

CHAPTER VIII: COHEN - MACAULAY RINGS AND MODULES

§1: REGULAR SEQUENCES

(8.1) Definition: Let R be a ring and M an R -module.

(a) An element $a \in R$ is called M -regular if $am \neq 0$ for all $m \in M - (0)$.

(b) A sequence of elements $a_1, \dots, a_n \in R$ is called an M -sequence if the following conditions are satisfied:

(i) a_1 is M -regular and for all $2 \leq i \leq n$ the element a_i is $(M/\sum_{j=1}^{i-1} a_j M)$ -regular.

(ii) $M \neq \sum_{i=1}^n a_i M$.

(8.2) Remark: A permutation of an M -sequence may not be an M -sequence (Homework).

(8.3) Theorem: Let R be a ring, M an R -module and $a_1, \dots, a_n \in R$ an M -sequence. For all $v_i \in \mathbb{N} - (0)$, where $1 \leq i \leq n$, the sequence $a_1^{v_1}, \dots, a_n^{v_n}$ is an M -sequence.

Proof: It suffices to show: if a_1, \dots, a_n is an M -sequence and $v \in \mathbb{N} - (0)$ then a_1^v, a_2, \dots, a_n is an M -sequence. Condition (ii) of definition (8.1) is obvious. The proof of condition (i) is by induction on v . We first show:

Claim: Let $b_1, \dots, b_n \in R$ be an M -sequence and $m_1, \dots, m_n \in M$ with $b_1 m_1 + \dots + b_n m_n = 0$.

Then for all $1 \leq i \leq n$: $m_i \in \sum_{j=1}^n b_j M$.

Pf of cl.: by induction on n . Since b_n is $(M/\sum_{j=1}^{n-1} b_j M)$ -regular and $b_n m_n = 0$ in $M/\sum_{j=1}^{n-1} b_j M$ there are elements $l_j \in M$ so that $m_n = \sum_{j=1}^{n-1} b_j l_j$. Thus

$$\sum_{j=1}^n b_j m_j = \sum_{j=1}^{n-1} b_j m_j + b_n \sum_{j=1}^{n-1} b_j l_j = \sum_{j=1}^{n-1} b_j (m_j + b_n l_j) = 0.$$

By induction hypothesis for all $1 \leq i \leq n-1$: $m_i + b_n l_i \in \sum_{j=1}^{n-1} b_j M$. This proves the claim.

In order to prove the theorem we show by induction on $v \in \mathbb{N} - (0)$ that

a_1^v, a_2, \dots, a_n is an M -sequence. For $v=1$ there is nothing to show. If $v > 1$, note that a_1^v is M -regular and suppose that for some $2 \leq i \leq n$ and some $w \in M$:

$$a_i w = a_i v_1 m_1 + \dots + a_{i-1} m_{i-1} \quad \text{where } m_j \in M.$$

By induction hypothesis $a_1^{v-1}, a_2, \dots, a_n$ is an M -sequence and there are $n_j \in M$ so that

$$w = a_1^{v-1} n_1 + \dots + a_{i-1} n_{i-1}. \quad \text{Thus}$$

$$\begin{aligned} a_i v_1 m_1 + \dots + a_{i-1} m_{i-1} - a_i w &= a_i v_1 m_1 + \dots + a_{i-1} m_{i-1} - a_i (a_1^{v-1} n_1 + \dots + a_{i-1} n_{i-1}) \\ &= a_i^{v-1} (a_1 m_1 - a_i n_1) + a_2 (m_2 - a_2 n_2) + \dots + a_{i-1} (m_{i-1} - a_i n_{i-1}) = 0 \end{aligned}$$

By the claim: $a_1 m_1 - a_i n_1 \in a_1^{v-1} M + \sum_{j=2}^{i-1} a_j M$ and therefore $a_i n_1 \in \sum_{j=1}^{i-1} a_j M$.

Since a_1, \dots, a_n is M -regular, $n_1 \in \sum_{j=1}^{i-1} a_j M$ implying that $w \in a_1^v M + \sum_{j=2}^{i-1} a_j M$.

In the following we want to show: If R is a local Noetherian ring, M a finite R -module, and $a_1, \dots, a_n \in R$ an M -sequence then for all $\sigma \in S_n$ $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ is an M -sequence.

(8.4) Remark and Definition: Let R be a ring, x_1, \dots, x_n variables over R , and M an R -module.

We consider elements of $M \otimes_R R[x_1, \dots, x_n]$ as 'polynomials' in the x_i with coefficients in M

and write $\sum_{|\alpha| \leq t} m_{(\alpha)} \otimes x_1^{\alpha_1} \dots x_n^{\alpha_n} = \sum_{|\alpha| \leq t} m_{(\alpha)} x_1^{\alpha_1} \dots x_n^{\alpha_n} = F(x_1, \dots, x_n)$ where $m_{(\alpha)} \in M$. We set $M[x_1, \dots, x_n] = M \otimes_R R[x_1, \dots, x_n]$. Obviously, $M[x_1, \dots, x_n]$ is an $R[x_1, \dots, x_n]$ -module and for all $b_1, \dots, b_n \in R$ there is an R -linear map $\varphi: M[x_1, \dots, x_n] \rightarrow M$ with $\varphi(F[x_1, \dots, x_n]) = F(b_1, \dots, b_n)$.

(8.5) Definition: Let R be a ring and $a_1, \dots, a_n \in R$. Set $I = (a_1, \dots, a_n)$ and let M be an R -module with $IM \neq M$. The sequence a_1, \dots, a_n is called M -quasiregular if for all $v \in \mathbb{N} - \{0\}$ the following condition (*) is satisfied:

(*) If $F(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree v with $F(a_1, \dots, a_n) \in I^{v+1} M$, then all coefficients of F are in IM .

(8.6) Remark: Condition (*) is independent of the order of the sequence a_1, \dots, a_n .

(8.7) Lemma: For all $v \in \mathbb{N} - \{0\}$ condition (*) is equivalent to:

(**) If $F(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree v with

$F(a_1, \dots, a_n) = 0$ then the coefficients of $F(x_1, \dots, x_n)$ are in IM .

Proof: Obviously, (*) implies (**). Conversely, let $F(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ be a homogeneous polynomial of degree ν with $F(a_1, \dots, a_n) \in I^{\nu+1}M$. Then there is a homogeneous polynomial $G(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ of degree $\nu+1$ with $F(a_1, \dots, a_n) = G(a_1, \dots, a_n)$. Write $G(x_1, \dots, x_n) = \sum_{i=1}^n x_i G_i(x_1, \dots, x_n)$ where $G_i(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ a homogeneous polynomial of degree ν for $1 \leq i \leq n$. Then $F^*(x_1, \dots, x_n) = F(x_1, \dots, x_n) - \sum_{i=1}^n a_i G_i(x_1, \dots, x_n)$ is a homogeneous polynomial of degree ν with $F^*(a_1, \dots, a_n) = 0$. By (**) the coefficients of F^* are in IM , hence the coefficients of F are in IM .

(8.8) Lemma: Let R be a ring, M an R -module and $a_1, \dots, a_n \in R$ an M -quasiregular sequence. Set $I = (a_1, \dots, a_n) \subseteq R$ and let $a \in R$ be such that $IM : a = IM$. Then for all $\nu \in \mathbb{N} - \{0\}$: $I^\nu M : a = I^\nu M$.

Proof: by induction on ν : For the induction step assume that $I^{\nu-1}M : a = I^{\nu-1}M$ and let $m \in M$ be so that $am \in I^\nu M$. Then $m \in I^{\nu-1}M$ and we may write $m = F(a_1, \dots, a_n)$ where $F(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree $\nu-1$. Since $am = aF(a_1, \dots, a_n) \in I^\nu M$, the homogeneous polynomial $aF(x_1, \dots, x_n)$ has coefficients in IM by (*). Write $F(x_1, \dots, x_n) = \sum_{|i|=\nu-1} m_{(i)} x_1^{i_1} \dots x_n^{i_n}$ with $m_{(i)} \in M$. Then $am_{(i)} \in IM$ and by assumption $m_{(i)} \in IM$. But then $m = F(a_1, \dots, a_n) \in I^\nu M$.

(8.9) Theorem: Let R be a ring, M an R -module and $a_1, \dots, a_n \in R$ an M -sequence. Then a_1, \dots, a_n is M -quasiregular.

Proof: by induction on n : For $n=1$ let $F(x) \in M[x]$ be a homogeneous polynomial of degree ν . Then $F(x) = mx^\nu$ for some $m \in M$. Suppose that $F(a) = ma^\nu \in a^{\nu+1}M$. Since a is M -regular, $m = a\alpha$ for some $\alpha \in M$.

