CHAPTER VII : HOMOLOGICAL ALGEBRA II

§1: COMPLEXES

(7.1) Definition: A complex of R-modules $(C_\bullet, \partial_\bullet)$ is a sequence of R-modules and R-linear maps:

$$C_\bullet: \quad \cdots\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots\cdots$$

so that $\partial_i \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$. $\partial_\bullet$ is called the differential of the complex. The R-modules $Z_i(C_\bullet) = \ker \partial_i$ are the cycles of $C_\bullet$, $B_i(C_\bullet) = \operatorname{im} \partial_{i+1}$ the boundaries of $C_\bullet$, and $H_i(C_\bullet) = \ker\partial_i / \operatorname{im}\partial_{i+1} = Z_i(C_\bullet)/B_i(C_\bullet)$ are the homology modules of $C_\bullet$.

(7.2) Remark: (a) $(C_\bullet, \partial_\bullet)$ is exact if and only if $H_i(C_\bullet) = 0$ for all $i \in \mathbb{Z}$.
(b) In order to avoid negative indices one often writes $(C^\bullet, \partial^\bullet)$ for the complex:

$$C^\bullet: \quad \cdots\cdots \longrightarrow C^{i-1} = C_{-i+1} \xrightarrow{\partial^{i-1}} C^i = C_{-i} \xrightarrow{\partial^i} C^{i+1} = C_{-i-1} \longrightarrow \cdots\cdots$$

and $H^i(C^\bullet) = H_{-i}(C^\bullet)$ for the homology. $H^\bullet(C^\bullet)$ is called the cohomology of $(C^\bullet, \partial^\bullet)$.

(7.3) Definition: (a) A morphism of complexes $u_\bullet : C_\bullet \longrightarrow C_\bullet'$ is a sequence of R-linear maps $u_i : C_i \longrightarrow C_i'$ so that $u_i \partial_{i+1} = \partial_{i+1}' u_{i+1}$ for all $i \in \mathbb{Z}$, that is, for all $i \in \mathbb{Z}$ the diagram:

$$\begin{array}{ccc} C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i \\ {\scriptstyle u_{i+1}}\big\downarrow & & \big\downarrow{\scriptstyle u_i} \\ C_{i+1}' & \xrightarrow{\partial_{i+1}'} & C_i' \end{array}$$

commutes.

(b) A sequence of morphisms of complexes $0 \longrightarrow C_\bullet' \xrightarrow{u_\bullet} C_\bullet \xrightarrow{v_\bullet} C_\bullet'' \longrightarrow 0$ is exact if $0 \longrightarrow C_i' \xrightarrow{u_i} C_i \xrightarrow{v_i} C_i'' \longrightarrow 0$ is exact for all $i \in \mathbb{Z}$.
(c) The direct sum $C_\bullet \oplus C_\bullet'$ of two complexes $(C_\bullet, \partial_\bullet)$ and $(C_\bullet', \partial_\bullet')$ is the complex with $(C_\bullet \oplus C_\bullet')_i = C_i \oplus C_i'$ and differential $\partial_i^{C_\bullet \oplus C_\bullet'} = \partial_i \oplus \partial_i'$.

(7.4) Remark: Let $u_\bullet : C_\bullet \longrightarrow C_\bullet'$ be a morphism of complexes. Then for all $i \in \mathbb{Z}$ $u_i(Z_i(C_\bullet)) \subseteq Z_i$ and $u_i(B_i(C_\bullet)) \subseteq B_i(C_\bullet')$. Hence $u_\bullet$ induces a sequence of R-linear maps $H_\bullet(u_\bullet) : H_\bullet(C_\bullet) \longrightarrow H_\bullet(C_\bullet')$ given by $H_i(u_\bullet)(z + B_i) = u_i(z) + B_i'$ where

$z \in Z_i(C_\cdot)$, $B_i = B_i(C_\cdot)$ and $B_i' = B_i(C_\cdot')$.

(7.5) Theorem: (Snake Lemma) Let

$$\begin{tikzcd} & M' \arrow[r, "\varphi"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r, "\psi"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\lambda"] & N \arrow[r, "\tau"] & N'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. Then there is a long exact sequence of induced maps:

$$\ker f' \xrightarrow{\overline{\varphi}} \ker f \longrightarrow \ker f'' \xrightarrow{\Delta} \operatorname{coker} f' \longrightarrow \operatorname{coker} f \xrightarrow{\overline{\psi}} \operatorname{coker} f''$$

Moreover, if φ is injective then so is $\overline{\varphi}$, if ψ is surjective so is $\overline{\psi}$.

Proof: Δ is defined as follows: Let $z'' \in \ker f''$. Since ψ is surjective, there is a $z \in M$ with $\psi(z) = z''$. Then $\tau f(z) = f''\psi(z) = 0$ and $f(z) \in \ker \tau = \operatorname{im} \lambda$. Hence there is a $y' \in N'$ with $\lambda(y') = f(z)$. Set $\Delta(z'') = y' + \operatorname{im} f$. Δ is well defined. The rest is diagram chasing.

(7.6) Proposition: Let $0 \longrightarrow C_\cdot' \xrightarrow{u_\cdot} C_\cdot \xrightarrow{v_\cdot} C_\cdot'' \longrightarrow 0$ be an exact sequence of complexes.

(a) For every i there is an R-linear map, called the connecting homomorphism,

$$\Delta_i : H_i(C_\cdot'') \longrightarrow H_{i-1}(C_\cdot')$$

defined by $\Delta_i(z'' + B_i'') = u_{i-1}^{-1}\, \partial_i\, v_i^{-1}(z'') + B_{i-1}'$.

