CHAPTER IX: THE KOSZUL COMPLEX

§1: REVIEW OF THE EXTERIOR ALGEBRA

Let R be a commutative ring and M an R-module. Consider R as a graded ring by the trivial grading: $R = \bigoplus_{n=0}^{\infty} R_n$ where $R = R_0$ and $R_n = 0$ for all $n > 0$. Let $M^{\otimes i}$ denote the ith tensor power of M, i.e. $M^{\otimes i} = M \otimes \cdots \otimes M$ with i factors of M if $i > 0$ and $M^{\otimes 0} = R$. Then $\bigotimes M = \bigoplus_{i=0}^{\infty} M^{\otimes i}$ is a graded R-module. The bilinear map $M^{\otimes i} \times M^{\otimes j} \longrightarrow M^{\otimes i+j}$ induced by $(x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_j) \longmapsto x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j$ extends linearly to a multiplication on $\bigotimes M$. With this definition $\bigotimes M$ becomes a graded associative R-algebra which is not commutative in general. $\bigotimes M$ is called the <u>tensor algebra</u> of M. The tensor algebra is characterized by the

(9.1) <u>Universal property</u>: Let S be an R-algebra (not necessarily commutative) and $\varphi : M \longrightarrow S$ an R-linear map. Then there is a unique R-algebra homomorphism $\psi : \bigotimes M \longrightarrow S$ extending φ, i.e. $\psi|_{M^{\otimes 1} = M} = \varphi$.

The <u>exterior algebra</u> ΛM is the residue class ring $\Lambda M = \bigotimes M / \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $\{x \otimes x \mid x \in M\}$. Since $\mathcal{I}$ is generated by homogeneous elements, ΛM is a graded R-algebra. The product in ΛM is denoted by $x \wedge y$. In general, ΛM is not commutative; it is alternating: if $x, y \in \Lambda M$ homogeneous, then $x \wedge y = (-1)^{(\deg x)(\deg y)}\, y \wedge x$ and $x \wedge x = 0$ for x homogeneous and $\deg x$ odd. (Proof: consider $(x+y) \wedge (x+y)$).

(9.2) <u>Universal property</u>: Let S be an R-algebra (not necessarily commutative) and $\varphi : M \to S$ an R-linear map with $\varphi(x)^2 = 0$ for all $x \in M$. Then there is a unique R-algebra homomorphism $\psi : \Lambda M \longrightarrow S$ extending φ, i.e. $\psi|_{\Lambda^1 M = M} = \varphi$.

(9.3) <u>Remark</u>: (a) The ith graded component of ΛM is denoted by $\Lambda^i M$ and is called the ith <u>exterior power</u> of M. Obviously, $\Lambda^0 M = R$, $\Lambda^1 M = M$, and for $i \geq 2$ $\Lambda^i M = M^{\otimes i} / (x_1 \otimes \cdots \otimes x_i \mid x_j = x_k \text{ for some } j \neq k)$.

(b) Let $x_1, \ldots, x_n \in M$. If τ is a permutation of $\{1, \ldots, n\}$ then $x_{\tau(1)} \wedge \cdots \wedge x_{\tau(n)} = \operatorname{sgn}(\tau)\, x_1 \wedge \cdots \wedge x_n$. If $I \subseteq \{1, \ldots, n\}$ set $x_I = x_{\nu_1} \wedge \cdots \wedge x_{\nu_i}$ if $I = \{\nu_1, \ldots, \nu_i\}$ with $\nu_1 < \cdots < \nu_i$. If $I, J \subseteq \{1, \ldots, n\}$ with $I \cap J = \emptyset$ set $\operatorname{sgn}(I, J) = (-1)^\ell$ where ℓ is the number of $(i,j) \in I \times J$ with $i > j$; if $I \cap J \neq \emptyset$ set $\operatorname{sgn}(I, J) = 0$. Then $x_I \wedge x_J = \operatorname{sgn}(I, J)\, x_{I \cup J}$.

(c) Let $\{x_g\}_{g \in G}$ be a system of generators of M. Then $\Lambda^n M$ is generated by exterior products x_I with $I \subseteq G$ and $|I| = n$. In particular, if $|G| = m < \infty$, then $\Lambda^i M = 0$ for all $i > m$.

(9.4) <u>Remark</u>: Let $\varphi: M \longrightarrow N$ be an R-linear map.

(a) There is a unique R-algebra morphism $\Lambda\varphi: \Lambda M \longrightarrow \Lambda N$ so that the diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle nat}\downarrow & & \downarrow{\scriptstyle nat} \\ \Lambda M & \xrightarrow{\Lambda\varphi} & \Lambda N \end{array}$$

commutes. $\Lambda\varphi$ is homogeneous of degree 0 with $\Lambda\varphi(x_1 \wedge \ldots \wedge x_n) = \varphi(x_1) \wedge \ldots \wedge \varphi(x_n)$. (This follows immediately from the universal property of the exterior product).

(b) For all $i > 0$ the sequence $\Lambda^{i-1} M \otimes \ker\varphi \longrightarrow \Lambda^i M \longrightarrow \Lambda^i N \longrightarrow 0$ is exact. In particular, $\ker \Lambda\varphi$ is generated by $\ker\varphi$ (without proof).