$n-1 \Rightarrow n$: Suppose that a_1, \dots, a_{n-1} is an M -quasiregular sequence. Let $F(x_1, \dots, x_n) \in$

$M[x_1, \dots, x_n]$ be a homogeneous polynomial of degree v with $F(a_1, \dots, a_n) = 0$. We want to show by induction on v that the coefficients of $F(x_1, \dots, x_n)$ are in IM . For $v=0$ $F(x_1, \dots, x_n) = m \in M$ and $m = F(a_1, \dots, a_n) = 0$. For the step $v-1 \Rightarrow v$ note that Lemma (8.7) shows that $(x) \Leftrightarrow (x^*)$ for a fixed $v \in \mathbb{N} - \{0\}$. Thus by induction hypothesis, if $H(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree $v-1$ with $H(a_1, \dots, a_n) \in I^v M$, then the coefficients of H are in IM . Write $F(x_1, \dots, x_n) = G(x_1, \dots, x_{n-1}) + x_n H(x_1, \dots, x_n)$ where $G(x_1, \dots, x_{n-1}) \in M[x_1, \dots, x_{n-1}]$ is a homogeneous polynomial of degree v and $H(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is homogeneous of degree $v-1$. Since $F(a_1, \dots, a_n) = 0$, $H(a_1, \dots, a_n) \in (a_1, \dots, a_{n-1})^v M : a_n$. Since $a_1, \dots, a_n \in M$ is M -regular, $(a_1, \dots, a_{n-1})M : a_n = (a_1, \dots, a_{n-1})M$ and thus by Lemma (8.8) $H(a_1, \dots, a_n) \in (a_1, \dots, a_{n-1})^v M$. Since $H(x_1, \dots, x_n)$ is homogeneous of degree $v-1$, by induction hypothesis the coefficients of $H(x_1, \dots, x_n)$ are in IM . Using $H(a_1, \dots, a_n) \in (a_1, \dots, a_{n-1})^v M$ there is a homogeneous polynomial $h(x_1, \dots, x_{n-1}) \in M[x_1, \dots, x_{n-1}]$ of degree v with $h(a_1, \dots, a_{n-1}) = H(a_1, \dots, a_n)$. Consider the homogeneous polynomial in $M[x_1, \dots, x_n]$ of degree v : $g(x_1, \dots, x_{n-1}) = G(x_1, \dots, x_{n-1}) + a_n h(x_1, \dots, x_{n-1})$. Then $g(a_1, \dots, a_{n-1}) = F(a_1, \dots, a_n) = 0$. By induction hypothesis on n the sequence a_1, \dots, a_{n-1} is M -quasiregular. Thus the coefficients of g are in IM and so are the coefficients of G and F .

Recall: If R is a Noetherian ring, $I \subseteq \text{Jac}(R)$ an ideal in the Jacobson radical of R , and M a finite R -module, then $\bigcap_{n \in \mathbb{N}} I^n M = 0$.

(8.10) Theorem: Let R be a Noetherian ring, M a finite R -module and $a_1, \dots, a_n \in R$ with $I = (a_1, \dots, a_n) \subseteq \text{Jac}(R)$. The following are equivalent:

- (a) a_1, \dots, a_n is M -regular
- (b) a_1, \dots, a_n is M -quasiregular.

Proof: By (8.9) (a) \Rightarrow (b). For (b) \Rightarrow (a) we will show: (i) a_1 is M -regular and (ii) the sequence a_2, \dots, a_n is $M/a_1 M$ -quasiregular. Then the statement follows by induction on n .

(i) Suppose that $a_1 m = 0$ for some $m \in M$. The polynomial $F = mx_1 \in M[x_1, \dots, x_n]$ is

homogeneous of degree 1 with $F(a_1) = F(a_1, \dots, a_n) = a_1 m = 0$. By (**), $m \in IM$, that is, $m = \sum_{i=1}^n a_i w_i$ for some $w_i \in M$. The polynomial $G(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_i$ is homogeneous of degree 2 with $G(a_1, \dots, a_n) = 0$. Again by (**), $w_i \in IM$ and thus $m \in I^2 M$. Continuing like this yields that $m \in \bigcap_{n \in \mathbb{N}} I^n M = (0)$.

(ii) Set $\bar{M} = M/a_1 M$ and let $f(x_2, \dots, x_n) \in \bar{M}[x_2, \dots, x_n]$ be a homogeneous polynomial of degree ν with $f(a_2, \dots, a_n) = 0$. Let $F(x_2, \dots, x_n) \in M[x_2, \dots, x_n]$ be a homogeneous polynomial of degree ν with $F + a_1 M[x_2, \dots, x_n] = f$. Then $F(a_2, \dots, a_n) \in a_1 M$, say $F(a_2, \dots, a_n) = a_1 w$ for some $w \in M$. Let $i \in \mathbb{N}$ with $i+1 \leq \nu$ be maximal with $w \in I^i M$. Then there is a homogeneous polynomial $G(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ of degree i with $G(a_1, \dots, a_n) = w$ and $F(a_2, \dots, a_n) = a_1 G(a_1, \dots, a_n) \in I^{i+1} M$ (since F is of degree ν). The polynomial $H(x_1, \dots, x_n) = x_1 G(x_1, \dots, x_n)$ is homogeneous of degree $i+1 \leq \nu$.

Case 1: $i+1 < \nu$: Since $H(a_1, \dots, a_n) = a_1 G(a_1, \dots, a_n) \in I^{i+1} M$, by (*) the coefficients of H and G are in IM . This implies that $w = G(a_1, \dots, a_n) \in I^{i+1} M$, a contradiction.

Case 2: $i+1 = \nu$: $F^* = F(x_2, \dots, x_n) - H(x_1, \dots, x_n) \in M[x_1, \dots, x_n]$ is a homogeneous polynomial of degree ν with $F^*(a_1, \dots, a_n) = 0$. By (**), the coefficients of F^* are in IM . Since the monomial terms of $F(x_2, \dots, x_n)$ do not involve x_1 , it follows that the coefficients of F are in IM and the coefficients of f are in $I\bar{M} = (a_2, \dots, a_n)\bar{M}$.

(8.11) Definition: Let R be a ring, M an R -module and $I \subseteq R$ an ideal. A sequence of elements $a_1, \dots, a_n \in I$ is called a maximal M -sequence in I if

- a_1, \dots, a_n is an M -sequence
- for all $b \in I$ the sequence a_1, \dots, a_n, b is not an M -sequence.

(8.12) Corollary: Let (R, \mathfrak{m}) be a local Noetherian ring and M a finite R -module. If $a_1, \dots, a_n \in \mathfrak{m}$ is an M -sequence so is the sequence $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ for all $\sigma \in S_n$.

(8.13) Remarks: Let R be a Noetherian ring and M an R -module. For every ideal $I \subseteq R$ maximal M -sequences in I exist (possibly of length 0).

Consider the exact sequence $0 \rightarrow M \xrightarrow{f_1} M \rightarrow M/f_1 M \rightarrow 0$. By left exactness the sequence $0 \rightarrow \text{Hom}_R(N, M) \xrightarrow{f_{1*}} \text{Hom}_R(N, M)$ is exact and f_{1*} is multiplication by f_1 . Since $f_1 \in I \subseteq \text{rad}(\text{ann}_R(N))$ and N finite, $f_1^r N = 0$ for some $r \in \mathbb{N}$. Thus $f_1^r \text{Hom}_R(N, M) = 0$ and, since f_{1*} is injective, $\text{Hom}_R(N, M) = 0$. By induction hypothesis $\text{Ext}_R^i(N, M_i) = 0$ for all $i < n-1$. This implies that for all $i < n$ the sequence $0 \rightarrow \text{Ext}_R^i(N, M) \xrightarrow{f_1} \text{Ext}_R^i(N, M)$ is exact, where $*$ is multiplication by f_1 . Compute $\text{Ext}_R^i(N, M)$ by using an injective resolution of M : $0 \rightarrow M \rightarrow Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_s \rightarrow \dots$. Since $f_1^r N = 0$ for some $r \in \mathbb{N}$, it follows that $f_1^r \text{Hom}_R(N, Q_j) = 0$ and therefore $f_1^r \text{Ext}_R^i(N, M) = 0$. Since $*$ is injective, $\text{Ext}_R^i(N, M) = 0$ for $i < n$.

(8.15) Corollary: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module with $IM \neq M$. If $a_1, \dots, a_n \in I$ is a maximal M -sequence in I , then $\text{Ext}_R^n(R/I, M) \neq 0$.

Proof: First note that by (8.14) $\text{Hom}_R(R/I, M) \neq 0$ if $n=0$. For $0 < i \leq n$ set $M_i = M/(a_1, \dots, a_i)M$. We want to show by induction on $n-i$ that $\text{Ext}_R^{n-i}(R/I, M_i) \neq 0$. Suppose that $\text{Ext}_R^{n-i-1}(R/I, M_{i+1}) \neq 0$ and consider the exact sequence $0 \rightarrow M_i \xrightarrow{a_{i+1}} M_i \rightarrow M_{i+1} \rightarrow 0$. This yields a long exact sequence: $\dots \rightarrow \text{Ext}_R^{n-i-1}(R/I, M_i) \rightarrow \text{Ext}_R^{n-i-1}(R/I, M_{i+1}) \rightarrow \text{Ext}_R^{n-i}(R/I, M_i) \xrightarrow{a_{i+1}*} \text{Ext}_R^{n-i}(R/I, M_i) \rightarrow \dots$. Note that by (8.14) $\text{Ext}_R^{n-i-1}(R/I, M_i) = 0$ and that $*$ is the zero map. Thus $\text{Ext}_R^{n-i-1}(R/I, M_{i+1}) \cong \text{Ext}_R^{n-i}(R/I, M_i)$ and the statement follows.

(8.16) Theorem: Let R, I , and M be as in (8.15). The length of a maximal M -sequence in I is independent of the choice of the sequence. In particular, the length n of a maximal M -sequence in I is given by:

$$\text{Ext}_R^i(R/I, M) = \begin{cases} 0 & \text{if } i < n \\ \neq 0 & \text{if } i = n. \end{cases}$$

§ 2: DEPTH

(8.17) Definition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module with $IM \neq M$.