(b) There is an exact sequence of R-modules, called the long exact sequence of homology:

$$\cdots\cdots \longrightarrow H_i(C_\cdot') \xrightarrow{H_i(u_\cdot)} H_i(C_\cdot) \xrightarrow{H_i(v_\cdot)} H_i(C_\cdot'') \xrightarrow{\Delta_i} H_{i-1}(C_\cdot') \xrightarrow{H_{i-1}(u_\cdot)} H_{i-1}(C_\cdot) \longrightarrow \cdots\cdots$$

(c) (Naturality of $\Delta_\cdot$.) Let

$$\begin{tikzcd} 0 \arrow[r] & C_\cdot' \arrow[r, "u_\cdot"] \arrow[d, "f_\cdot'"'] & C_\cdot \arrow[r, "v_\cdot"] \arrow[d, "f_\cdot"'] & C_\cdot'' \arrow[r] \arrow[d, "f_\cdot''"'] & 0 \\ 0 \arrow[r] & D_\cdot' \arrow[r] & D_\cdot \arrow[r] & D_\cdot'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of morphisms of complexes with exact rows. Then the diagram:

$$\begin{tikzcd} \cdots\cdots \arrow[r] & H_i(C_\cdot') \arrow[r] \arrow[d, "H_i(f_\cdot')"] & H_i(C_\cdot) \arrow[r] \arrow[d, "H_i(f_\cdot)"] & H_i(C_\cdot'') \arrow[r, "\Delta_i"] \arrow[d, "H_i(f_\cdot'')"] & H_{i-1}(C_\cdot') \arrow[r] \arrow[d, "H_{i-1}(f_\cdot')"] & H_{i-1}(C_\cdot) \arrow[r] \arrow[d, "H_{i-1}(f_\cdot)"] & \cdots \\ \cdots\cdots \arrow[r] & H_i(D_\cdot') \arrow[r] & H_i(D_\cdot) \arrow[r] & H_i(D_\cdot'') \arrow[r, "\widetilde{\Delta}_i"] & H_{i-1}(D_\cdot') \arrow[r] & H_{i-1}(D_\cdot) \arrow[r] & \cdots \end{tikzcd}$$

commutes and has exact rows.

Proof: By the Snake Lemma (7.5) the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & C'_{i+1} \arrow[r] \arrow[d, "\partial'_{i+1}"] & C_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"] & C''_{i+1} \arrow[r] \arrow[d, "\partial''_{i+1}"] & 0 \\ 0 \arrow[r] & C'_i \arrow[r] & C_i \arrow[r] & C''_i \arrow[r] & 0 \end{tikzcd}$$

with exact rows induces an exact sequence:

$$C'_i/B'_i \xrightarrow{\overline{u}_i} C_i/B_i \xrightarrow{\overline{v}_i} C''_i/B''_i \longrightarrow 0$$

where $B'_\cdot, B_\cdot, B''_\cdot$ denote boundaries. Likewise, by the Snake Lemma, there is an induced exact sequence of cycles: $0 \longrightarrow Z'_{i-1} \xrightarrow{\widetilde{u}_{i-1}} Z_i \xrightarrow{\widetilde{v}_{i-1}} Z''_{i-1}$. The differentials $\partial'_i, \partial_i, \partial''_i$ induce a commutative diagram with exact rows:

$$\begin{tikzcd} & C'_i/B'_i \arrow[r, "\overline{u}_i"] \arrow[d, "\overline{\partial}'_i"] & C_i/B_i \arrow[r, "\overline{v}_i"] \arrow[d, "\overline{\partial}_i"] & C''_i/B''_i \arrow[r] \arrow[d, "\overline{\partial}''_i"] & 0 \\ 0 \arrow[r] & Z'_{i-1} \arrow[r, "\widetilde{u}_{i-1}"] & Z_{i-1} \arrow[r, "\widetilde{v}_{i-1}"] & Z''_{i-1} & \end{tikzcd}$$

Note that $\ker \overline{\partial}_i = H_i(C_\cdot)$ and $\operatorname{coker} \overline{\partial}_i = H_{i-1}(C_\cdot)$ (likewise for $\overline{\partial}'_i$ and $\overline{\partial}''_i$). Then (a) and (b) are an immediate consequence of the Snake Lemma. (c) is easy to see.

(7.7) Definition: A morphism of complexes $u_\cdot : C_\cdot \longrightarrow C'_\cdot$ is called null-homotopic if there exists a sequence of R-linear maps $s_i : C_i \longrightarrow C'_{i+1}$

$$\begin{tikzcd} \cdots\cdots \arrow[r] & C_{i+1} \arrow[r] & C_i \arrow[r, "\partial_i"] \arrow[dl, "s_i"'] \arrow[d, "u_i"] & C_{i-1} \arrow[r] \arrow[dl, "s_{i-1}"'] & \cdots\cdots \\ \cdots\cdots \arrow[r] & C'_{i+1} \arrow[r, "\partial'_{i+1}"] & C'_i \arrow[r] & C'_{i-1} \arrow[r] & \cdots\cdots \end{tikzcd}$$

so that $u_i = \partial'_{i+1} s_i + s_{i-1} \partial_i$. Notation: $u_\cdot \sim 0$. Two morphism of complexes $u_\cdot, v_\cdot : C_\cdot \longrightarrow C'_\cdot$ are homotopic, $u_\cdot \sim v_\cdot$, if $u_\cdot - v_\cdot \sim 0$. The sequence of maps s_i is called a homotopy.

(7.8) Proposition: Let $u_\cdot, v_\cdot : C_\cdot \longrightarrow C'_\cdot$ be morphisms of complexes. If $u_\cdot \sim v_\cdot$ then $H_\cdot(u_\cdot) = H_\cdot(v_\cdot)$.