(9.5) <u>Remark</u>: Let $\varphi: R \longrightarrow S$ be a morphism of rings and M an R-module. Then there is a natural isomorphism of graded S-algebras: $(\Lambda M) \otimes_R S \cong \Lambda(M \otimes_R S)$.

Let M and N be R-modules. Define:

(a) a grading on $(\Lambda M) \otimes_R (\Lambda N)$ by $[(\Lambda M) \otimes_R (\Lambda N)]_n = \bigoplus_{i+j=n} (\Lambda^i M) \otimes_R (\Lambda^j N)$.

(b) a multiplication on $(\Lambda M) \otimes_R (\Lambda N)$ by $(x \otimes y)(x' \otimes y') = (-1)^{(\deg y)(\deg x')} (x \wedge x') \otimes (y \wedge y')$ for all homogeneous elements $x, x' \in \Lambda M$; $y, y' \in \Lambda N$. Then $(\Lambda M) \otimes (\Lambda N)$ is an alternating graded R-algebra with degree 1 component $(M \otimes R) \oplus (R \otimes N) \cong M \oplus N$. The natural map

$M \oplus N \longrightarrow (\Lambda M) \otimes (\Lambda N)$ extends to an R-algebra morphism $\varphi : \Lambda(M \oplus N) \longrightarrow (\Lambda M) \otimes (\Lambda N)$. Conversely, the natural maps $M \hookrightarrow \Lambda(M \oplus N)$ and $N \hookrightarrow \Lambda(M \oplus N)$ extend to morphisms of R-algebras $\psi_1 : \Lambda M \longrightarrow \Lambda(M \oplus N)$ and $\psi_2 : \Lambda N \longrightarrow \Lambda(M \oplus N)$. By the universal property of tensor products ψ_1 and ψ_2 induce an R-algebra morphism $\psi : (\Lambda M) \otimes (\Lambda N) \longrightarrow \Lambda(M \oplus N)$. Since $[\psi \circ \varphi]_1 = \mathrm{id}_{M \oplus N} = [\varphi \circ \psi]_1$, by linear extension ψ and φ are inverse to each other. Thus:

(9.6) <u>Proposition</u>: $(\Lambda M) \otimes_R (\Lambda N) \cong \Lambda(M \oplus N)$ as alternating graded R-algebras.

(9.7) <u>Proposition</u>: (a) If $\{x_1, \ldots, x_n\}$ is a generating set of M, then $\{x_I \mid |I| = i\}$ is a generating set of $\Lambda^i M$.

(b) If $\{e_1, \ldots, e_n\}$ is a basis of a free R-module F, then $\{e_I \mid |I| = i\}$ is a basis of $\Lambda^i F$. In particular, $\Lambda^i F$ is free of rank $\binom{n}{i}$.

<u>Proof</u>: (b) Notice that $\Lambda Re_n = R \oplus Re_n$ where $Re_n \cong R$. By (9.6) $\Lambda F = \Lambda(Re_1 \oplus \ldots \oplus Re_{n-1}) \otimes \Lambda Re_n$. Thus by induction on n, ΛF has an R-basis $\{e_I, e_I \wedge e_n \mid I \subseteq \{1, \ldots, n-1\}\}$.

§2: BASIC PROPERTIES OF THE KOSZUL COMPLEX

Let R be a ring, L an R-module, and $f: L \longrightarrow R$ an R-linear map. The map $\tilde{f}^{(n)}: L^n \longrightarrow \Lambda^{n-1}L$ defined by $\tilde{f}^{(n)}(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{i+1} f(x_i)\, x_1 \wedge \dots \hat{x}_i \dots \wedge x_n$ is an alternating n-linear map. Thus $\tilde{f}^{(n)}$ factors through an R-linear map $d_f^{(n)}: \Lambda^n L \longrightarrow \Lambda^{n-1}L$ with

$$d_f^{(n)}(x_1 \wedge \dots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} f(x_i)\, x_1 \wedge \dots \hat{x}_i \dots \wedge x_n$$

for all $x_1, \dots, x_n \in L$. The collection of all maps $d_f^{(n)}$ defines a graded R-homomorphism $d_f: \Lambda L \longrightarrow \Lambda L$.

(9.8) Remark: (a) d_f has the following properties:

(i) $d_f \circ d_f = 0$

(ii) For all homogeneous $x, y \in \Lambda L$: $d_f(x \wedge y) = d_f(x) \wedge y + (-1)^{\deg x}\, x \wedge d_f(y)$.

(b) Since $d_f \circ d_f = 0$ we obtain a complex:

$$(*) \quad \dots \longrightarrow \Lambda^n L \xrightarrow{d_f} \Lambda^{n-1} L \xrightarrow{d_f} \dots \longrightarrow \Lambda^2 L \xrightarrow{d_f} \Lambda^1 L = L \xrightarrow{f} R \longrightarrow 0$$

and (ii) implies that d_f is an antiderivation (of degree -1).

(9.9) Definition: $(*)$ is the Koszul complex of f, denoted $K.(f)$. If M is an R-module, then $K.(f, M)$ is the complex $K.(f) \otimes_R M$, called the Koszul complex of f with coefficients in M. Its differential is denoted by $d_{f,M}$.