(a) $\text{depth}_I M = I$ -depth of $M = \text{grade}(I, M) = \text{grade of } I \text{ on } M = \text{maximal length of an } M\text{-sequence in } I$. If $IM = M$ we set $\text{depth}_I M = \infty$.

(b) $\text{grade } I = \text{grade}(I, R)$

(c) $\text{grade } M = \text{grade ann}(M)$

(d) If (R, \mathfrak{m}) is local, then $\text{depth } M = \text{depth}_{\mathfrak{m}} M$.

(8.18) Proposition: Let R be a Noetherian ring, $I \subseteq R$ an ideal and M a finite R -module.

(a) $\text{depth}_I M = \min \{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}$

(b) $\text{grade } M = \min \{i \mid \text{Ext}_R^i(M, R) \neq 0\}$.

Proof: (a) If $IM \neq M$, the assertion follows by (8.16). If $IM = M$, then $\text{Ext}_R^i(R/I, M) = 0$ if and only if $\text{Ext}_R^i(R/I, M)_P = \text{Ext}_{R_P}^i(R_P/I_P, M_P) = 0$ for all $P \in \text{Spec } R$. If $I \subseteq P$, then $I_P M_P = M_P$ and by Nakayama $M_P = 0$, if $I \not\subseteq P$, then $I_P = R_P$. Thus $\text{Ext}_R^i(R/I, M) = 0$ for all i .

(b) By (8.14) $\text{grade } M = \text{depth}_{\text{ann}(M)} R = \min \{i \mid \text{Ext}_R^i(M, R) \neq 0\}$.

(8.19) Corollary: Let R be a local Noetherian ring with residue field k and M a finite R -module.

Then $\text{depth } M = \min \{i \mid \text{Ext}_R^i(k, M) \neq 0\}$.

(8.20) Theorem: Let (R, \mathfrak{m}, k) be a local Noetherian ring, M and N nonzero finite R -modules.

Then $\text{Ext}_R^i(N, M) = 0$ for all $i < \text{depth } M - \dim N$.

Proof: by induction on $r = \dim N$. If $r = 0$, then $\text{Supp}_R(N) = \{\mathfrak{m}\}$, since $N \neq 0$. The assertion follows by (8.14). If $r > 0$ consider a filtration of N : $N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_r = 0$ with $N_j/N_{j+1} \cong R/P_j$ for some $P_j \in \text{Spec } R$.

Claim: Set $s = \text{depth } M$ and suppose that $(*) \text{Ext}_R^i(N_j/N_{j+1}, M) = 0$ for all $i < s - r$ and all

$j=0, \dots, n-1$. Then $\text{Ext}_R^i(N, M) = 0$ for all $i < s-r$.

Pf of Cl.: We show by decreasing induction on j that $(*)$ implies $\text{Ext}_R^i(N_j, M) = 0$ for all $i < s-r$.

By assumption $(*)$ for $j=n-1$: $\text{Ext}_R^i(N_{n-1}, M) = \text{Ext}_R^i(N_{n-1}/N_n, M) = 0$ for all $i < s-r$. For the induction step $j \Rightarrow j-1$ consider the exact sequence $0 \rightarrow N_j \rightarrow N_{j-1} \rightarrow N_{j-1}/N_j \rightarrow 0$, which yields a long exact sequence: $\dots \rightarrow \text{Ext}_R^i(N_{j-1}/N_j, M) \rightarrow \text{Ext}_R^i(N_{j-1}, M) \rightarrow \text{Ext}_R^i(N_j, M) \rightarrow \dots$

By $(*)$ $\text{Ext}_R^i(N_{j-1}/N_j, M) = 0$ and by induction hypothesis $\text{Ext}_R^i(N_j, M) = 0$.

Suppose now that the statement holds for all finite R -modules of dimension $\leq r-1$ and let N be a finite R -module of dimension r . Since $\dim(N_{j-1}/N_j) \leq \dim N = r$, by the claim it remains to show that $\text{Ext}_R^i(R/P, M) = 0$ for all $i < s-r$ and all $P \in \text{Spec } R$ with $\dim R/P = r$.

For such a P let $x \in \mathfrak{m} - P$ and consider the exact sequence $0 \rightarrow R/P \xrightarrow{x} R/P \rightarrow R/P + xR \rightarrow 0$.

Since $\dim(R/P + xR) \leq r-1$, by induction hypothesis $\text{Ext}_R^i(R/P + xR, M) = 0$ for all $i < s-r+1$. For $i < s-r$ consider the long exact sequence:

$$\dots \rightarrow 0 = \text{Ext}_R^i(R/P + xR, M) \rightarrow \text{Ext}_R^i(R/P, M) \xrightarrow{x} \text{Ext}_R^i(R/P, M) \rightarrow \text{Ext}_R^{i+1}(R/P + xR, M) = 0 \rightarrow \dots$$

Thus multiplication by x is an isomorphism and $\text{Ext}_R^i(R/P, M) \cong x \text{Ext}_R^i(R/P, M)$. Since $\text{Ext}_R^i(R/P, M)$ is a finite R -module, by Nakayama $\text{Ext}_R^i(R/P, M) = 0$.

(8.21) Corollary: Let R be a local Noetherian ring, M a finite R -module, and $P \in \text{Ass}_R(M)$. Then $\dim(R/P) \geq \text{depth } M$.

Proof: Suppose that $P \in \text{Ass}_R(M)$ with $\dim(R/P) < \text{depth } M$. By (8.20) $\text{Hom}_R(R/P, M) = 0$, a contradiction.

(8.22) Proposition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ an exact sequence of finite R -modules. Then

- $\text{depth}_I M \geq \min \{ \text{depth}_I M', \text{depth}_I M'' \}$
- $\text{depth}_I M' \geq \min \{ \text{depth}_I M, \text{depth}_I M'' + 1 \}$
- $\text{depth}_I M'' \geq \min \{ \text{depth}_I M, \text{depth}_I M' - 1 \}$.

Proof: (8.18) and the long exact sequence for $\text{Ext}_R^i(R/I, -)$.

(8.23) Theorem: (Auslander-Buchsbaum formula) Let R be a local Noetherian ring and $M \neq 0$ a finite R -module with $\text{projdim } M < \infty$. Then:

$$\text{projdim } M + \text{depth } M = \text{depth } R.$$

Proof: by induction on $\text{projdim } M$. If $\text{projdim } M = 0$, then M is projective, hence free and $M \cong R^n$.

Thus $\text{depth } M = \text{depth } R$ by (8.18). If $\text{projdim } M = 1$, then there is an exact sequence

$$0 \rightarrow R^s \xrightarrow{\varphi} R^n \rightarrow M \rightarrow 0 \text{ with } s \geq 1 \text{ and all entries of } \varphi \text{ are in } \mathfrak{m}, \text{ the maximal ideal of } R.$$

$$\text{The map } \bar{\varphi}_i = \text{Ext}_R^i(k, \varphi): \begin{array}{ccc} \text{Ext}_R^i(k, R^s) & \longrightarrow & \text{Ext}_R^i(k, R^n) \\ \text{Ext}_R^i(k, R) \otimes_R R^s & & \text{Ext}_R^i(k, R) \otimes_R R^n \end{array}$$

is simply $\text{id}_{\text{Ext}_R^i(k, R)} \otimes \varphi$, which is zero since all entries of φ are in \mathfrak{m} and $\mathfrak{m} \text{Ext}_R^i(k, R) = 0$.

By the long exact sequence for $\text{Ext}_R^*(k, -)$ and (8.18) we obtain that $\text{depth } M = \text{depth } R - 1$.

If $\text{projdim } M \geq 2$, there is an exact sequence $0 \rightarrow N \rightarrow R^n \rightarrow M \rightarrow 0$ with $\text{projdim } N = \text{projdim } M - 1$. By induction hypothesis $\text{depth } N = \text{depth } R - \text{projdim } N < \text{depth } R$. Thus by (8.22) $\text{depth } M = \text{depth } N - 1$, and we are done.

(8.24) Remark: If R is a Noetherian ring and M a finite R -module then by (8.18) $\text{grade } M \leq \text{projdim } M$.

(8.25) Definition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module.

(a) M is called perfect if $\text{grade } M = \text{projdim } M$ ($\Rightarrow M \neq 0$ and $\text{projdim } M < \infty$).

(b) I is called perfect if R/I is perfect as an R -module.

(8.26) Lemma: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module, $M \neq 0$.

If I consists of zero divisors of M then $I \subseteq \mathfrak{P}$ for some $\mathfrak{P} \in \text{Ass}_R(M)$.

Proof: Since $\bigcup_{\mathfrak{P} \in \text{Ass}(M)} \mathfrak{P}$ is the set of zero divisors of M , $I \subseteq \bigcup_{\mathfrak{P} \in \text{Ass}(M)} \mathfrak{P}$. The assertion follows since $\text{Ass}_R(M)$ is finite.

(8.27) Proposition: Let R be a Noetherian ring, $I, J \in R$ ideals, and M a finite R -module.

(a) $\text{depth}_I M = \min \{ \text{depth}_{M_P} \mid P \in V(I) \}$

(b) $\text{depth}_I M = \text{depth}_{\text{rad}(I)} M$

(c) $\text{depth}_{I \cap J} M = \min \{ \text{depth}_I M, \text{depth}_J M \}$

(d) If $\underline{a} = a_1, \dots, a_n$ is an M -sequence in I , then $\text{depth}_{I/(\underline{a})} M/(\underline{a})M = \text{depth}_I M/(\underline{a})M = \text{depth}_I M - n$.