Proof: We have to show that $H_\cdot(u_\cdot) = 0$ if $u_\cdot \sim 0$. Let $z \in Z_i(C_\cdot)$. Then
$u_i(z) + B'_i = \partial'_{i+1} s_i(z) + s_{i-1}\partial_i(z) + B'_i = \partial'_{i+1} s_i(z) + B'_i = B'_i$.

(7.9) Definition: A complex $C_\cdot$ has contracting homotopy if its identity map $1_{C_\cdot}: C_\cdot \longrightarrow C_\cdot$ is null-homotopic.

(7.10) Proposition: A complex $C_\cdot$ with contracting homotopy is an exact sequence.

Proof: If $1_{C_\cdot} \sim 0$, then by (7.8) $H_i(1_{C_\cdot}) = H_i(0)$ for all $i \in \mathbb{Z}$. $H_i(1_{C_\cdot}): H_i(C_\cdot) \longrightarrow H_i(C_\cdot)$ is the identity map while $H_i(0)$ is the zero map.

(7.11) Example: Consider

$$\begin{array}{cccccccc} C_\cdot: & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 \\ & & & {\scriptstyle id}\downarrow & & \downarrow & & \\ C'_\cdot: & & 0 & \longrightarrow & & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

and notice that $H_1(u_\cdot) = 0$, but $u_\cdot \not\sim 0$.

(7.12) Definition: (a) A complex $C_\cdot: \ldots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ ($C^\cdot: 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \ldots$) is called acyclic if $H_i(C_\cdot) = 0$ ($H^i(C^\cdot) = 0$) for all $i \neq 0$.

(b) A projective resolution of a module M is an acyclic complex $P_\cdot$ with P_i projective modules for all i together with an isomorphism $H_0(P_\cdot) \cong M$ (or equivalently: $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is exact and P_i projective for all i).

(c) An injective resolution of a module M is an acyclic complex $I^\cdot$ with I^i injective modules for every i together with an isomorphism $H^0(I^\cdot) \cong M$ (or equivalently: $0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots$ is exact and I^i is injective for all i).

(7.13) Remark: Every module has a projective and an injective resolution.

(7.14) Proposition: (a) Let $C_\cdot: \ldots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ and $C'_\cdot: \ldots \longrightarrow C'_1 \longrightarrow C'_0 \longrightarrow 0$ be complexes where C_i are projective modules for all i and $C'_\cdot$ is acyclic. Then for every R-linear map $\varphi: H_0(C_\cdot) \longrightarrow H_0(C'_\cdot)$ there is a morphism of complexes $u_\cdot: C_\cdot \longrightarrow C'_\cdot$ with $H_0(u_\cdot) = \varphi$. Moreover, $u_\cdot$ is unique up to homotopy.

(b) Let $C^\cdot: 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots$ and $C'^\cdot: 0 \longrightarrow C'^0 \longrightarrow C'^1 \longrightarrow \dots$ be complexes where $C^\cdot$ is acyclic and C'^i are injective modules for all i. Then for every R-linear map $\varphi: H^0(C^\cdot) \longrightarrow H^0(C'^\cdot)$ there is a morphism of complexes $u^\cdot: C^\cdot \longrightarrow C'^\cdot$ with $H^0(u^\cdot) = \varphi$. Moreover, $u^\cdot$ is unique up to homotopy.

<u>Proof</u>: We construct u_i inductively. For $i=0$ consider:

$$\begin{tikzcd} C_0 \arrow[r, "\pi"] \arrow[d, dotted, "u_0"'] \arrow[dr, "\varphi\pi"] & H_0(C_\cdot) \arrow[d, "\varphi"] \\ C_0' \arrow[r, "\pi'"] & H_0(C_\cdot') \end{tikzcd}$$

$\varphi\pi$ can be lifted to an R-linear map $u_0: C_0 \longrightarrow C_0'$ since π' is surjective and C_0 is a projective module.

For the induction step assume that $u_0, \text{—}, u_i$ have been constructed. This yields a commutative diagram:

$$\begin{tikzcd} C_{i+1} \arrow[r, "\partial_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "u_i"] & C_{i-1} \arrow[d, "u_{i-1}"] \\ C_{i+1}' \arrow[r, "\partial_{i+1}'"] & C_i' \arrow[r, "\partial_i'"] & C_{i-1}' \end{tikzcd}$$

where to bottom row is exact. (for $i=0$ set $u_{-1} = \varphi: H_0(C_\cdot) \longrightarrow H_0(C_\cdot')$. Thus $u_i(\operatorname{im}\partial_{i+1}) \subseteq u_i(\ker\partial_i) \subseteq \ker\partial_i' = \operatorname{im}\partial_{i+1}'$. Since C_{i+1} is projective, we may lift the map $u_i\partial_{i+1}$ to an R-linear map $u_{i+1}: C_{i+1} \longrightarrow C_{i+1}'$ with $\partial_{i+1}' u_{i+1} = u_i \partial_{i+1}$.