(9.10) Proposition: Let R be a ring, L an R-module, and $f: L \longrightarrow R$ an R-linear map.

(a) The Koszul complex $K.(f)$ carries the structure of an associative graded alternating algebra, namely that of ΛL.

(b) Its differential d_f is an antiderivation of degree -1.

(c) For every R-module M the complex $K.(f, M)$ is a $K.(f)$-module in a natural way.

(d) One has $d_{f,M}(x.y) = d_f(x).y + (-1)^{\deg x}\, x.d_{f,M}(y)$ for all homogeneous x of $K.(f)$ and all elements $y \in K.(f, M)$.

Proof: (a) and (b) follow from the definition.
(c) clear, if S is an R-algebra and M an R-module, then $S \otimes_R M$ is a (left) S-module in the natural way.
(d) It is enough to show the claim for $y = w \otimes z$ with $w \in K.(f)$, $z \in M$. Then $d_{f,M}(x.w \otimes z) = d_{f,M}((x \wedge y) \otimes z) = d_f(x \wedge y) \otimes z$ and the rest follows since d_f is an antiderivation.

Set $Z.(f) = \ker d_f$, $Z.(f,M) = \ker d_{f,M}$, and $B.(f) = \operatorname{im} d_f$, $B.(f,M) = \operatorname{im} d_{f,M}$.

(9.11) Definition: The homology $H.(f) = Z.(f)/B.(f)$ is the Koszul homology of f. For every R-module M the homology $Z.(f,M)/B.(f,M)$ is denoted by $H.(f,M)$ and called the Koszul homology of f with coefficients in M.

For a subset $S \subseteq K.(f)$ and a subset $U \subseteq K.(f,M)$ let $S.U$ denote the R-submodule of $K.(f,M)$ generated by $\{s.u \mid s \in S, u \in U\}$. Then $Z.(f).Z.(f,M) \subseteq Z.(f,M)$, $Z.(f).B.(f,M) \subseteq B.(f,M)$ and $B.(f).Z.(f,M) \subseteq B.(f,M)$. Notice that $K.(f) \cong K.(f,R)$. The first inclusion shows that $Z.(f)$ is a graded R-subalgebra of $K.(f)$. The second and third inclusions show that $B.(f)$ is a twosided ideal in $Z.(f)$.

(9.12) Proposition: Let R be a ring, L an R-module, and $f: L \longrightarrow R$ an R-linear map.
(a) The Koszul homology $H.(f)$ carries the structure of an associative graded alternating R-algebra.
(b) For every R-module M the homology $H.(f,M)$ is an $H.(f)$-module in a natural way.

Proof: (a) $H.(f)$ is an R-algebra since $Z.(f)$ is an R-algebra and $B.(f)$ is an ideal of $Z.(f)$. $Z.(f)$ is an associative graded alternating R-algebra and $B.(f)$ is homogeneous.
(b) Since $Z.(f)Z.(f,M) \subseteq Z.(f,M)$, $Z(f,M)$ is a $Z.(f)$-module. Since $Z.(f)B.(f,M) \subseteq B.(f,M)$, $B.(f,M)$ is a $Z.(f)$-submodule of $Z.(f,M)$ and since $B.(f)Z.(f,M) \subseteq B.(f,M)$, $H.(f,M)$ is annihilated by $B.(f)$.

(9.13) Corollary: With $I = \operatorname{im} f$ the Koszul homology $H.(f,M)$ is an R/I-module. In particular, $H_0(f) = R/I$ and $H_0(f,M) = M/IM$.

Define $K^\cdot(f) = \operatorname{Hom}_R(K.(f), R)$, $K^\cdot(f,M) = \operatorname{Hom}_R(K.(f), M)$, and $H^\cdot(f) = H^\cdot(K^\cdot(f))$, $H^\cdot(f,M) = H^\cdot(K^\cdot(f,M))$. $H^\cdot(f)$, $H^\cdot(f,M)$ are called the Koszul cohomology of f (with coefficients in M).

(9.14) Proposition: Let R be a ring, L an R-module, and $f: L \to R$ an R-linear map. Set $I = \operatorname{im} f$.
(a) For all $a \in I$ multiplication by a on $K.(f), K.(f,M), K^\cdot(f), K^\cdot(f,M)$ is null-homotopic.
(b) I annihilates $H.(f), H.(f,M), H^\cdot(f), H^\cdot(f,M)$.
(c) If $I = R$, the complexes $K.(f), K.(f,M), K^\cdot(f), K^\cdot(f,M)$ are null-homotopic. In particular, their (co)homology vanishes.

Proof: (a) Let $x \in L$ with $f(x) = a$ and let ϑ_a denote multiplication by a on $K.(f)$ and λ_x left multiplication by x on $K.(f)$. Then $\vartheta_a = d_f \circ \lambda_x + \lambda_x \circ d_f$. Thus multiplication by a is null-homotopic and so are $\vartheta_a \otimes M$ and $\operatorname{Hom}_R(\vartheta_a, M)$, the multiplication by a on $K.(f,M), K^\cdot(f,M)$.
(b) is a general fact.
(c) choose $a=1$ and apply (a) and (b).