Proof: (a) By Nakayama's Lemma $IM = M$ if and only if $M_P = 0$ for all $P \in V(I)$. Thus we may assume $IM \neq M$. ' \leq ' is clear by (8.14). In order to prove ' \geq ' let a_1, \dots, a_n be an M -regular sequence of maximal length in I . By (8.26) $I \subseteq P$ for some $P \in \text{Ass}_R(M/(\underline{a})M)$. Then $PR_P \in \text{Ass}_{R_P}(M_P/(\underline{a})M_P)$ and a_1, \dots, a_n form a maximal M_P -regular sequence contained in PR_P . Therefore $\text{depth}_{M_P} = n = \text{depth}_I M$.

(b) and (c) follow from (a) since $V(I) = V(\text{rad}(I))$ and $V(I \cap J) = V(I) \cup V(J)$.

(d) follows from (8.16).

(8.28) Definition: Let R be a local Noetherian ring and M a finite R -module.

(a) M is called Cohen-Macaulay (CM) if $M=0$ or if $\text{depth } M = \dim M$.

(b) M is called maximal Cohen-Macaulay (MCM) if $\text{depth } M = \dim R$ ($\Leftrightarrow M \neq 0$ is CM with $\dim M = \dim R$).

(c) R is called a Cohen-Macaulay ring if R is CM as module over itself.

(8.29) Theorem: Let R be a local Noetherian ring and M a finite R -module.

(a) If M is a CM-module, then for all $P \in \text{Ass}_R(M)$: $\dim R/P = \dim M = \text{depth } M$. In particular M has no embedded prime ideals.

(b) Let $a_1, \dots, a_n \in \mathfrak{m}$ be an M -sequence and $M' = M/(\underline{a})M$. Then M is CM if and only if M' is CM.

(c) If M is CM and $P \in \text{Spec } R$, then the R_P -module M_P is CM. In particular, if $M_P \neq 0$, then $\text{depth}_P M = \text{depth}_{R_P}(M_P)$.

Proof: (a) $\dim M = \sup \{ \dim (R/P) \mid P \in \text{Ass}(M) \} \geq \inf \{ \dim (R/P) \mid P \in \text{Ass}_R(M) \} \geq (8.21) \text{depth } M$.

(b) Since all maximal M -sequences have the same length $\text{depth } M' = \text{depth } M - n$. By (4.48) we know if $a \in \mathfrak{m}$ and $a \notin P$ for all minimal primes $P \in \text{Supp}_R(M)$ then $\dim M/aM \leq \dim M - 1$.

This implies that $\dim M' \leq \dim M - n$ and hence $\dim M' \leq \text{depth } M'$. By (8.21) $\dim M' \geq \text{depth } M'$.

(c) Let $P \in \text{Spec } R$ with $M_P \neq 0$. Then $\text{ann}_R(M) \subseteq P$ and $\dim M_P \geq \text{depth } M_P \geq \text{depth}_P M$. We show by induction on $\text{depth}_P M$ that $\dim M_P = \text{depth}_P M$. If $\text{depth}_P M = 0$, then P is contained in the set of zero divisors of M and there is a $Q \in \text{Ass}_R(M)$ with $P \subseteq Q$ (8.26).

Since $M_P \neq 0$, $P \in \text{Supp}_R(M)$ and there is a prime $Q' \in \text{Ass}_R(M)$ with $Q' \subseteq P$. M is CM and by (a) M has no embedded prime ideals. Thus $Q' = P = Q \in \text{Ass}_R(M)$ and P is minimal in $\text{Supp}_R(M)$. Hence $\dim M_P = 0$.

If $\text{depth}_P M \neq 0$, let $a \in P$ be an M -regular element and put $M' = M/aM$. By (b) M' is a CM-module with $\text{depth}_P M' = \text{depth}_P M - 1$. Note that $M'_P \neq 0$. By induction hypothesis $\dim M'_P = \text{depth}_P M' = \text{depth}_P M - 1$ and by (4.48) $\dim M'_P = \dim M_P - 1$, since a is regular on M_P . Thus $\dim M_P = \text{depth}_P M$.

Recall: A Noetherian ring R is called catenary if for all prime ideals $P, Q \in R$ with $P \subseteq Q$ all saturated chains of prime ideals $P \subsetneq P_1 \subsetneq \dots \subsetneq P_s \subsetneq Q$ have the same length.

(8.30) Definition: Let (R, \mathfrak{m}) be a local Noetherian ring with $\dim R = n$. Elements $a_1, \dots, a_n \in \mathfrak{m}$ are called a system of parameters (SOP) if $I = (a_1, \dots, a_n)$ is an ideal of definition of R , that is, if $\text{rad}(I) = \mathfrak{m}$. Elements $b_1, \dots, b_r \in \mathfrak{m}$ are called part of a system of parameters of R if there are elements $b_{r+1}, \dots, b_n \in \mathfrak{m}$ so that b_1, \dots, b_n is a SOP.

Note that in a local Noetherian ring systems of parameters always exist.

(8.31) Theorem: Let (R, \mathfrak{m}) be a local Noetherian CM-ring.

(a) For elements $a_1, \dots, a_r \in \mathfrak{m}$ the following are equivalent:

(i) a_1, \dots, a_r is a regular sequence.

(ii) $\text{ht}(a_1, \dots, a_i) = i$ for all $1 \leq i \leq r$

(iii) $\text{ht}(a_1, \dots, a_r) = r$

(iv) a_1, \dots, a_r is part of a system of parameters of R .

(b) For an ideal $I \subseteq R$ the following holds:

(i) $\text{ht} I = \text{depth}_I R$

(ii) $\text{ht} I + \dim R/I = \dim R$

(c) R is catenary.

Proof: (a) (i) \Rightarrow (ii): Since a_1, \dots, a_r is regular, a_{i+1} is regular on $R/(a_1, \dots, a_i)$ and $a_{i+1} \notin P$ for all $P \in \text{Ass}_R(R/(a_1, \dots, a_i))$. In particular, a_{i+1} is not contained in any minimal prime ideal of (a_1, \dots, a_i) and $\text{ht}(a_1, \dots, a_{i+1}) \geq \text{ht}(a_1, \dots, a_i) + 1$. This implies that $\text{ht}(a_1, \dots, a_i) \geq i$. By Krull's generalized principal ideal theorem $\text{ht}(a_1, \dots, a_i) \leq i$, hence $\text{ht}(a_1, \dots, a_i) = i$ for all $1 \leq i \leq r$.

(ii) \Rightarrow (iii): trivial

(iii) \Rightarrow (iv): If $\dim R = r$, we are done. Suppose that $\dim R \neq r$. Since (a_1, \dots, a_r) is generated by r elements, by Krull's generalized principal ideal theorem the maximal ideal \mathfrak{m} is not a minimal prime ideal over (a_1, \dots, a_r) . Pick an element $a_{r+1} \in \mathfrak{m}$ which is not contained in any minimal prime ideal over (a_1, \dots, a_r) . Then $\text{ht}(a_1, \dots, a_{r+1}) = r+1$. Continue like this.

(iv) \Rightarrow (i): It suffices to show that any SOP of R is a regular sequence. The proof is by induction on $n = \dim R$. Let a_1, \dots, a_n be an SOP of R . Since R is CM, by (8.29) every prime ideal $P \in \text{Ass}(R)$ is a minimal prime ideal of R and $\dim R/P = \dim R$ for all $P \in \text{Ass}(R)$. Since $a_1 + P, \dots, a_n + P$ is a SOP of R/P for all $P \in \text{Ass}(R)$, $a_i \notin P$ for all $P \in \text{Ass}(R)$ and a_1 is regular on R . Set $R' = R/a_1R$. By (8.29) R' is a local Noetherian CM-ring with $\dim R' = n-1$. Moreover, $a_2 + a_1R, \dots, a_n + a_1R$ is a SOP of R' . By induction hypothesis, $a_2 + a_1R, \dots, a_n + a_1R$ is a regular sequence of R' .

(b) (i) Suppose $\text{ht} I = r$. Then there are elements $a_1, \dots, a_r \in I$ with $\text{ht}(a_1, \dots, a_r) = r$. By (a) the sequence a_1, \dots, a_r is regular and $\text{depth}_I R \geq \text{ht} I = r$. Let $P \subseteq R$ be a prime ideal with $I \subseteq P$ and $\text{ht} P = r$. By (8.29) R_P is a CM-ring and $\text{depth}_P R = \text{depth} R_P = \dim R_P = r$. Since $\text{depth}_I R \leq \text{depth}_P R$ the statement follows.

(ii) Let $S = \{P \in \text{Spec } R \mid I \subseteq P \text{ minimal}\}$ be the set of all prime ideals which are minimal over I .

By definition: $\text{ht } I = \inf \{\text{ht } P \mid P \in S\}$ and $\dim R/I = \sup \{\dim R/P \mid P \in S\}$.

Claim: For all $P \in \text{Spec } R$, $\dim R/P + \text{ht } P = \dim R$.