Uniqueness: We show that if $u_\cdot$ is a morphism of complexes (with the properties of (a)) and with $H_0(u_\cdot) = 0$, then $u_\cdot \sim 0$. We construct the homotopy maps s_i inductively. For $i=0$, since $H_0(u_\cdot)=0$, we have $\operatorname{im} u_0 \subseteq \operatorname{im}\partial_1'$:

$$\begin{tikzcd} & C_0 \arrow[d, "u_0"] \\ C_1' \arrow[r, "\partial_1'"] & C_0' \end{tikzcd}$$

Since C_0 is projective, u_0 can be lifted to $s_0: C_0 \longrightarrow C_1'$ with $u_0 = \partial_1' s_0$. For the induction step assume that $s_0, \text{—}, s_i$ have been constructed

$$\begin{tikzcd} C_{i+2} \arrow[r, "\partial_{i+2}"] & C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "u_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "u_i"] \arrow[dl, "s_i"'] & C_{i-1} \arrow[d, "u_{i-1}"] \arrow[dl, "s_{i-1}"'] \\ C_{i+2}' \arrow[r, "\partial_{i+2}'"] & C_{i+1}' \arrow[r, "\partial_{i+1}'"] & C_i' \arrow[r, "\partial_i'"] & C_{i-1}' \end{tikzcd}$$

Then $\partial_{i+1}'(u_{i+1} - s_i\partial_{i+1}) = \partial_{i+1}' u_{i+1} - (\partial_{i+1}' s_i)\partial_{i+1} = \partial_{i+1}' u_{i+1} - (u_i - s_{i-1}\partial_i)\partial_{i+1} = \partial_{i+1}' u_i - u_i \partial_{i+1} = 0$.

Thus $\operatorname{im}(u_{i+1} - s_i\partial_{i+1}) \subseteq \ker\partial_{i+1}' = \operatorname{im}\partial_{i+1}'$, where the last equality follows since $i+1 > 0$

and $C'_\bullet$ is acyclic. Since C_{i+1} is projective there exists an R-linear map $s_{i+1}: C_{i+1} \longrightarrow C'_{i+2}$ so that $u_{i+1} - s_i \partial_{i+1} = \partial'_{i+2} s_{i+1}$.
(b) follows by similar arguments.

(7.15) Corollary: Let C and C' be projective (injective) resolutions of a module M. Then there exist morphisms of complexes $u: C \longrightarrow C'$ and $v: C' \longrightarrow C$ with $uv \sim \mathrm{id}$ and $vu \sim \mathrm{id}$.

(7.16) Definition: Let M be an R-module.
(a) If M has a finite projective resolution $0 \longrightarrow P_n \longrightarrow \dots \longrightarrow P_0 \longrightarrow 0$, then M is said to have finite projective dimension. In this case the smallest possible n is called the projective dimension of M. Notation: $\mathrm{projdim}_R M = \mathrm{projdim}\, M$.
(b) If M has a finite injective resolution $0 \longrightarrow I_0 \longrightarrow \dots \longrightarrow I_n \longrightarrow 0$, then M is said to have finite injective dimension. In this case the smallest possible n is called the injective dimension of M. Notation: $\mathrm{injdim}_R M = \mathrm{injdim}\, M$.

(7.17) Definition: (a) A free resolution of a module M is a projective resolution $F_\bullet$ of M with F_i free for all i.
(b) Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. A minimal free resolution of M is a free resolution $(F_\bullet, \partial_\bullet)$ of M with F_i a finite R-module for all i and $\mathrm{im}\, \partial_{i+1} \subseteq \mathfrak{m} F_i$ for all i.

(7.18) Remark: Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finite R-module. By (6.52) M has a minimal free resolution.

(7.19) Proposition: Let $(R, \mathfrak{m})$ be a local Noetherian ring, M a finite R-module, and $F_\bullet$ a minimal free resolution of M. Then:
(a) $F_\bullet$ is unique up to isomorphism.
(b) If $P_\bullet$ is a projective resolution of M, then $F_\bullet$ is isomorphic to a direct

summand of $P_\bullet$

Proof: (a) follows from (b).

(b) By (7.15) there are morphisms of complexes $v_\bullet : F_\bullet \longrightarrow P_\bullet$ and $w_\bullet : P_\bullet \longrightarrow F_\bullet$ so that $w_\bullet v_\bullet \sim id_{F_\bullet}$. Write $u_\bullet = w_\bullet v_\bullet$. We claim that $u_\bullet$ is an isomorphism. Since $u_\bullet \sim id_{F_\bullet}$, for all i: $u_i = id_{F_i} + \partial_{i+1} s_i + s_{i-1}\partial_i$. Since $F_\bullet$ is a minimal resolution, $\operatorname{im}(\partial_{i+1} s_i + s_{i-1}\partial_i) \subseteq \mathfrak{m} F_i$. Thus $F_i = \operatorname{im} u_i + \mathfrak{m} F_i$ and by Nakayama's Lemma $F_i = \operatorname{im} u_i$. Thus $u_i : F_i \longrightarrow F_i$ is surjective and hence an isomorphism (1.36). This implies that $w_i : P_i \longrightarrow F_i$ is surjective for all i and that the complex $F_\bullet$ is a direct summand of the complex $P_\bullet$.

Let M be an R-module. In the following we denote by $E_R(M)$ or $E(M)$ the injective hull of M (6.75).

(7.20) Definition: A minimal injective resolution of a module M is an injective resolution $(E^\bullet, \partial^\bullet)$ with $E^0 = E(M)$ and $E^{i+1} = E(\operatorname{coker} \partial^{i-1})$.

(7.21) Remark: Let M be an R-module. Then M has a minimal injective resolution.

(7.22) Proposition: Let M be an R-module and $E^\bullet$ a minimal injective resolution of M.
(a) $E^\bullet$ is unique up to isomorphism.
(b) If $I^\bullet$ is an injective resolution of M then $E^\bullet$ is isomorphic to a direct summand of $I^\bullet$.

Proof: similar to (7.19).