Let L_1 and L_2 be R-modules and $f_1: L_1 \to R$, $f_2: L_2 \to R$ R-linear maps. f_1 and f_2 induce a linear form $f: L_1 \oplus L_2 \to R$ by $f(x_1 \oplus x_2) = f_1(x_1) + f_2(x_2)$.

(9.15) Proposition: There is an isomorphism of complexes $K.(f_1) \otimes_R K.(f_2) \cong K.(f)$.

Proof: Since $(\Lambda L_1) \otimes_R (\Lambda L_2) \cong L$ by (9.6) $K.(f_1) \otimes_R K(f_2) \cong K.(f)$ as graded R-algebras. The nth graded component of $K.(f_1) \otimes_R K.(f_2)$ is $\bigoplus_{i=0}^{n} (\Lambda^i L_1) \otimes (\Lambda^{n-i} L_2)$ and the differential $d_{f_1} \otimes d_{f_2}$ is defined by $(*)$ $d_{f_1} \otimes d_{f_2}(x \otimes y) = d_{f_1}(x) \otimes y + (-1)^i x \otimes d_{f_2}(y)$ for $x \in \Lambda^i L_1$ and $y \in \Lambda^{n-i} L_2$. d_f is an antiderivation on ΛL which coincides with $d_{f_1} \otimes d_{f_2}$ on the degree one component $L = L_1 \oplus L_2$. $(*)$ implies that $d_{f_1} \otimes d_{f_2}$ is also an antiderivation on

$\Lambda L \cong (\Lambda L_1) \otimes (\Lambda L_2)$. Since an antiderivation on ΛL is uniquely determined by its values on L, $d_f = d_{f_1} \otimes d_{f_2}$ (provided ΛL and $(\Lambda L_1) \otimes (\Lambda L_2)$ are identified).

(9.16) <u>Proposition</u>: Let R be a ring, L an R-module, and $f: L \longrightarrow R$ an R-linear map. Suppose that $\varphi: R \longrightarrow S$ is a homomorphism of rings.

(a) There is a natural isomorphism $K.(f) \otimes_R S \cong K.(f \otimes S)$.

(b) If φ is flat, then $H.(f, M) \otimes_R S \cong H.(f \otimes S, M \otimes_R S)$ for all R-modules M.

<u>Proof</u>: (a) By (9.5) $(\Lambda L) \otimes_R S \cong \Lambda(L \otimes_R S)$ and $d_f \otimes S$ and $d_{f \otimes S}$ are antiderivations on $\Lambda(L \otimes_R S)$ which coincide in degree one. Thus $d_f \otimes S = d_{f \otimes S}$ by the same argument as in the proof of (9.15).

(b) If $C_\bullet$ is any complex of R-modules and S a flat R-algebra, then $H.(C. \otimes S) = H.(C.) \otimes S$.

Let L and L' be R-modules with linear forms $f: L \longrightarrow R$ and $f': L' \longrightarrow R$. By (9.4) every R-linear map $\varphi: L \longrightarrow L'$ induces a unique homomorphism of R-algebras $\Lambda\varphi: \Lambda L \longrightarrow \Lambda L'$. If $f = f' \circ \varphi$, then $\Lambda\varphi$ is a homomorphism of Koszul complexes, i.e. for all n the diagram:

$$\begin{array}{ccc} \Lambda^n L & \xrightarrow{d_f} & \Lambda^{n-1} L \\ \Lambda\varphi \downarrow & & \downarrow \Lambda\varphi \\ \Lambda^n L' & \xrightarrow{d_{f'}} & \Lambda^{n-1} L' \end{array}$$

commutes. We just showed:

(9.17) <u>Proposition</u>: With the notation as above, if $f = f' \circ \varphi$ then $\Lambda\varphi$ is a homomorphism of complexes.

§3: THE KOSZUL COMPLEX OF A SEQUENCE

Let L be a finitely generated free R-module with basis $e_1, \dots, e_n$. A linear form $f: L \to R$ is uniquely determined by the values $f(e_i) = x_i$ for $1 \le i \le n$. Conversely, given a sequence $\underline{x} = x_1, \dots, x_n \in R$ there is a linear form $f: L \to R$ with $f(e_i) = x_i$ for all $1 \le i \le n$. We set $K_\cdot(\underline{x}) = K_\cdot(f)$, $H_\cdot(\underline{x}) = H_\cdot(f)$, $K_\cdot(\underline{x}, M) = K_\cdot(f, M)$, etc if $f: L \to R$ is defined by $f(e_i) = x_i$.

Since $L = \bigoplus_{i=1}^n Re_i$ and $f = \oplus f_i$ where $f_i : Re_i \to R$ with $f_i(e_i) = x_i$ by (9.15) $K_\cdot(\underline{x}) = K_\cdot(\underline{x}') \otimes K_\cdot(x_n) = K_\cdot(x_1) \otimes \dots \otimes K_\cdot(x_n)$ where $\underline{x}' = x_1, \dots, x_{n-1}$. Moreover, $K_\cdot(\underline{x})$ is essentially invariant under a permutation of $\underline{x}$.