Pf of cl: Set $\dim R = n$ and $\text{ht } P = \dim R_P = r$. By (8.29) R_P is a CM-ring and $\text{depth } R_P = \text{depth}_P R = \dim R_P = r$. Let $a_1, \dots, a_r \in P$ be a maximal regular sequence in P . Then $R/(a_1, \dots, a_r)$ is a CM-ring of dimension $n-r$. By (a) $\text{ht } P = r = \text{ht}(a_1, \dots, a_r)$ and P is a minimal prime over (a_1, \dots, a_r) . Thus $P \in \text{Ass}(R/(a_1, \dots, a_r))$ and by (8.29)(a) $\dim R/P = \dim(R/(a_1, \dots, a_r)) = n-r$.

In order to finish the proof of (ii) let $P \in S$ with $\text{ht } P = \text{ht } I$. Then $\dim R/I \geq \dim R/P$ and $\text{ht } I + \dim R/I \geq \text{ht } P + \dim R/P = \dim R$. Let $Q \in S$ be such that $\dim R/I = \dim R/Q$. Then $\text{ht } I \leq \text{ht } Q$ and $\text{ht } I + \dim R/I \leq \text{ht } Q + \dim R/Q = \dim R$. Thus $\text{ht } I + \dim R/I = \dim R$.

(c) Consider prime ideals $P \subseteq Q$ of R . Since R_Q is a CM-ring, by (b) $\dim R_Q = \text{ht } PR_Q + \dim (R/I)_Q$. Note that $\dim R_Q = \text{ht } Q$, $\text{ht } PR_Q = \text{ht } P$, and $\dim (R/I)_Q = \text{ht}(Q/P)$. This shows that $\text{ht}(Q/P) = \text{ht } Q - \text{ht } P$. Consider an intermediate prime ideal $P \subseteq W \subseteq Q$. Then $\text{ht}(Q/W) = \text{ht } Q - \text{ht } W$ and $\text{ht}(W/P) = \text{ht } W - \text{ht } P$. Thus $\text{ht}(Q/P) = \text{ht } Q - \text{ht } P = \text{ht}(Q/W) + \text{ht}(W/P)$ and R is catenary.

(8.32) Definition: Let R be a Noetherian ring.

(a) An ideal $I \subseteq R$ is called unmixed if for all $P \in \text{Ass}(R/I)$: $\text{ht } P = \text{ht } I$.

(b) The unmixedness theorem holds for the ring R if every ideal $I = (a_1, \dots, a_r) \subseteq R$ with $\text{ht } I = r$ is unmixed.

(8.33) Remark: (a) Let R be a Noetherian ring and $I = (a_1, \dots, a_r) \subseteq R$ an ideal with $\text{ht } I = r$. I is unmixed if and only if I has no embedded primes.

(b) By (8.29) the zero ideal (0) in a local CM-ring R is unmixed.

(8.34) Definition: Let R be a Noetherian ring and M a finite R -module.

(a) R is a Cohen-Macaulay ring if R_m is CM for all $m \in \text{m Spec } R$.

(b) M is a Cohen-Macaulay module if M_m is CM (over R_m) for all $m \in \text{mSpec } R$.

(8.35) Theorem: A Noetherian ring R is CM if and only if the unmixedness theorem holds for R .

Proof: " \Rightarrow ": Let R be a CM-ring and $I = (a_1, \dots, a_r) \in R$ an ideal with $\text{ht } I = r$. For all $P \in \text{Ass}(R/I)$ the ring R_P is CM and a_1, \dots, a_r is an R_P -sequence by (8.33). By (8.29) the ideal $I_P = (a_1, \dots, a_r)R_P$ has no embedded prime ideals. Thus P is minimal over I and $\text{ht } P = r = \text{ht } I$.

" \Leftarrow ": Suppose that the unmixedness theorem holds for R and let $P \in \text{Spec } R$. We claim that R_P is a CM-ring. Suppose $\text{ht } P = r$ and let $a_1, \dots, a_r \in P$ with $\text{ht}(a_i) = i$ for $1 \leq i \leq r$. By the unmixedness theorem all $Q \in \text{Ass}(R/(a_1, \dots, a_r))$ have height i . This implies that a_{i+1} is regular on $R/(a_1, \dots, a_i)$ and a_1, \dots, a_r is a regular sequence in R_P . Thus $\dim R_P = \text{ht } P = r = \text{depth } R_P$ and R_P is CM.

(8.36) Theorem: Let R be a CM-ring. The polynomial ring $R[x_1, \dots, x_n]$ in finitely many variables over R is a CM-ring.

Proof: We only need to show that the polynomial ring $R[x]$ in one variable is CM. Let $P \subseteq R[x]$ be a prime ideal and $\mathfrak{m} = P \cap R$ its contraction to R . The ring $R[x]_P$ is a localization of $R_{\mathfrak{m}}[x]$ and we may assume that R is a local CM-ring with maximal ideal \mathfrak{m} and that $P \subseteq R[x]$ is a prime ideal with $P \cap R = \mathfrak{m}$. We claim that $R[x]_P$ is CM. Suppose $\dim R = n$ and let $a_1, \dots, a_n \in \mathfrak{m}$ be a regular sequence in R . Since $R[x]_P$ is flat over R , a_1, \dots, a_n is a regular sequence in $R[x]_P$, in particular, $\text{depth } R[x]_P \geq n$. If $P = \mathfrak{m}R[x]$, then $\dim R[x]_P = n$ and $R[x]_P$ is CM. If $P \neq \mathfrak{m}R[x]$ set $K = R/\mathfrak{m}$ and note that $R[x]_{\mathfrak{m}R[x]} \cong K[x]$. Thus $\bar{P} = P/\mathfrak{m}R[x]$ is a principal ideal generated by a monic irreducible polynomial \bar{f} . Let $f \in R[x]$ be monic polynomial with $f + \mathfrak{m}R[x] = \bar{f}$. Since f is monic, f is regular on $R[x]/(a_1, \dots, a_n) \cong (R/(a_1, \dots, a_n))[x]$. Hence $\text{depth } R[x]_P \geq n+1 = \dim R[x]_P$. $R[x]_P$ is a CM-ring.

(8.37) Corollary: Every algebra of finite type over a CM-ring is catenary.

§3: REGULAR RINGS

(8.38) Definition: Let (R, \mathfrak{m}, k) be a local Noetherian ring. R is regular if $\dim R = \mu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

(8.39) Examples: Let R be a local Noetherian ring.

(a) If $\dim R = 0$, then R is regular if and only if R is a field.

(b) If $\dim R = 1$, then R is regular if and only if R is a DVR.

(8.40) Theorem (Tate): Let (R, \mathfrak{m}, k) be a local Noetherian ring and $p \in \mathfrak{m}$ an element such that (p) is a prime ideal of height one. Then R is a domain.

Proof: We first show that for every element $a \in R - (0)$ there is an element $a' \in R - (p)$ and an integer $r \in \mathbb{N}$ so that $a = p^r a'$. Since R is local Noetherian, $\bigcap_{r \in \mathbb{N}} (p^r) = (0)$ and there is an $r \in \mathbb{N}$ with $a \in (p^r)$ and $a \notin (p^{r+1})$. Then $a = p^r a'$ for some $a' \in R - (p)$. Next let $a, b \in R - (0)$, $a', b' \in R - (p)$ and $r, s \in \mathbb{N}$ with $a = p^r a'$ and $b = p^s b'$. Then $ab = p^{r+s} a' b'$. Since (p) is a prime ideal, $a' b' \notin (p)$ and there is a minimal prime ideal $Q \subseteq R$ with $a' b' \notin Q$. Since $\text{ht}(p) = 1$, $(p) \not\subseteq Q$ and $ab = p^{r+s} a' b' \notin Q$. Thus $ab \neq 0$.

(8.41) Corollary: Let R be a local Noetherian ring which is not a domain. Every principal prime ideal of R is minimal.

(8.42) Theorem: A regular local ring (R, \mathfrak{m}, k) is a domain.

Proof: by induction on $\dim R$: If $\dim R = 0$, then R is a field. Assume $\dim R = n$ and let P_1, \dots, P_r be the minimal prime ideals of R . We claim that if R is not a domain then $\mathfrak{m} \subseteq \mathfrak{m}^2 \cup P_1 \cup \dots \cup P_r$. Suppose R is not a domain and let $x_1, x_2, \dots, x_n \in \mathfrak{m} - \mathfrak{m}^2$ so that $x_1 + \mathfrak{m}^2, x_2 + \mathfrak{m}^2, \dots, x_n + \mathfrak{m}^2$ is a basis of the k -vector space $\mathfrak{m}/\mathfrak{m}^2$. In $R/(x_1)$ the maximal ideal $\mathfrak{m}/(x_1)$ is generated by $x_2 + (x_1), \dots, x_n + (x_1)$. By (4.48) (a) $\dim R/(x_1) \geq n-1$.

If $x \notin P_i$ for all $1 \leq i \leq r$, then $\dim R/(x) \leq n-1$ and hence $\dim R/(x) = n-1$. In this case $R/(x)$ is a regular local ring and a domain by induction hypothesis. This implies that $(x) \subseteq R$ is a principal prime ideal of height one. By (8.40) R is a domain, a contradiction. Thus $x \in P_i$ for some $1 \leq i \leq r$ and the claim follows.

Thus if R is not a domain then $m \subseteq m^2 \cup P_1 \cup \dots \cup P_r$. If $m \neq (0)$, by Nakayama $m \not\subseteq m^2$ and $m \subseteq P_i$ for some $1 \leq i \leq r$. But then $\dim R = 0$, a contradiction.

