v_1, v_2, v_3\} = \{v_1, v_2\}$ is linearly independent.

Theorem 1.6. *Let $S = \{v_1, \ldots, v_n\}$ be a finite set of vectors in V. The following are equivalent:*

(1) *S is linearly independent.*
(2) *Every vector in span(S) has a unique expression as a linear combination of vectors in S.*
(3) *No vector in S is a linear combination of the other vectors in S.*

Proof. (1) $\Rightarrow$ (2) : Let $v \in \operatorname{span}(S)$ and assume that

$$v = r_1 v_1 + \ldots + r_n v_n = s_1 v_1 + \ldots + s_n v_n$$

where $r_i, s_i \in F$. Then

$$0 = v - v = (r_1 - s_1)v_1 + \ldots + (r_n - s_n)v_n.$$

Since the vectors $v_1, \ldots, v_n$ are linearly independent:

$$r_1 - s_1 = r_2 - s_2 = \ldots r_n - s_n = 0 \quad \Rightarrow \quad r_1 = s_1, r_2 = s_2, \ldots, r_n = s_n.$$

(2) $\Rightarrow$ (3) : Suppose that $v_i \in \mathrm{span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. Then we can write v_i in two different ways as a linear combination of the vectors $v_1, \ldots, v_n$, namely:

$$v_i = r_1 v_1 + \ldots + r_{i-1}v_{i-1} + r_{i+1}v_{i+1} + \ldots + r_n v_n \quad \text{and} \quad v_i = v_i,$$

a contradiction

(3) $\Rightarrow$ (1) : Suppose that $0 = r_1 v_1 + \ldots + r_n v_n$. If $r_i \neq 0$ then

$$r_i v_i = (-r_1)v_1 + \ldots + (-r_{i-1})v_{i-1} + (-r_{i+1})v_{i+1} + \ldots + (-r_n)v_n.$$

Thus

$$v_i = (-r_1/r_i)v_1 + \ldots + (-r_{i-1}/r_i)v_{i-1} + (-r_{i+1}/r_i)v_{i+1} + \ldots + (-r_n/r_i)v_n$$

and $v_i \in \mathrm{span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, a contradiction

Notation. A set A is called a *proper* subset of a set B, if $A \subseteq B$ and $A \neq B$. We may also say that A is *properly* contained in B.

Theorem 1.7. *Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in V. The following are equivalent:*

(1) *S is linearly independent and spans V.*
(2) *For every $v \in V$ there are unique scalars $r_1, \ldots, r_n \in F$ so that $v = r_1 v_1 + \ldots + r_n v_n$.*
(3) *S is a minimal spanning set, that is, S spans V and no proper subset of S spans V.*
(4) *S is a maximal linear independent set, that is, S is linearly independent and any subset T of V that properly contains S is linearly dependent.*

Proof. (1) $\Rightarrow$ (2) : Let $v \in V$. Since S spans V there are scalars $r_1, \ldots, r_n \in F$ so that

$$v = r_1 v_1 + \ldots + r_n v_n.$$

If there is another list of scalars $s_1, \ldots, s_n \in F$ with

$$v = s_1 v_1 + \ldots + s_n v_n$$

then

$$0 = v - v = (r_1 - s_1)v_1 + \ldots + (r_n - s_n)v_n.$$

Since $v_1, \ldots, v_n$ are linearly independent, $r_1 = s_1, r_2 = s_2, \ldots, r_n = s_n$.

(2) $\Rightarrow$ (3) : By contradiction: Suppose that $A \subseteq S$ is a subset with $A \neq S$ and suppose that $V = \mathrm{span}(A)$. Since A is properly contained in S there is at least one v_i with $v_i \notin A$. After renumbering the $v's$ - if necessary - we may assume that

$v_1 \notin A$. Since A spans V any set containing A spans V. Thus we may assume that $A = \{v_2, v_3, \ldots, v_n\}$. Since $V = \text{span}(A)$ there are scalars $r_2, \ldots, r_n \in F$ so that

$$v_1 = r_2 v_2 + \ldots + r_n v_n.$$

This is one way to write v_1 as a linear combination of vectors $v_1, \ldots, v_n$. Another is $v_1 = v_1$, a contradiction to assumption (2).