Set $I = (\underline{x})$ and let $F_\cdot$ be a free resolution of R/I. Since $H_0(\underline{x}) = R/I$, by (7.14) there is a morphism of complexes $\varphi_\cdot : K_\cdot(\underline{x}) \to F_\cdot$ with $H_0(\varphi_\cdot) = \mathrm{id}_{R/I}$. $\varphi_\cdot$ is unique up to homotopy.

(9.18) <u>Proposition</u>: Let R be a ring, $\underline{x} = x_1, \dots, x_n$ a sequence in R, and $I = (\underline{x})$. For all i there are natural R-linear maps $H_i(\underline{x}, M) \to \mathrm{Tor}_i^R(R/I, M)$ and $\mathrm{Ext}_R^i(R/I, M) \to H^i(\underline{x}, M)$.

<u>Proof</u>: The morphism of complexes $\varphi_\cdot : K_\cdot(\underline{x}) \to F_\cdot$ yields morphisms of complexes $\varphi_\cdot \otimes M : K_\cdot(x, M) \to F_\cdot \otimes M$ and $\mathrm{Hom}_R(\varphi_\cdot, M) : \mathrm{Hom}_R(F_\cdot, M) \to K^\cdot(\underline{x}, M)$. Apply (7.4).

(9.19) <u>Remark</u>: Let $\underline{x} = x_1, \dots, x_n \in R$, $I = (\underline{x})$, and M an R-module. Then $H_0(\underline{x}, M) \cong M/IM$ and $H_n(\underline{x}, M) \cong 0 :_M I$.

<u>Proof</u>: For the second claim notice that $K_\cdot(\underline{x}, M) : 0 \to M \xrightarrow{d_n} M^n \to \dots$ with $d_n(m) = (x_1 m, -x_2 m, \dots, (-1)^{n+1} x_n m)$. Thus $H_n(\underline{x}, M) = \ker d_n = \{ m \in M \mid x_i m = 0 \text{ for all } 1 \le i \le n \}$.

Let M be an R-module. Then $M^* = \mathrm{Hom}_R(M, R)$ denotes the <u>dual</u> of M. Let L be a free R-module with basis $e_1, \dots, e_n$. Then $e_1 \wedge \dots \wedge e_n$ is a basis of $\Lambda^n L$. Thus there is an R-isomorphism $\omega_n : \Lambda^n L \xrightarrow{\sim} R$ with $\omega_n(e_1 \wedge \dots \wedge e_n) = 1$. ($\omega_n$ is an <u>orientation</u> of L.)

For all $0 \le i \le n$ the bilinear forms $\Lambda^i L \times \Lambda^{n-i} L \xrightarrow{\wedge} \Lambda^n L \xrightarrow{\omega_n} R$ induce an R-linear map $\omega_i : \Lambda^i L \longrightarrow (\Lambda^{n-i} L)^*$ defined by $(\omega_i(x))(y) = \omega_n(x \wedge y)$ for all $x \in \Lambda^i L$, $y \in \Lambda^{n-i} L$. For $I \subseteq \{1, -, n\}$ with $|I| = i$ write $I' = \{1, -, n\} - I$ and let $J \subseteq \{1, -, n\}$ with $|J| = n-i$. Then:

$$e_I \wedge e_J = \operatorname{sgn}(I,J)\, e_{I \cup J} = \begin{cases} 0 & \text{if } J \neq I' \\ \operatorname{sgn}(I,I')\, e_{\{1,-,n\}} & \text{if } J = I'. \end{cases}$$

This implies that for all $0 \le i \le n$ $\omega_i : \Lambda^i L \longrightarrow (\Lambda^{n-i} L)^*$ is an isomorphism. Set $u_i = (-1)^{\binom{i}{2}} \omega_i$ and $u_\bullet = \bigoplus u_i : \Lambda L \longrightarrow (\Lambda L)^*$.

(9.20) Proposition: (Self-duality) Let $\underline{x} = x_1, -, x_n \subseteq R$ and M an R-module.

(a) $u_\bullet : K_\bullet(\underline{x}) \longrightarrow (K^\bullet(\underline{x}))(-n)$ is an isomorphism of complexes.

(b) $K_\bullet(\underline{x}, M) \cong (K^\bullet(\underline{x}, M))(-n)$

(c) $H_i(\underline{x}, M) \cong H^{n-i}(\underline{x}, M)$.

Proof: (a) We need to show that $u_\bullet$ is a morphism of complexes. This follows once we have shown that $\omega_{i-1} \circ d_i = (-1)^{i-1} d^*_{n-i+1} \circ \omega_i$ where $d_\bullet = d_{\underline{x}}$. The latter follows from the identity $\operatorname{sgn}(\nu, I - \{\nu\}) \cdot \operatorname{sgn}(I - \{\nu\}, I' - \{\nu\}) = (-1)^{i-1} \operatorname{sgn}(\nu, I') \operatorname{sgn}(I, I')$ where $\nu \in I \subseteq \{1, -, n\}$ and $|I| = i$.

(b) follows from (a) and (c) follows from (b).