(8.43) Theorem: Let (R, m, k) be a local Noetherian ring and $x_1, \dots, x_r \in m$ a minimal system of generators of m . The following are equivalent:

(a) R is regular

(b) Let $k[z_1, \dots, z_r]$ be the polynomial ring in r variables over k . The homomorphism of rings $\varphi: k[z_1, \dots, z_r] \rightarrow \hat{g}_m(R) = \bigoplus_{i=0}^{\infty} m^i/m^{i+1}$ defined by $\varphi(z_i) = x_i + m^2 \in m/m^2$ is bijective.

Proof. (b) \Rightarrow (a): Let $S_n = \{f \in k[z_1, \dots, z_r] \mid f \text{ homogeneous of degree } n\}$ be the k -vector space of homogeneous polynomials of degree n . If φ is bijective, $S_n \cong m^n/m^{n+1}$ as k -vector spaces. Thus $\ell_R(m^n/m^{n+1}) = \binom{r+n-1}{r-1}$ and $\ell_R(R/m^n) = \sum_{i=0}^{n-1} \ell_R(m^i/m^{i+1}) = \binom{r+n-1}{r}$. $\ell_R(R/m^n)$ is a polynomial of degree r in n and therefore $\dim R = r = \dim_k(m/m^2)$.

(a) \Rightarrow (b): Note that φ is always surjective and $I = \ker \varphi$ is a homogeneous ideal since φ is a homogeneous morphism. Thus $I = \bigoplus_{n \in \mathbb{N}} I_n$ where $I_n \subseteq S_n$. Suppose $I \neq (0)$ and let $u \in I_h$ with $u \neq 0$. Then for all $n \geq h$: $u S_{n-h} \subseteq I_n$ and $\ell_R(m^n/m^{n+1}) = \ell(S_n/I_n) \leq \binom{r+n-1}{r-1} - \binom{r+n-h-1}{r-1}$. $\binom{r+n-1}{r-1}$ and $\binom{r+n-h-1}{r-1}$ are polynomials in n of degree $r-1$ with the same leading coefficient. This implies that $\ell_R(m^n/m^{n+1})$ is a polynomial in n of degree $\leq r-2$. Thus $\dim R \leq r-1$, a contradiction.

(8.44) Proposition: Let (R, m, k) be a regular local ring. Every minimal system of generators of m is a regular sequence.

Proof: by induction on $n = \dim R$. If $n=0$, then R is a field. If $n \geq 1$, let $m = (x_1, \dots, x_n)$.

Since R is a domain, x_1 is a regular element of R . Then $\dim R/(x_1) = n-1$ and $R/(x_1)$ is a regular local ring of dimension $n-1$. By induction hypothesis $x_2+(x_1), \dots, x_n+(x_1)$ is a regular sequence of $R/(x_1)$.

(8.45) Corollary: Every regular local ring is Cohen-Macaulay.

(8.46) Proposition: Let (R, \mathfrak{m}, k) be a local Noetherian ring and $x_1, \dots, x_n \in \mathfrak{m}$ a regular sequence of R with $\mathfrak{m} = (x_1, \dots, x_n)$. Then:

- (a) R is a regular local ring.
- (b) x_1, \dots, x_n is a minimal system of generators of \mathfrak{m} .

Proof: Since $n \geq \text{edim } R = \dim_k (\mathfrak{m}/\mathfrak{m}^2) \geq \dim R \geq \text{depth } R = n$.

(8.47) Definition: Let (R, \mathfrak{m}, k) be a local Noetherian ring and $x_1, \dots, x_n \in \mathfrak{m}$ a regular sequence of R . x_1, \dots, x_n is called a regular system of parameters (RSOP) if $\mathfrak{m} = (x_1, \dots, x_n)$.

(8.48) Remark: A local Noetherian ring R has a RSOP if and only if R is regular.

(8.49) Corollary: Let (R, \mathfrak{m}, k) be a regular local ring and $x_1, \dots, x_r \in \mathfrak{m}$. The following are equivalent:

- (a) x_1, \dots, x_r are part of an RSOP.
- (b) $x_1 + \mathfrak{m}^2, \dots, x_r + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k .
- (c) $R/(x_1, \dots, x_r)$ is a regular local ring with $\dim R/(x_1, \dots, x_r) = \dim R - r$.

Proof: (a) \Leftrightarrow (b): trivial.

Set $I = (x_1, \dots, x_r)$ and $S = R/I$. S is a local Noetherian ring with maximal ideal $\mathfrak{m}_S = \mathfrak{m}/I$.

Consider the exact sequence of k -vector spaces: $0 \rightarrow I/I \cap \mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 + I \cong \mathfrak{m}_S/\mathfrak{m}_S^2 \rightarrow 0$.

Let $s = \dim_k (I/I \cap \mathfrak{m}^2)$. Since R is regular, $n = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ and $\dim_k (\mathfrak{m}_S/\mathfrak{m}_S^2) = n - s$.

(b) \Rightarrow (c): (b) implies that $s = r$ and therefore $\text{edim } S = \dim_k (\mathfrak{m}_S/\mathfrak{m}_S^2) = n - r$. Since R is regular,

R is a CM-ring and by (8.31): $\dim R/I = \dim R - \text{ht } I$. Since $\text{ht } I \leq r$, $\dim S \geq n-r$ and S is regular.

(c) \Rightarrow (b): $\dim S = \text{edim } S = n-r$ implies that $\dim_k (I/I \cap m^2) = r$. The sequence $x_1 + m^2, \dots, x_r + m^2$ is linearly independent over k .

(8.50) Theorem: Let (R, m, k) be a regular local ring and $I \subseteq R$ an ideal. The ring R/I is regular if and only if I is generated by part of an RSOP.

Proof: " \Leftarrow ": by (8.49)

" \Rightarrow ": Suppose that R/I is regular. Let $x_1, \dots, x_s \in m$ so that $x_1 + I, \dots, x_s + I$ is an RSOP of R/I . In particular, $m = (x_1, \dots, x_s) + I$. Let $y_1, \dots, y_r \in I$ so that $y_1 + m^2, \dots, y_r + m^2$ is a basis of $m^2 + I / m^2 \cong I / I \cap m^2$. From the exact sequence of k -vector spaces: $0 \rightarrow m^2 + I / m^2 \rightarrow m / m^2 \rightarrow m / m^2 + I \rightarrow 0$ we obtain that $x_1, \dots, x_s, y_1, \dots, y_r$ is an RSOP of R . Let $J = (y_1, \dots, y_r) \subseteq I$. By (8.49) R/J is a regular local ring with $\dim R/J = s = \dim R/I$. Since R/J is a domain, $I = J$.

Homological description of regular local rings

(8.51) Lemma: Let (R, m, k) be a local Noetherian ring and M a finite R -module. If (F, ∂) is a minimal free resolution of M , then:

- (a) $\dim_k \text{Tor}_i^R(M, k) = \text{rank } F_i$ for all i
 (b) $\text{projdim } M = \sup \{i \mid \text{Tor}_i^R(M, k) \neq 0\} \leq \text{projdim } k$.

Proof: First note that (F, ∂) is a minimal free resolution of M if i) every F_i is a finite free R -module, ii) $\partial_i F_i \subseteq m F_{i-1}$ for all $i > 0$, and iii) $F_0 / m F_0 \cong M / mM$.

(a) Because of condition ii), the boundary maps of the complex $(F \otimes k, \partial \otimes k)$ are all zero. Thus $\text{Tor}_i^R(M, k) = H_i(F \otimes k) = F_i \otimes k$. The dimension of the k -vector space $F_i \otimes k$ is the rank of F_i .

(b) follows from (a).

(8.52) Theorem: Let (R, \mathfrak{m}, k) be a local Noetherian ring, M a finite R -module and $x \in \mathfrak{m}$ an M -regular element. Then $\text{projdim } M + 1 = \text{projdim } M/xM$.

Proof: The exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ induces a long exact sequence:

$$\dots \rightarrow \text{Tor}_{i+1}^R(k, M) \xrightarrow{x} \text{Tor}_{i+1}^R(k, M) \rightarrow \text{Tor}_{i+1}^R(k, M/xM) \rightarrow \text{Tor}_i^R(k, M) \xrightarrow{x} \text{Tor}_i^R(k, M) \rightarrow \dots$$

For all $i \in \mathbb{N}$ $\text{Tor}_i^R(k, M)$ is annihilated by \mathfrak{m} , thus multiplication by x is the zero map and the sequence $0 \rightarrow \text{Tor}_{i+1}^R(k, M) \rightarrow \text{Tor}_{i+1}^R(k, M/xM) \rightarrow \text{Tor}_i^R(k, M) \rightarrow 0$ is exact for all $i \in \mathbb{N}$. If $\text{projdim } M = r < \infty$, then $\text{Tor}_{r+1}^R(k, M/xM) \neq 0$ and $\text{Tor}_i^R(k, M/xM) = 0$ for all $i > r+1$. If $\text{projdim } M = \infty$, then $\text{Tor}_i^R(k, M) \neq 0$ for all $i \in \mathbb{N}$ and $\text{Tor}_i^R(k, M/xM) \neq 0$ for all $i \in \mathbb{N}$.

(8.53) Corollary: Let (R, \mathfrak{m}, k) be a local Noetherian ring, M a finite R -module, and x_1, \dots, x_n an M -sequence. Then $\text{projdim } M = \text{projdim } M/(x_1, \dots, x_n)M + n$.

(8.54) Corollary: Let (R, \mathfrak{m}, k) be a regular local ring with $\dim R = n$. Then $\text{gldim } R = n$.