§2: DERIVED FUNCTORS

Similar to definition (6.5) we call a functor (contravariant functor) $F: \mathfrak{M}(R) \longrightarrow \mathfrak{M}(S)$ additive if for any two R-modules M, M' the induced map $\mathrm{Hom}_R(M,M') \longrightarrow \mathrm{Hom}_S(F(M),F(M'))$ ($\mathrm{Hom}_R(M,M') \longrightarrow \mathrm{Hom}_S(F(M'),F(M))$, respectively) is a homomorphism of abelian groups.

(7.23) Examples: Let N be an R-module and $I \subseteq R$ an ideal.

(a) $F = - \otimes_R N : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(R)$ given by $F(M) = M \otimes_R N$ and $F(f) = f \otimes_R \mathrm{id}_N$ is an additive functor which is right exact. F is exact if and only if N is flat.

(b) $F = \mathrm{Hom}_R(N, -) : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(R)$ given by $F(M) = \mathrm{Hom}_R(N, M)$ and $F(f) = \mathrm{Hom}_R(N, f)$ is an additive left exact functor. It is exact if and only if N is projective.

(c) $F = \mathrm{Hom}_R(-, N) : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(R)$ given by $F(M) = \mathrm{Hom}_R(M, N)$ and $F(f) = \mathrm{Hom}_R(f, N)$ is an additive contravariant left exact functor. It is exact if and only if N is injective.

(d) $F = \Gamma_I : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(R)$ given by $F(M) = \Gamma_I(M)$ and $F(f) = \Gamma_I(f)$ is an additive left exact functor.

Note that in all 4 examples the induced map on the Hom's is R-linear.

Let $F: \mathfrak{M}(R) \longrightarrow \mathfrak{M}(S)$ be an additive functor. For a complex $(C_\bullet, \partial_\bullet)$ of R-modules $(F(C_\bullet), F(\partial_\bullet))$ is a complex of S-modules with $F(C_\bullet)_i = F(C_i)$ and $\partial_i^{F(C_\bullet)} = F(\partial_i)$. For a morphism of complexes $u_\bullet : C_\bullet \longrightarrow C'_\bullet$ let $F(u_\bullet) : F(C_\bullet) \longrightarrow F(C'_\bullet)$ be given by $F(u)_i = F(u_i)$. $F(u_\bullet)$ is a morphism of complexes. If $u_\bullet, v_\bullet$ are morphisms of complexes with $u_\bullet \sim v_\bullet$, then $F(u_\bullet) \sim F(v_\bullet)$ since F is additive. In particular, if the complex $C_\bullet$ has contracting homotopy then so has $F(C_\bullet)$.

Let $F: \mathfrak{M}(R) \longrightarrow \mathfrak{M}(S)$ be an additive functor. For every R-module M fix a projective resolution $P_{M\bullet}$ of M and define $L_i F(M) = H_i(F(P_{M\bullet}))$. If $\varphi: M \longrightarrow M'$ is an R-linear map then by (7.14) there is a morphism of complexes $u_\bullet : P_{M\bullet} \longrightarrow P_{M'\bullet}$ with $H_0(u_\bullet) = \varphi$. Define $L_i F(\varphi) : L_i F(M) \longrightarrow L_i F(M')$ by $L_i F(\varphi) = H_i(F(u_\bullet))$. $L_i F(\varphi)$

is well defined. If $v_\cdot : P_{M\cdot} \longrightarrow P_{M'\cdot}$ is another morphism of complexes with $H_0(v_\cdot) = \varphi$ then by (7.14) $u_\cdot \sim v_\cdot$. Thus $F(u_\cdot) \sim F(v_\cdot)$ and by (7.8) $H_i(F(u_\cdot)) = H_i(F(v_\cdot))$. One easily checks that $L_iF: \mathcal{M}(R) \longrightarrow \mathcal{M}(S)$ are additive functors.

(7.24) <u>Definition</u>: The functors L_iF are called <u>left derived functors</u> of F.

(7.25) <u>Definition</u>: Two functors $F, G: \mathcal{M}(R) \longrightarrow \mathcal{M}(S)$ are <u>naturally equivalent</u>, $F \cong G$, if for every R-module M there is an isomorphism $t_M : F(M) \xrightarrow{\sim} G(M)$ so that for every $f \in \mathrm{Hom}_R(M, M')$ the following diagram commutes:

$$\begin{array}{ccc} F(M) & \xrightarrow[\sim]{t_M} & G(M) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(M') & \xrightarrow[\sim]{t_{M'}} & G(M') \end{array}$$

(similarly for contravariant functors)

For every R-module M fix some other projective resolution $\widehat{P}_M$ and use these to define $\widehat{L}_iF$.

(7.26) <u>Proposition</u>: $L_iF \simeq \widehat{L}_iF$

<u>Proof</u>: For every R-module M by (7.14) there are morphisms of complexes $u_\cdot\, P_{M\cdot} \longrightarrow \widehat{P}_{M\cdot}$ and $v_\cdot : \widehat{P}_{M\cdot} \longrightarrow P_{M\cdot}$ with $u_\cdot v_\cdot \sim \mathrm{id}_{\widehat{P}_M}$ and $v_\cdot u_\cdot \sim \mathrm{id}_M$. Hence $F(u_\cdot)F(v_\cdot) \sim \mathrm{id}_{F(\widehat{P}_M)}$ and $F(v_\cdot)F(u_\cdot) \sim \mathrm{id}_{F(P_M)}$. By (7.8) $H_i(F(u_\cdot))H_i(F(v_\cdot)) = \mathrm{id}$ and $H_i(F(v_\cdot))\, H_i(F(u_\cdot)) = \mathrm{id}$ and $t_M = H_i(F(u_\cdot)) : L_iF(M) \xrightarrow{\sim} \widehat{L}_iF(M)$ is an isomorphism of S-modules. Let $\varphi: M \longrightarrow M'$ be an R-linear map. Using (7.14) and (7.8) again one shows that the diagram:

$$\begin{array}{ccc} L_iF(M) & \xrightarrow{t_M} & \widehat{L}_iF(M) \\ {\scriptstyle L_iF(\varphi)}\downarrow & & \downarrow{\scriptstyle \widehat{L}_iF(\varphi)} \\ L_iF(M') & \xrightarrow{t_{M'}} & \widehat{L}_iF(M') \end{array}$$

commutes.