(9.21) Proposition: Let $\underline{x} = x_1, -, x_n \subseteq R$ and $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ an exact sequence of R-modules. The induced sequence $0 \longrightarrow K_\bullet(\underline{x}, M') \longrightarrow K_\bullet(\underline{x}, M) \longrightarrow K_\bullet(\underline{x}, M'') \longrightarrow 0$ is an exact sequence of complexes. This induces a long exact sequence of homology:

$$\cdots \longrightarrow H_i(\underline{x}, M') \longrightarrow H_i(\underline{x}, M) \longrightarrow H_i(x, M'') \longrightarrow H_{i-1}(\underline{x}, M') \longrightarrow \cdots$$

Proof: Exactness follows from the fact that $\Lambda^i R^n$ are free R-modules. Then apply (7.6).

(9.22) Proposition: Let $x \in R$ and $C_\bullet$ a complex of R-modules.

(a) There is an exact sequence of morphisms of complexes:

$$0 \longrightarrow C_\cdot \longrightarrow C_\cdot \otimes K_\cdot(x) \longrightarrow C_\cdot(-1) \longrightarrow 0$$

(b) The induced long exact sequence of homology is:

$$\cdots \longrightarrow H_i(C_\cdot) \longrightarrow H_i(C_\cdot \otimes K_\cdot(x)) \longrightarrow H_{i-1}(C_\cdot) \xrightarrow{(-1)^{i-1}x} H_{i-1}(C_\cdot) \longrightarrow \cdots$$

(c) If x is a NZD on $C_\cdot$, $H_i(C_\cdot \otimes K_\cdot(x)) \cong H_\cdot(C_\cdot \otimes R/(x))$.

<u>Proof</u>: (a) The commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "id"] \arrow[d, "x"] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "id"] & R \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

yields an exact sequence of complexes $0 \to R \to K_\cdot(x) \to R(-1) \to 0$. Tensor with $C_\cdot \otimes_R -$.

(b) We only have to show that the connecting homomorphism is multiplication by $\pm x$.

Let $\partial_\cdot$ be the differential on $C_\cdot$. In degrees i and $i-1$ the exact sequence of (a) looks like:

$$\begin{tikzcd}
0 \arrow[r] & C_i \arrow[r, "nat"] \arrow[d, "\partial_i"'] & C_i \oplus C_{i-1} \arrow[r, "nat"] \arrow[d, "\varphi_i"] & C_{i-1} \arrow[r] \arrow[d, "\partial_{i-1}"] & 0 \\
0 \arrow[r] & C_{i-1} \arrow[r, "nat"] & C_{i-1} \oplus C_{i-2} \arrow[r, "nat"] & C_{i-2} \arrow[r] & 0
\end{tikzcd}$$

where φ_i is given by the matrix $\begin{bmatrix} \partial_i & (-1)^{i-1}x \\ 0 & \partial_{i-1} \end{bmatrix}$. The claim follows from the construction of the connecting homomorphism (7.6).

(c) Since x is a NZD, $0 \to C_\cdot \xrightarrow{x} C_\cdot \to C_\cdot \otimes R/(x)$ is exact. On the other hand, the commutative diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d, equal] & K_\cdot(x) \arrow[d] \\
R \arrow[r] & H_0(x) = R/(x)
\end{tikzcd}$$

induces a commutative diagram with with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_\cdot \arrow[r] \arrow[d, equal] & C_\cdot \otimes K_\cdot(x) \arrow[r] \arrow[dl] \arrow[d] & C_\cdot(-1) \arrow[r] & 0 \\
0 \arrow[r] & C_\cdot \arrow[r, "x"] & C_\cdot & C_\cdot \otimes R/(x)
\end{tikzcd}$$

which gives

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(C_\cdot) \arrow[r, "(-1)^{i+1}x"] \arrow[d, "\cong\ (-1)^{i+1}"'] & H_i(C_\cdot) \arrow[r] \arrow[d, equal] & H_i(C_\cdot \otimes K_\cdot(x)) \arrow[r] \arrow[d, "\psi_i"] & H_{i-1}(C_\cdot) \arrow[r, "(-1)^i x"] \arrow[d, "\cong\ (-1)^i"'] & H_{i-1}(C_\cdot) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_i(C_\cdot) \arrow[r, "x"] & H_i(C_\cdot) \arrow[r] & H_i(C_\cdot \otimes R/(x)) \arrow[r] & H_{i-1}(C_\cdot) \arrow[r, "x"] & H_{i-1}(C_\cdot) \arrow[r] & \cdots
\end{tikzcd}$$

It is easy to check that the middle diagram commutes. Thus φ_i is an isomorphism by the Five Lemma.

(9.23) Corollary: Let $\underline{x} = x_1, \ldots, x_n \in R$ and M an R-module.

(a) With $\underline{x}' = x_1, \ldots, x_{n-1}$ there is an exact sequence:

$$\cdots \longrightarrow H_i(\underline{x}', M) \longrightarrow H_i(\underline{x}, M) \longrightarrow H_{i-1}(\underline{x}', M) \xrightarrow{\pm x_n} H_{i-1}(\underline{x}', M) \longrightarrow \cdots$$

(b) Let $\underline{x}' = x_1, \ldots, x_s$; $\underline{x}'' = x_{s+1}, \ldots, x_n$ and assume that $\underline{x}'$ is weakly M-regular. Then $H.(\underline{x}, M) \cong H.(\underline{x}'', M \otimes R/(\underline{x}'))$.