Proof: Let x_1, \dots, x_n be a RSOP of R . By (8.46) x_1, \dots, x_n is a regular sequence of R and by (8.53) $\text{projdim } k = \text{projdim } R/(x_1, \dots, x_n) = \text{projdim } R + n = n$. By (7.44) $\text{gldim } R = \sup \{ \text{projdim } R/\mathfrak{I} \mid \mathfrak{I} \text{ an } R\text{-ideal} \}$ and by (8.51) $\text{projdim } M \leq \text{projdim } k$ for every finite R -module M . Thus $\text{gldim } R = n$.

(8.55) Proposition: Let (R, \mathfrak{m}, k) be a local Noetherian ring, M a finite R -module, and $x \in \mathfrak{m}$ an element which is R -regular and M -regular. Then $\text{projdim}_{R/xR} M/xM \leq \text{projdim}_R M$.

Proof: If $\text{projdim } M = \infty$, we are done. If $\text{projdim } M = n < \infty$ we proceed by induction on n . If $n = 0$, M is a projective R -module. Since R is local, M is free and M/xM is a free R/xR -module. If $n > 0$, consider an exact sequence $0 \rightarrow N \rightarrow R^n \rightarrow M \rightarrow 0$. Then $\text{projdim } N = n-1$ and, since $N \subseteq R^n$, x is also an N -regular element. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & R^r & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & R^r & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & N/xN & \xrightarrow{\alpha} & R^r/xR^r & \longrightarrow & M/xM \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

By the Snake Lemma α is injective and therefore $\text{projdim}_{R/xR} N/xN = \text{projdim}_{R/xR} M/xM - 1$. By induction hypothesis $\text{projdim}_{R/xR} N/xN \leq \text{projdim}_R N$ and hence $\text{projdim}_{R/xR} M/xM \leq \text{projdim}_R M$.

(8.56) Lemma: Let R be a Noetherian ring and M_1, M_2 finite R -modules with $\text{projdim}(M_1 \oplus M_2) = r < \infty$. Then $\text{projdim} M_i \leq r$ for $i=1, 2$.

Proof: For $i=1, 2$ consider exact sequences: $0 \rightarrow K_i \rightarrow P_{i,r-1} \rightarrow \dots \rightarrow P_{i,0} \rightarrow M_i \rightarrow 0$ with $P_{i,j}$ projective R -modules. The direct sum of the two sequences $0 \rightarrow K_1 \oplus K_2 \rightarrow P_{1,r-1} \oplus P_{2,r-1} \rightarrow \dots \rightarrow P_{1,0} \oplus P_{2,0} \rightarrow M_1 \oplus M_2 \rightarrow 0$ is exact. Since $\text{projdim} M_1 \oplus M_2 = r$, the R -module $K_1 \oplus K_2$ is projective. Thus K_i is a projective R -module and $\text{projdim} M_i \leq r$.

(8.57) Theorem: Let (R, \mathfrak{m}, k) be a local Noetherian ring. R is regular if and only if $\text{gldim} R < \infty$. Moreover, if R is regular then $\text{gldim} R = \dim R$.

Proof: By (8.54) if R is regular then $\text{gldim} R = \dim R$. Suppose that $\text{gldim} R < \infty$ and proceed by induction on $n = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \text{edim} R$. If $n=0$ then $\mathfrak{m} = \mathfrak{m}^2$ and by Nakayama $\mathfrak{m} = (0)$. R is a field. Suppose $n > 0$. If every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zerodivisor of R then $\mathfrak{m} \subseteq \mathfrak{m}^2 \cup \bigcup_{P \in \text{Ass}(R)} P$ and $\mathfrak{m} \in \text{Ass}(R)$. Thus $\text{depth} R = 0$. Since $\text{projdim}_R k < \infty$, by Auslander-Buchsbaum (8.23) $\text{projdim} k = 0$ and k is a free R -module, a contradiction to $n > 0$. Thus $\mathfrak{m} - \mathfrak{m}^2$ contains an R -regular element.

Let $x \in \mathfrak{m} - \mathfrak{m}^2$ be an R -regular element. The ring $S = R/xR$ is local Noetherian with maximal ideal $\mathfrak{m}_S = \mathfrak{m}/xR$ and embedding dimension $\dim_k \mathfrak{m}_S/\mathfrak{m}_S^2 = n-1$. Moreover,

$\dim S = \dim R - 1$ by (4.48). If $\text{gldim } S < \infty$, then S is regular by induction hypothesis and $\dim_k m_S/m_S^2 = n-1 = \dim S$. In this case, $\dim S = n = \dim_k m/m^2$ and R is regular.

Thus it remains to show that $\text{gldim } S < \infty$. By (8.51) we know that $\text{gldim } S = \text{projdim}_S k$. The exact sequence $0 \rightarrow m_S \rightarrow S \rightarrow k \rightarrow 0$ yields that $\text{projdim}_S k = \text{projdim}_S m_S + 1$ and it suffices to show that $\text{projdim}_S m_S < \infty$. Note that $m_S = m/xR$. By (8.55) $\text{projdim}_S (m/xm) \leq \text{projdim}_R m < \infty$ and we need to compare $\text{projdim}_S (m/xm)$ and $\text{projdim}_S (m/xR)$.

Claim: $m_S = m/xR$ is a direct summand of the S -module m/xm .

Pr of cl: Extend x to a minimal system of generators x, x_2, \dots, x_n of m and set $I = mx + (x_2, \dots, x_n)$. Obviously, $Rx + I = m$. Let $a \in R$ with $ax \in Rx \cap I$. Then $ax = \ell x + a_2 x + \dots + a_n x_n$ where $\ell \in m$ and $a_i \in R$. Since $x \notin m^2, x_2 \notin m^2, \dots, x_n \notin m^2$ is a basis of the k -vector space m/m^2 , we have that $a \in m$ and $Rx \cap I = mx$. Thus $m_S = m/xR = xR + I/xR \cong I/Rx \cap I = I/mx$. Consider the S -linear map: $\varphi: I/xm \oplus xR/xm \rightarrow m/xm$ defined by $\varphi(a, b) = a + b$. Obviously, φ is surjective. Moreover, $\varphi(a + xm, b + xm) = 0 \iff a + b \in xm = I \cap Rx \iff a, b \in xm$. φ is injective. Thus $m_S = m/xR$ is a direct summand of m/xm . By (8.56) $\text{projdim}_S m_S < \infty$.

(8.58) Corollary: Let (R, m, k) be a regular local ring and $P \in \text{Spec } R$. Then R_P is regular.

Proof: Let $P \in \text{Spec } R$ and $k(P) = (R/P)_P = R_P/PR_P$. We have to show that $\text{gldim } R_P = \text{projdim}_{R_P} k(P) < \infty$. Since $\text{projdim}_R R/P < \infty$, there is an exact sequence of R -modules $0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \dots \rightarrow P_0 \rightarrow R/P \rightarrow 0$ where P_i is a projective R -module for all i . Localizing at P yields an exact sequence of R_P -modules: $0 \rightarrow (P_m)_P \rightarrow \dots \rightarrow (P_0)_P \rightarrow k(P) \rightarrow 0$. Since $(P_i)_P$ are projective R_P -modules, the assertion follows.

(8.59) Definition: A Noetherian ring R is called regular if R_P is a regular local ring for all $P \in \text{Spec } R$.

(8.60) Remark: Let R be a Noetherian ring.

(a) R is regular if and only if R_m is regular for all $m \in \text{Spec } R$, m maximal.

(b) If R is a domain of dimension one then R is regular if and only if R is a Dedekind domain.

(8.61) Theorem: Let $\varphi: (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, k')$ be a faithfully flat morphism of local Noetherian rings.

(a) If S is regular, so is R .

(b) If R and $S/\mathfrak{m}S$ are regular, S is regular and $\dim S = \dim R + \dim S/\mathfrak{m}S$.

Proof: (a) Suppose that $\dim S = n$. Let M be a finite R -module and $0 \rightarrow K \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$ an exact sequence with finite free R -modules F_i . Since S is flat over R , the sequence $0 \rightarrow K \otimes_R S \rightarrow F_{n-1} \otimes_R S \rightarrow \dots \rightarrow F_0 \otimes_R S \rightarrow M \otimes_R S \rightarrow 0$ is exact and $F_i \otimes_R S$ is a finite free S -module for all $0 \leq i \leq n-1$. Since S is regular of dimension n , $K \otimes_R S$ is a free S -module. Thus $K \otimes_R S$ is a flat R -module. Let $0 \rightarrow N' \rightarrow N$ be an exact sequence of R -modules. Then $0 \rightarrow N' \otimes_R (K \otimes_R S) \rightarrow N \otimes_R (K \otimes_R S)$ is exact. Since S is faithfully flat over R , the sequence $0 \rightarrow N' \otimes_R K \rightarrow N \otimes_R K$ is exact and K is a flat R -module. Since K is finite and R is local, K is a free R -module and $\text{projdim } M < \infty$. R is a regular local ring.