(3) $\Rightarrow$ (4) : Let $T \subseteq V$ be a subset with $S \subseteq T$ and $S \neq T$. Let $v \in T - S$. We know by (3) that $V = \text{span}(S)$. Thus there are scalars $r_1, \ldots, r_n \in F$ so that

$$v = r_1 v_1 + \ldots + r_n v_n.$$

This gives a nontrivial linear combination of 0:

$$0 = r_1 v_1 + \ldots + r_n v_n + (-1)v$$

and the vectors $v, v_1, \ldots, v_n$ are linearly dependent. Thus T is linearly dependent.

(4) $\Rightarrow$ (1) : By assumption (4) the set S is linearly independent. We have to show that S spans V. Let $v \in V$. If $v \in S$ then $v = v_i$ for some $i = 1, 2, \ldots, n$ and, in particular, $v = v_i \in \text{span}(S)$. Let $v \notin S$. By assumption the set $S \cup \{v\}$ is linearly dependent and there are scalars $t, r_1, \ldots, r_n \in F$, not all 0, so that

$$0 = r_1 v_1 + \ldots + r_n v_n + tv.$$

If $t = 0$ then not all of the r_i are 0 and $0 = r_1 v_1 + \ldots + r_n v_n$, a contradiction to S a linearly independent set. Thus $t \neq 0$ and

$$v = (-r_1/t)v_1 + \ldots + (-r_n/t)v_n \in \text{span}(S).$$

Definition. A set of vectors in V that satisfies any of the equivalent conditions of Theorem 1.7 is called a *basis* of V.

Corollary 1.8. *A finite set $S = \{v_1, \ldots, v_n\}$ of vectors of V is a basis of V if and only if*

$$V = \langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle.$$

Proof. Homework

Example 1.6. The ith *Standard vector* of F^n is the vector e_i that has 0s in all coordinate positions except the ith, where it has 1. Thus

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1).$$

The set $\{e_1, \ldots, e_n\}$ is called the *standard basis* of F^n.

Definition. A vector space V is called *finite dimensional* if there is a finite subset $S = \{v_1, \ldots, v_n\} \subseteq V$ so that $V = \text{span}(S)$.

Linear Dependence Lemma. *If $v_1, \ldots, v_n \in V$ are linearly dependent vectors in V with $v_1 \neq 0$, then there is a $j \in \{2, \ldots, n\}$ so that*

(a) $v_j \in span(v_1, \ldots, v_{j-1})$
(b) $span(v_1, \ldots, v_n) = span(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

Proof. (a) There are $r_1, \ldots, r_n \in F$, not all 0, so that:

$$0 = r_1 v_1 + \ldots + r_n v_n.$$

Since $v_1 \neq 0$, not all $r_2, \ldots, r_n$ are 0. Let $j \geq 2$ be the largest integer with $r_j \neq 0$. Then

$$v_j = (-r_1/r_j)v_1 + \ldots + (-r_{j-1}/r_j)v_{j-1}.$$

This shows that $v_j \in \text{span}(v_1, \ldots, v_{j-1})$.

(b) Obviously, $\text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \subseteq \text{span}(v_1, \ldots, v_n)$. In order to show the other inclusion we use the fact that $v_j \in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$ and write

$$v_j = r_1 v_1 + \ldots r_{j-1} v_{j-1} + r_{j+1} v_{j+1} + \ldots + r_n v_n$$

for some $r_i \in F$. Let $v \in \text{span}(v_1, \ldots, v_n)$ then

$$v = t_1 v_1 + \ldots + t_n v_n$$

for some $t_i \in F$. Substituting the first equation for v_j into the second equation yields:

$$v = (t_1 + t_j r_1)v_1 + \ldots + (t_{j-1} + t_j r_{j-1})v_{j-1} + (t_{j+1} + t_j r_{j+1})v_{j+1} + \ldots (t_n + t_j r_n)v_n$$

and $v \in \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$.