Proof: (a) $K.(\underline{x}, M) = K.(\underline{x}) \otimes M \cong K.(\underline{x}') \otimes K.(x_n) \otimes M \cong K.(\underline{x}', M) \otimes K.(x_n)$. Apply (9.22).

(b) By induction on s we may assume that $s=1$. We may permute x_1 in the sequence $\underline{x}$ and $K.(\underline{x}, M) = K.(\underline{x}'', x_1, M) \cong K.(\underline{x}'', M) \otimes K.(x_1)$. Thus $H.(\underline{x}, M) = H.(K.(\underline{x}'', M) \otimes K.(x_1)) \cong H.(K.(\underline{x}'', M) \otimes R/(x_1))$ by (9.22)(c), since x_1 is a NZD on $K.(\underline{x}'', M) \cong \oplus M$. But $H.(K.(\underline{x}'', M) \otimes R/(x_1)) \cong H.(\underline{x}'', M \otimes_R R/(x_1))$.

(9.24) Corollary: Let $\underline{x} = x_1, \ldots, x_n \in R$ and M an R-module.

(a) If $\underline{x}$ is weakly M-regular, then $K.(\underline{x}, M)$ is acyclic.

(b) If $\underline{x}$ is R-regular, then $K.(\underline{x})$ is a free R-resolution of $R/(\underline{x})$.

Proof: Use (9.23)(b).

(9.25) Proposition: Let $\underline{x} = x_1, \ldots, x_n \in R$, $I = (\underline{x})$, $\underline{y} = y_1, \ldots, y_m \in I$, and M an R-module. If $\underline{y}$ is M-regular, then $H_i(\underline{x}, M) = 0$ for $i > n-m$ and $H_{n-m}(\underline{x}, M) \cong \operatorname{Hom}_R(R/I, M/(\underline{y})M) \cong \operatorname{Ext}_R^m(R/I, M)$.

Proof: The last isomorphism follows by (11.14), the rest will be shown by induction on m. If $m=0$ then $H_i(\underline{x}, M) = 0$ for $i > n$ and by (9.19) $H_n(\underline{x}, M) \cong 0 :_M I \cong \operatorname{Hom}_R(R/I, M)$. Let $m > 0$ and write $\overline{M} = M/y_1M$. The exact sequence $0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow \overline{M} \longrightarrow 0$ induces by (9.21) an exact sequence $0 \longrightarrow K.(\underline{x}, M) \xrightarrow{y_1} K.(\underline{x}, M) \longrightarrow K.(\underline{x}, \overline{M}) \longrightarrow 0$ and thus a long exact sequence of homology:

$$H_{i+1}(\underline{x},M) \xrightarrow{y_1} H_{i+1}(\underline{x},M) \longrightarrow H_{i+1}(\underline{x},\overline{M}) \longrightarrow H_i(\underline{x},M) \xrightarrow{y_1} H_i(\underline{x},M).$$

Since $y_1 \in I$ annihilates $H_\cdot(\underline{x},M)$ by (9.14), this long exact sequence breaks up into short exact sequences $0 \longrightarrow H_{i+1}(\underline{x},M) \longrightarrow H_{i+1}(\underline{x},\overline{M}) \longrightarrow H_i(\underline{x},M) \longrightarrow 0$. If $i > n-m$ then $i+1 > n-(m-1)$, hence $H_{i+1}(\underline{x},\overline{M}) = 0$ by induction hypothesis, and therefore $H_i(\underline{x},M) = 0$.
Since $H_{n-m+1}(\underline{x},M) = 0$, $H_{n-m+1}(\underline{x},\overline{M}) \cong H_{n-m}(\underline{x},M)$ and by induction hypothesis:
$H_{n-m+1}(\underline{x},\overline{M}) \cong \operatorname{Hom}_R(R/I, \overline{M}/(y_2,\dots,y_m)\overline{M}) \cong \operatorname{Hom}_R(R/I, M/(\underline{y})M)$.

(9.26) <u>Theorem</u>: Let R be a Noetherian ring, M a finite R-module, $\underline{x} = x_1,\dots,x_n \in R$, $I = (\underline{x})$, and $g = \operatorname{grade}(I,M) = \operatorname{depth}_I M$.

(a) $K_\cdot(\underline{x},M)$ is exact if and only if $M = IM$.

(b) If $K_\cdot(\underline{x},M)$ is not exact, then $\max\{i \mid H_i(\underline{x},M) \neq 0\} = n-g$.

<u>Proof</u>: (a) "$\Rightarrow$": clear since $M/IM = H_0(\underline{x},M)$. "$\Leftarrow$" By (9.14) $\operatorname{Supp}_R(H_\cdot(\underline{x},M)) \subseteq V(I) \cap \operatorname{Supp}(M) \overset{(*)}{=} \operatorname{Supp}(M/IM) = \emptyset$, where $(*)$ follows by Nakayama's lemma.
(b) If $K_\cdot(\underline{x},M)$ is not exact, by (a) $IM \neq M$ and $g < \infty$. By (9.25) $H_i(\underline{x},M) = 0$ for $i > n-g$ and $H_{n-g}(\underline{x},M) \cong \operatorname{Ext}^g_R(R/I,M)$ and $\operatorname{Ext}^g_R(R/I,M) \neq 0$ by (8.16).