(b) Let $x_1, \dots, x_t \in \mathfrak{m}$ be a RSOP of R and $y_1, \dots, y_s \in \mathfrak{n}$ with $y_1 + \mathfrak{m}S, \dots, y_s + \mathfrak{m}S$ a RSOP of $S/\mathfrak{m}S$. Obviously, $\mathfrak{n} = (x_1, \dots, x_t, y_1, \dots, y_s)$. We claim that $x_1, \dots, x_t, y_1, \dots, y_s$ is an S -regular sequence. For $1 \leq i \leq t$: $0 \rightarrow R/(x_1, \dots, x_i) \xrightarrow{x_{i+1}} R/(x_1, \dots, x_i)$ is injective and, since S is flat over R , $0 \rightarrow S/(x_1, \dots, x_i) \xrightarrow{x_{i+1}} S/(x_1, \dots, x_i)$ is exact. Hence x_1, \dots, x_t is an S -regular sequence. For $1 \leq j \leq s$: $S/(x_1, \dots, x_t, y_1, \dots, y_j)S \cong (S/\mathfrak{m}S)/(y_1 + \mathfrak{m}S, \dots, y_j + \mathfrak{m}S)(S/\mathfrak{m}S)$ and $x_1, \dots, x_t, y_1, \dots, y_s$ is an S -regular sequence. Thus $\dim R + \dim S/\mathfrak{m}S = t + s \leq \dim_{k'} (\mathfrak{n}/\mathfrak{n}^2) \geq \dim S \geq \text{depth } S \geq t + s$. S is a regular ring.

(8.62) Theorem: Let R be a regular ring. The polynomial ring $R[x_1, \dots, x_n]$ in n variables over R is a regular ring.

Proof: We have to show that $R[x]$ is regular. Let $Q \in \text{Spec } R[x]$ and $P = Q \cap R \in \text{Spec } R$. Since $R \rightarrow R[x]$ is flat, the induced morphism $R_P \rightarrow R[x]_Q$ is faithfully flat. Moreover,

$(R[X]/PR[X])_{\mathcal{Q}} \cong k(P)[X]_{\mathcal{Q}}$ where $k(P) = (R/P)_P$. Since $k(P)[X]$ is regular, by (8.61) $R[X]_{\mathcal{Q}}$ is regular.

Regular local rings are factorial

(8.63) Lemma: Let R be a domain with field of quotients $K = Q(R)$, $n \in \mathbb{N}$ an integer, and $I \subseteq R$ an ideal with $I \oplus R^n \cong R^{n+1}$. Then I is principal.

Proof: Let e_0, \dots, e_n denote the canonical basis of R^{n+1} and let $\varphi: R \oplus R^n \xrightarrow{\cong} I \oplus R^n \subseteq R \oplus R^n$ be an isomorphism. Considering φ as an R -linear map from R^{n+1} to R^{n+1} we write $\varphi(e_i) = \sum_{j=0}^n a_{ij} e_j$ with $a_{ij} \in R$ for all $0 \leq i \leq n$. Let $M = (a_{ij})$ be the matrix of φ (considered as a map from R^{n+1} to R^{n+1}) and $d = \det M = \det (a_{ij})$ its determinant. Then $M\tilde{M} = \tilde{M}M = dJ$ where \tilde{M} is the adjoint matrix of M and J is the $(n+1) \times (n+1)$ identity matrix. Let M_{i0} be the $n \times n$ -matrix obtained from M by eliminating the i th row and 0 th column. The first row of \tilde{M} is $(d_0, -d_1, \dots, (-1)^n d_n)$ where $d_i = \det M_{i0}$. From $\tilde{M}M = J$ we obtain that $\sum_{i=0}^n a_{i0} (-1)^i d_i = d$ and $\sum_{i=0}^n a_{ij} (-1)^i d_i = 0$ for all $1 \leq j \leq n$. Since φ is injective, φ extends to an isomorphism on K^{n+1} and $d \neq 0$. With $f_0 = \sum_{i=0}^n (-1)^i d_i e_i$:

$$\begin{aligned} \varphi(f_0) &= \varphi\left(\sum_{i=0}^n (-1)^i d_i e_i\right) = \sum_{i=0}^n (-1)^i d_i \varphi(e_i) = \\ &= \sum_{i=0}^n (-1)^i d_i \left(\sum_{j=0}^n a_{ij} e_j\right) = \sum_{j=0}^n \left(\sum_{i=0}^n a_{ij} (-1)^i d_i\right) e_j = d e_0 \end{aligned}$$

and $d \in I$. Since φ is surjective onto $I \oplus R^n$, for all $1 \leq j \leq n$ there is an $f_j \in R^{n+1}$ with $\varphi(f_j) = e_j$. Write $f_j = \sum_{k=0}^n c_{jk} e_k$ where $c_{jk} \in R$ for $1 \leq j \leq n$ and $c_{0k} = (-1)^k d_k$. The $(n+1) \times (n+1)$ matrix $C = (c_{jk})$ defines an R -linear map $\psi: R^{n+1} \rightarrow R^{n+1}$ with $\psi\varphi(e_0) = \varphi(f_0) = d e_0$ and $\psi\varphi(e_j) = \varphi(f_j) = e_j$ for $1 \leq j \leq n$. Thus

$$MC = \begin{bmatrix} d & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

and $\det MC = d = \det M \det C$. Thus $\det C = 1$ and ψ is an isomorphism on R^{n+1} . In particular, f_0, \dots, f_n is a basis of R^{n+1} . This implies that $I = (d)$.

(8.64) Definition: An R -module M is called stably free if there are finite free R -modules

F and F' with $M \oplus F \cong F'$.

(8.65) Remark: Let M be a stably free R -module. Then

(a) M is finitely generated.

(b) M is projective.

(c) There is an exact sequence $0 \rightarrow F \rightarrow F' \rightarrow M \rightarrow 0$ with F and F' finite free R -modules.

(8.66) Lemma: Let M be a finite projective R -module and $0 \rightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$ an exact sequence with finite free R -modules F_i . Then M is stably free.

Proof: We proceed by induction on n . If $n=1$, $0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ is exact and $F_0 \cong F_1 \oplus M$ since M is projective. If $n > 1$ consider the exact sequences $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow K \rightarrow 0$ and $0 \rightarrow K \rightarrow F_0 \rightarrow M \rightarrow 0$. Since M is projective, $F_0 \cong K \oplus M$ and K is projective. By induction hypothesis K is stably free. Thus $F^* \oplus K \cong F'$ for finite free R -modules F^* and F' . Therefore $F^* \oplus F_0 \cong F^* \oplus K \oplus M \cong F' \oplus M$. M is stably free.

Note that a Noetherian domain R is factorial if and only if every height one prime ideal of R is principal.

(8.67) Theorem: Let R be a Noetherian domain, $\Gamma \subseteq R$ a set of prime elements of R , and $S \subseteq R$ the multiplicative set generated by Γ , i.e. $S = \{1\} \cup \{p_1^{e_1} \dots p_n^{e_n} \mid n, e_i \in \mathbb{N}, p_i \in \Gamma\}$. If $S^{-1}R$ is factorial, R is factorial.

Proof: Let $P \subseteq R$ be a height one prime ideal. If $P \cap S \neq \emptyset$ then there is an element $p \in \Gamma$ with $p \in P$ and $P = pR$. If $P \cap S = \emptyset$ consider the set of ideals $\Delta = \{pR \mid p \in P \text{ and } pS^{-1}R = pS^{-1}R\}$. Since $S^{-1}R$ is factorial, $\Delta \neq \emptyset$, and Δ contains a maximal element pR . If $q \in \Gamma$ with $q \mid p$ then $p = qt$ for some $t \in R$. Thus $t \in P$ with $pS^{-1}R = tS^{-1}R = pS^{-1}R$. Therefore $tR \in \Delta$ and $pR = tR$ by the maximality of pR . This implies that q is a unit in R , a contradiction.

Thus for all $q \in \mathcal{P}$: $q \mid p$. Let $x \in \mathcal{P}$ and $s \in S$ with $sx = py$ for some $y \in R$. Write $s = p_1 \cdots p_n$ with $p_i \in \mathcal{P}$. Then $p_i \mid y$ for all $1 \leq i \leq n$ and $x \in pR$. Hence $\mathcal{P} = pR$.

(6.68) Theorem: A regular local ring is factorial.

Proof: Let (R, \mathfrak{m}, k) be a regular local ring. The proof is by induction on $\dim R = n$. If $n=0$, R is a field and if $n=1$, R is a discrete valuation ring. Let $n > 1$ and $x \in \mathfrak{m} - \mathfrak{m}^2$. Since xR is a prime ideal (8.46), x is a prime element of R . By (8.67) we have to show that R_x is factorial. Let $\mathcal{P} \subseteq R_x$ be a prime ideal of height one and set $Q = R \cap \mathcal{P}$. Then $\mathcal{P} = QR_x$. Since R is a regular local ring, there is an exact sequence $0 \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow Q \rightarrow 0$ with F_i finite free R -modules for all $0 \leq i \leq n$. Localization at x yields that $0 \rightarrow F_{nx} \rightarrow \cdots \rightarrow F_{0x} \rightarrow \mathcal{P} \rightarrow 0$ is an exact sequence with F_{ix} finite free R_x -modules. We claim that \mathcal{P} is a projective R_x -module. Every prime ideal $W \in \text{Spec } R_x$ corresponds to a prime ideal $W_0 = W \cap R$ with $W_0 \neq \mathfrak{m}$. Moreover, $R_{W_0} \cong (R_x)_W$ and $(R_x)_W$ is factorial by induction hypothesis. The height one prime ideal $\mathcal{P}(R_x)_W$ is principal, thus projective. This implies that \mathcal{P} is a projective R_x -module. By (8.66) \mathcal{P} is stably free and by (8.63) \mathcal{P} is principal.