Remark. Note that the linear dependence lemma also shows: *If $v_1, \ldots, v_n$ is a list of vectors in V with $v_1 \neq 0$ and $v_j \notin span(v_1, \ldots, v_{j-1})$ for all $2 \leq j \leq n$ then $v_1, \ldots, v_n$ is a linearly independent list of vectors in V.*

Theorem 1.E. *Let V be a finite vector space with $V = span(v_1, \ldots, v_n)$. If $u_1, \ldots, u_m$ is a list of linearly independent vectors then $m \leq n$.*

Proof. Since $v_1, \ldots, v_n$ is a spanning set of V the list of vectors $u_1, v_1, \ldots, v_n$ is linearly dependent. By the linear dependence theorem we can remove one of the $v's$ and still have a spanning set of lenght n of V.

Suppose we have added $u's$ and removed $v's$ so that the set

$$(*) \qquad u_1, \ldots, u_{j-1}, v_{k_j}, \ldots, v_{k_n}$$

is a spanning set of V of lenght n. Then by adding one vector to this list we obtain a linearly dependent list. Thus

$$u_1, \ldots, u_j, v_{k_j}, \ldots, v_{k_n}$$

is a linearly dependent list of vectors. Since the $u's$ are linearly independent for all $2 \leq \ell \leq m$: $u_\ell \notin \text{span}(u_1, \ldots, u_{\ell-1})$ and by the linear dependence lemma we can remove one of the $v's$. The process stops when there are no $u's$ or no $v's$ left. If there are no $u's$ left then $m \leq n$, as desired. If there are no $v's$ left then $u_1, \ldots, u_n$ is a linearly independent spanning set of V. If $m > n$ then $u_{n+1} \in \text{span}(u_1, \ldots, u_n)$, a contradiction to the linear independence of the $u's$. Thus in this case $n = m$.

Corollary. *Every subspace of a finite dimensional vector space V is finite dimensional.*

Proof. Let $U \subseteq V$ be a subspace of V. If $U = \{0\}$ we are done. Suppose $U \neq \{0\}$ and take $u_1 \in U$ with $u_1 \neq 0$. If $U = \text{span}(u_1)$ we are done. If $U \neq \text{span}(u_1)$ take $u_2 \in U - \text{span}(u_1)$. Again, if $U = \text{span}(u_1, u_2)$ we are done. If $U \neq \text{span}(u_1, u_2)$ take $u_3 \in U - \text{span}(u_1, u_2)$ etc. This way we obtain a list of vectors $u_1, \ldots, u_n \in U$ with $u_j \notin \text{span}(u_1, \ldots, u_{j-1})$. Since $u_1 \neq 0$ by the linear dependence lemma the list $u_1, \ldots, u_n$ is linearly independent. Theorem 1.E tells us that the process must stop after finitely many steps. Thus U has a finite spanning set.

Theorem 1.9. *Let V be a finite dimensional nonzero vector space. Then:*

(a) *Any linearly independent set in V is contained in a basis of V.*
(b) *Any spanning set of V contains a basis of V.*

Proof. (a) Let $u_1, \ldots, u_m \in V$ be a list of linearly independent vectors in V. If $V = \text{span}(u_1, \ldots, u_m)$ we are done. If not we expand the list by a vector $u_{m+1} \in V - \text{span}(u_1, \ldots, u_m)$. Again if $V = \text{span}(u_1, \ldots, u_{m+1})$ we are done. If not take $u_{m+2} \in V - \text{span}(u_1, \ldots, u_{m+1})$ etc. This way we create a list of vectors $u_1, \ldots, u_n$ with $u_j \notin \text{span}(u_1, \ldots, u_{j-1})$ for $2 \leq j \leq n$. By the linear dependence lemma this list of vectors is linearly independent. The process must stop after finitely many steps since there is an upper bound to the length of a linearly independent set in V.