(9.27) <u>Lemma</u>: Let (R,m) be a local Noetherian ring, M a finite R-module, and $\underline{x} = x_1,\dots,x_n \in m$. If $H_s(\underline{x},M) = 0$, then $H_i(x_1,\dots,x_j,M) = 0$ for all $i \geq s$, $j \leq n$.

<u>Proof</u>: By induction on n. The case $n=0$ is trivial. For $n>0$ write $\underline{x}' = x_1,\dots,x_{n-1}$ and consider the long exact sequence of homology from (9.23)(a): $\dots \longrightarrow H_i(\underline{x}',M) \longrightarrow H_i(\underline{x},M) \longrightarrow H_{i-1}(\underline{x}',M) \longrightarrow \dots \longrightarrow H_s(\underline{x}',M) \xrightarrow{\pm x_n} H_s(\underline{x}',M) \longrightarrow H_s(\underline{x},M) \longrightarrow \dots$. If $H_s(\underline{x},M) = 0$ then $H_s(\underline{x}',M) = x_n H_s(\underline{x}',M)$ and $H_s(\underline{x}',M) = 0$ by Nakayama since $x_n \in m$. By induction hypothesis $H_i(x_1,\dots,x_j,M) = 0$ for $i \geq s$, $j \leq n-1$. Since $H_i(\underline{x}',M) = 0$ for $i \geq s$, the above long exact sequence yields $H_i(\underline{x},M) = 0$ for $i \geq s$.

(9.28) <u>Corollary</u>: (Rigidity) Let R be a Noetherian ring, M a finite R-module, and

$\underline{x} = x_1, \ldots, x_n \in R$. If $H_s(\underline{x}, M) = 0$ then $H_i(\underline{x}, M) = 0$ for all $i \geq s$.

Proof: Let $I = (\underline{x})$. Since $\mathrm{Supp}_R(H_\cdot(\underline{x}, M)) \subseteq V(I)$, we may localize at $P \in V(I)$ to assume that (R, m) is local with $\underline{x} \subseteq m$. Use (9.27).

(9.29) Corollary: Let R be a Noetherian ring, M a finite R-module, $\underline{x} = x_1, \ldots, x_n \in R$, $I = (\underline{x})$, and assume that $IM \neq M$. The following are equivalent:

(a) $\underline{x}$ is M-quasi regular

(b) $K_\cdot(\underline{x}, M)$ is acyclic

(c) $H_1(\underline{x}, M) = 0$

(d) $\mathrm{grade}(I, M) = n$.

Proof: Home work

(9.30) Theorem: Let R be a local Noetherian ring and $M \neq 0$ a finite R-module of finite injective dimension. Then $\dim M \leq \mathrm{injdim}\, M = \mathrm{depth}\, R$.

Proof: $\dim M \leq \mathrm{injdim}\, M$. Let $d = \dim M$ and let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$ be a chain of primes in $\mathrm{Supp}(M)$. We want to show by induction on i that $\mu_i(P_i, M) \neq 0$. Then $\mu_d(P_d, M) \neq 0$ and $\mathrm{injdim}\, M \geq d$. For $i = 0$, $P_0R_{P_0}$ is minimal in $\mathrm{Supp}(M_{P_0})$. Thus $P_0R_{P_0}$ is an associated prime of M_{P_0}, hence $\mathrm{Hom}_{R_{P_0}}(k(P_0), M_{P_0}) \neq 0$ and $\mu_0(P_0, M) \neq 0$. Suppose $0 \leq i < d$ and $\mu_i(P_i, M) \neq 0$. If $\mu_{i+1}(P_{i+1}, M) = 0$ then $\mathrm{Ext}^{i+1}_{R_{P_{i+1}}}(k(P_{i+1}), M_{P_{i+1}}) = 0$ and by (11.11) $\mathrm{Ext}^{i}_{R_{P_{i+1}}}((R/P_i)_{P_{i+1}}, M_{P_{i+1}}) \neq 0$ since $\dim (R/P_i)_{P_{i+1}} = 1$. Localizing at $P_iR_{P_{i+1}}$ yields $\mu_i(P_i, M) = 0$, a contradiction.

$\mathrm{injdim}\, M = \mathrm{depth}\, R$: Let $t = \mathrm{depth}\, R$ and let $\underline{x} = x_1, \ldots, x_t$ be an R-regular sequence. By (9.24) $K_\cdot(\underline{x})$ is a free R-resolution of $N = R/(\underline{x})$ of length t. Thus $\mathrm{Ext}^t_R(N, M) = H^t(\mathrm{Hom}_R(K_\cdot(\underline{x}), M)) = H^t(\underline{x}, M) \cong H_0(\underline{x}, M) \cong M/(\underline{x})M \neq 0$ by (9.20) and (9.19). Hence $t = \sup\{n \mid \mathrm{Ext}^n_R(N, M) \neq 0\}$. Since $\mathrm{depth}\, N = 0$, by (11.13) $t = \mathrm{injdim}\, M$.