(7.27) <u>Proposition</u>: (a) If P is a projective module then $L_iF(P) = 0$ for all $i > 0$.
(b) If M has finite projective dimension then $L_iF(M) = 0$ for all $i > \mathrm{projdim}\, M$.

(c) If $0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is an exact sequence with P_j projective (K_n is called an <u>n-th syzygy module</u>) then $L_i F(M) \cong L_{i-n} F(K_n)$ for all $i > n$.
(d) If F is exact then $L_i F = 0$ for all $i > 0$.
(e) If F is right exact then $L_0 F \simeq F$.

<u>Proof</u>: (c) Let $\cdots P_{n+1} \longrightarrow P_n \longrightarrow 0$ be a projective resolution of K_n. Then $\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$ is a projective resolution of M. The statement follows from the definition of $L_i F$.
(d) Let $P_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$ be a projective resolution of a module M. Then $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is exact and $F(P_1) \longrightarrow F(P_0) \longrightarrow F(M) \longrightarrow 0$ is exact, since F is right exact. Thus $F(M) \cong H_0(F(P_\bullet)) = L_0 F(M)$. It is easy to see that this isomorphism is natural.

(17.28) <u>Lemma</u>: (Horseshoe Lemma) Let $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of R-modules and let $P'_\bullet$ and $P''_\bullet$ be projective resolutions of M' and M''. Then there is an exact sequence of morphisms of complexes $0 \longrightarrow P'_\bullet \xrightarrow{u_\bullet} P_\bullet \xrightarrow{v_\bullet} P''_\bullet \longrightarrow 0$ so that $P_\bullet$ is a projective resolution of M and $H_0(u_\bullet) = \varphi$ and $H_0(v_\bullet) = \psi$.

<u>Proof</u>: Consider the diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_1' \arrow[d] & & K_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\pi'"'] & & P_0'' \arrow[d, "\pi''"] & \\
0 \arrow[r] & M' \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Set $P_0 = P_0' \oplus P_0''$ and let $\pi : P_0 \longrightarrow M$ be the R-linear map with $\pi|_{P_0'} = \varphi\pi'$ and $\pi|_{P_0''}$ any lifting of the map π'' (such a lifting exists since P_0'' is projective and ψ is surjective. Let $u_0 : P_0' \to P_0$ and $v_0 : P_0 \to P_0''$ be the canonical maps. The diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0' & \xrightarrow{u_0} & P_0 & \xrightarrow{v_0} & P_0'' & \longrightarrow & 0 \\ & & \pi' \downarrow & & \pi \downarrow & & \pi'' \downarrow & & \\ 0 & \longrightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \end{array}$$

is commutative with exact rows. By the Snake Lemma (7.5) π is surjective and u_0 and v_0 induce an exact sequence $0 \to K_1' \xrightarrow{\varphi_1} K_1 = \ker(\pi) \xrightarrow{\psi_1} K_1'' \longrightarrow 0$. Continue.

(7.29) <u>Theorem</u>: Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of R-modules. Then there is a long exact sequence

$$\cdots \longrightarrow L_iF(M') \xrightarrow{L_iF(\varphi)} L_iF(M) \xrightarrow{L_iF(\psi)} L_iF(M'') \xrightarrow{\Delta_i} L_{i-1}F(M') \longrightarrow \cdots$$
$$\longrightarrow L_0F(M') \longrightarrow L_0F(M) \longrightarrow L_0F(M'') \longrightarrow 0$$

<u>Proof</u>: By (7.28) there is an exact sequence of morphisms of complexes:

$$0 \longrightarrow P_\bullet' \xrightarrow{u_\bullet} P_\bullet \xrightarrow{v_\bullet} P_\bullet'' \longrightarrow 0$$

where $P_\bullet', P_\bullet, P_\bullet''$ are projective resolutions of M', M, M'' and $H_0(u_\bullet) = \varphi$, $H_0(v_\bullet) = \psi$. For all i the sequence $0 \to P_i' \xrightarrow{u_i} P_i \xrightarrow{v_i} P_i'' \longrightarrow 0$ is split exact since P_i'' is projective. Hence $0 \to F(P_i') \xrightarrow{F(u_i)} F(P_i) \xrightarrow{F(v_i)} F(P_i'') \longrightarrow 0$ is exact and

$$0 \longrightarrow F(P_\bullet') \xrightarrow{F(u_\bullet)} F(P_\bullet) \xrightarrow{F(v_\bullet)} F(P_\bullet'') \longrightarrow 0$$

is an exact sequence of complexes. By (7.6) there is a long exact sequence of homology:

$$\cdots \longrightarrow H_i(F(P_\bullet')) \xrightarrow{H_i(F(u_\bullet))} H_i(F(P_\bullet)) \xrightarrow{H_i(F(v_\bullet))} H_i(F(P_\bullet'')) \xrightarrow{\Delta_i} H_{i-1}(F(P_\bullet')) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(F(P_\bullet'')) \longrightarrow H_{-1}(F(P_\bullet')) = 0$$

The assertion follows from the definition of the functors L_iF.