(b) Let $V = \text{span}(v_1, \ldots, v_n)$. We may remove any $v_i = 0$ from the spanning list and still have a spanning list. Thus we may assume that $v_1 \neq 0$. If $v_1, \ldots, v_n$ is a list of linearly independent vectors we are done. If not apply the linear dependence lemma and remove one of the v_j for $2 \leq j \leq n$ so that $V = \text{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$. Apply the same argument to the spanning list $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$ and so on. The process stops when the reduced spanning list of vectors is linearly independent.

The dimension of a vector space

Note that Theorem 1.10 is identical to Theorem 1.E.

Corollary. *Let V be a finite dimensional vector space. Any two bases of V have the same length.*

Definition. Let V be a finite dimensional vector space. If $V \neq \{0\}$ the *dimension* of V is the lenght of a basis of V. If $V = \{0\}$ we say that the dimension of V is 0. Notation: $\dim(V)$

Remark. If V is a finite dimensional vector space and if $S \subseteq V$ is a subspace of V then $\dim(S) \leq \dim(V)$. Moreover, $\dim(S) = \dim(V)$ if and only if $S = V$.

Theorem 1.13. *Let V be a finite dimensional vector space.*

(a) *If $\mathcal{B}$ is a basis of V and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ with $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ then*

$$V = \langle \mathcal{B}_1 \rangle \oplus \langle \mathcal{B}_2 \rangle.$$

(b) *Let $V = S \oplus T$ for subspaces S and T of V. If $\mathcal{B}_1$ is a basis of S and $\mathcal{B}_2$ a basis of T then $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of V.*

Proof. Homework

Theorem 1.14. *Let V be a finite dimensional vector space and $S, T \subseteq V$ subspaces of V. Then:*

$$dim(S+T) = dim(S) + dim(T) - dim(S \cap T).$$

Proof. Let $u_1, \ldots, u_s$ be a basis of $S \cap T$. We extend the $u's$ to a basis of S and T, so let:

$$\mathcal{B}_1 = \{u_1, \ldots, u_s, v_1, \ldots, v_t\} \quad \text{be a basis of } S$$

$$\mathcal{B}_2 = \{u_1, \ldots, u_s, w_1, \ldots, w_r\} \quad \text{be a basis of } T.$$

In particular, $\dim(S \cap T) = s$, $\dim(S) = s + t$, and $\dim(T) = s + r$. We claim that

$$\mathcal{B} = \{u_1, \ldots, u_s, v_1, \ldots, v_t, w_1, \ldots, w_r\}$$

is a basis of $S + T$.

(a) $\mathcal{B}$ is linearly independent.

Suppose that $a_i, b_i, c_i \in F$ with

$$a_1u_1 + \ldots + a_su_s + b_1v_1 + \ldots + b_tv_t + c_1w_1 + \ldots + c_rw_r = 0.$$

Then

$$v = a_1u_1 + \ldots + a_su_s + b_1v_1 + \ldots + b_tv_t = -(c_1w_1 + \ldots + c_rw_r) \in S \cap T$$

and thus there are $d_i \in F$ so that

$$v = d_1u_1 + \ldots + d_su_s.$$

But then

$$v = d_1u_1 + \ldots + d_su_s = -(c_1w_1 + \ldots + c_rw_r)$$

and by the linear independence of $\mathcal{B}_2$ we get that $d_1 = d_2 = \ldots = d_s = c_1 = c_2 = \ldots = c_r = 0$. In particular, $v = 0$ and therefore

$$0 = v = a_1u_1 + \ldots + a_su_s + b_1v_1 + \ldots + b_tv_t.$$

By the linear independence of $\mathcal{B}_1$ we have that $a_1 = a_2 = \ldots = a_s = b_1 = b_2 = \ldots = b_t = 0$. This shows that $\mathcal{B}$ is linear independent.

(b) $\mathcal{B}$ spans $S + T$.

Verify that if S is spanned by $\mathcal{B}_1$ and T is spanned by $\mathcal{B}_2$ then $S + T$ is spanned by $\mathcal{B}_1 \cup \mathcal{B}_2$.

This shows that $\dim(S + T) = s + t + r$ and the formula follows.

READ sections: ORDERED BASES AND COORDINATE MATRICES and THE ROW AND COLUMN SPACE OF A MATRIX.