(7.30) <u>Remark</u>: Let F be right exact. Then every exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence:

$$\cdots \longrightarrow L_1F(M') \longrightarrow L_1F(M) \longrightarrow L_1F(M'') \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0.$$

(7.31) <u>Theorem</u>: Consider a commutative diagram of R-linear maps with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\varphi"] \arrow[d, "\gamma'"] & M \arrow[r, "\psi"] \arrow[d, "\gamma"] & M'' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\varepsilon"] & N \arrow[r, "\mu"] & N'' \arrow[r] & 0 .
\end{tikzcd}$$

Then there is a commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & L_i F(M') \arrow[r] \arrow[d, "L_iF(\gamma')"] & L_i F(M) \arrow[r] \arrow[d, "L_iF(\gamma)"] & L_i F(M'') \arrow[r, "\Delta_i"] \arrow[d, "L_iF(\gamma'')"] & L_{i-1} F(M') \arrow[r] \arrow[d, "L_{i-1}F(\gamma')"] & \cdots \\
\cdots \arrow[r] & L_i F(N') \arrow[r] & L_i F(N) \arrow[r] & L_i F(N'') \arrow[r, "\tilde{\Delta}_i"] & L_{i-1} F(N') \arrow[r] & \cdots
\end{tikzcd}$$

<u>Proof</u>: The result follows from (7.6)(c), the naturality of the long exact sequence of homology, once we have shown the following.

(7.32) <u>Lemma</u>: Let

$$\begin{tikzcd}
0 \arrow[r] & P_\cdot' \arrow[r, "u_\cdot"] \arrow[d, "g_\cdot'"'] & P_\cdot \arrow[r, "v_\cdot"] \arrow[d, dotted] & P_\cdot'' \arrow[r] \arrow[d, "g_\cdot''"] & 0 \\
0 \arrow[r] & Q_\cdot' \arrow[r, "x_\cdot"] & Q_\cdot \arrow[r, "y_\cdot"] & Q_\cdot'' \arrow[r] & 0
\end{tikzcd}$$

be morphisms of complexes with exact rows so that $P_\cdot', P_\cdot, P_\cdot'', Q_\cdot', Q_\cdot, Q_\cdot''$ are projective resolutions of M', M, M'', N', N, N'' and morphisms $u_\cdot, v_\cdot, g_\cdot', g_\cdot'', x_\cdot, y_\cdot$ induced by maps $\varphi, \psi, \gamma', \gamma'', \varepsilon, \mu$. Then there is a morphism of complexes $g_\cdot : P_\cdot \longrightarrow Q_\cdot$ with $H_0(g_\cdot) = \gamma$ so that the above diagram commutes.

<u>Proof</u>: We construct g_i inductively. To define g_0 we may assume $P_0 = P_0' \oplus P_0''$, $Q_0 = Q_0' \oplus Q_0''$ and that u_0, v_0, x_0, y_0 are the natural embeddings and projections.

$$\begin{tikzcd}
 & P_0' \arrow[rr, "u_0"] \arrow[dl, "g_0'"'] \arrow[dd, "\pi'"] & & P_0 \arrow[rr] \arrow[dl, dotted, "g_0"'] \arrow[dd, "\pi"] & & P_0'' \arrow[dl, "g_0''"] \arrow[dd, "\pi''"] \\
Q_0' \arrow[rr, "x_0"] \arrow[dd, "\tau'"'] & & Q_0 \arrow[rr] \arrow[dd, "\tau"] & & Q_0'' \arrow[dd, "\tau''"] & \\
 & M' \arrow[rr, "\varphi"] \arrow[dl, "\gamma'"] & & M \arrow[rr, "\psi"] \arrow[dl, "\gamma"] & & M'' \arrow[dl, "\gamma''"] \\
N' \arrow[rr, "\varepsilon"] & & N \arrow[rr, "\mu"] & & N'' &
\end{tikzcd}$$

Write $p = \pi|_{P_0''}$ and $t = \tau|_{Q_0''}$. Define $g_0 : P_0 = P_0' \oplus P_0'' \longrightarrow Q_0 = Q_0' \oplus Q_0''$ by $g_0 = \begin{pmatrix} g_0' & f \\ 0 & g_0'' \end{pmatrix}$ where $f : P_0'' \longrightarrow Q_0'$ is yet to be determined. Note that the two rectangles on the top commute already. We have to determine f so that $\tau g_0 = \gamma\pi$. We have that $\tau g_0 u_0 = \tau x_0 g_0' = \varepsilon\tau' g_0'$ and $\gamma\pi u_0 = \gamma\varphi\pi' = \varepsilon\gamma'\pi' = \varepsilon\tau' g_0'$ and therefore $\tau g_0|_{P_0'} = \gamma\pi|_{P_0'}$. Thus $\tau g_0 = \gamma\pi$ if and only if $\tau g_0|_{P_0''} = \gamma\pi|_{P_0''}$, which means $\varepsilon\tau' f + t g_0'' = \gamma p$, or equivalently, $\varepsilon\tau' f = \gamma p - t g_0''$. (Note that $g_0'' = g_0|_{P_0''}$). Since P_0'' is projective it follows that such an f exists provided that $\operatorname{im}(\gamma p - t g_0'') \subseteq \operatorname{im}(\varepsilon\tau')$:

$$\begin{tikzcd} & P \arrow[dl, dotted, "f"'] \arrow[d, "\gamma p - t g_0''"] & \\ Q_0' \arrow[r, "\varepsilon\tau'"] & \operatorname{im}(\varepsilon\tau') \arrow[r] & 0 \end{tikzcd}$$

But $\mu\gamma p = \gamma''\psi p = \gamma''\pi'' = \tau'' g_0'' = \mu t g_0''$ and $\mu(\gamma p - t g_0'') = 0$. Hence $\operatorname{im}(\gamma p - t g_0'') \subseteq \ker\mu = \operatorname{im}\varepsilon = \operatorname{im}(\varepsilon\tau')$ where the last equality follows from the surjectivity of τ'. Continue (with the same argument) by replacing M', M, M'', N', N, N'' by $\ker\pi', \ker\pi, \ker\pi'', \ker\tau', \ker\tau, \ker\tau''$ etc.

(7.33) <u>Remark</u>: Theorem (7.31) and its proof also show that the maps Δ_i constructed in the proof of (7.29) are determined by the exact sequence $0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ and do not depend on $0 \longrightarrow P_\cdot' \xrightarrow{u_\cdot} P_\cdot \xrightarrow{v_\cdot} P_\cdot'' \longrightarrow 0$.

For every R-module M fix an injective resolution $I_M^\cdot$. Let $F : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(S)$ be an additive functor. Define $R^i F(M) = H^i(F(I_M^\cdot))$. Let $\varphi : M \longrightarrow M'$ be an R-linear map. By (7.14) there is a morphism of complexes $u^\cdot : I_M^\cdot \longrightarrow I_{M'}^\cdot$ with $H_0(u^\cdot) = \varphi$. Define $R^i F(\varphi) : R^i F(M) \longrightarrow R^i F(M')$ by $R^i F(\varphi) = H^i(F(u^\cdot))$. By (7.14) and (7.8) $R^i F(\varphi)$ is well defined. $R^i F : \mathfrak{M}(R) \longrightarrow \mathfrak{M}(S)$ are additive functors. One can show as in (7.26) that they are independent of the choices of injective resolutions.

(7.34) <u>Definition</u>: The functors $R^i F$ are called <u>right derived functors</u> of F.

(7.35) <u>Theorem</u>: (a) If E is an injective module the $R^i F(E) = 0$ whenever $i > 0$.

(b) If M has finite injective dimension then $R^iF(M)=0$ for all $i>\operatorname{injdim} M$.

(c) If $0 \to M \to I^0 \to \dots \to I^{n-1} \to L \to 0$ is exact with I^j injective then $R^iF(M) = R^{i-n}F(L)$ for all $i>n$.

(d) If F is left exact then $R^0F \simeq F$.

(7.36) Theorem: (a) Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be an exact sequence of R-modules. Then there is a long exact sequence:

$$0 \to R^0F(M') \to R^0F(M) \to R^0F(M'') \to R^1F(M') \to \dots$$
$$\dots \to R^{i-1}F(M'') \xrightarrow{\Delta^i} R^iF(M') \xrightarrow{R^iF(\varphi)} R_iF(M) \xrightarrow{R^iF(\psi)} R^iF(M'') \to \dots$$

(b) The long exact sequence of (a) is natural.

Let $F: \mathcal{M}(R) \to \mathcal{M}(S)$ be an additive contravariant functor. For every R-module M fix a projective resolution $P_{M\cdot}$ and an injective resolution $I^\cdot_M$. Define $R^iF(M) = H^i(F(P_{M\cdot}))$, $L_iF(M) = H_i(F(I^\cdot_M))$, and for an R-linear map $\varphi: M \to M'$ define $R^iF(\varphi): R^iF(M') \to R^iF(M)$, and $L_iF(\varphi): L_iF(M') \to L_iF(M)$ in the obvious way. R^iF and L_iF are additive contravariant functors whose definitions do not depend on the choices of projective, injective resolutions. The functors R^iF are called right derived functors and L_iF left derived functors of F.

(7.37) Theorem: Let F be an additive contravariant functor.

(a) If P is a projective module then $R^iF(P)=0$ for all $i>0$.

(b) If M has finite projective dimension then $R^iF(M)=0$ for all $i>\operatorname{projdim} M$.

(c) If K is an nth syzygy module of M then $R^iF(M)=R^{i-n}F(K)$ for all $i>n$.

(d) If F is left exact then $R^0F \simeq F$.

(e) If $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is an exact sequence then there is a long exact sequence:

$$0 \to R^0F(M'') \to R^0F(M) \to R^0F(M') \to R^1F(M'') \to \dots$$
$$\dots \to R^{i-1}F(M') \xrightarrow{\Delta^i} R^iF(M'') \xrightarrow{R^iF(\psi)} R^iF(M) \xrightarrow{R^iF(\varphi)} R^iF(M') \to \dots$$

This long exact sequence is natural.

§3: Tor and Ext

(7.38) Definition: Let N be an R-module.

(a) $\mathrm{Tor}_i^R(-,N) = L_i(-\otimes_R N)$

(b) $\mathrm{Tor}_i^R(N,-) = L_i(N\otimes_R -)$

(c) $\mathrm{Ext}_R^i(-,N) = R^i\mathrm{Hom}_R(-,N)$

(d) $\mathrm{Ext}_R^i(N,-) = R^i\mathrm{Hom}_R(N,-)$

$\mathrm{Tor}_i^R(-,N)$ and $\mathrm{Tor}_i^R(N,-)$ are additive functors. Since $-\otimes_R N \simeq N\otimes_R -$, $\mathrm{Tor}_i^R(N,-)\simeq \mathrm{Tor}_i^R(-,N)$. $\mathrm{Tor}_0^R(-,N)\simeq -\otimes_R N$, since $-\otimes_R N$ is right exact. Moreover, $\mathrm{Tor}_i^R(P,N)=0$ if P projective and $i>0$.

(7.39) Theorem: If $0\longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$ is an exact sequence, then there is a long exact sequence:

$$\cdots\longrightarrow \mathrm{Tor}_i(M',N)\xrightarrow{\mathrm{Tor}_i(\varphi,N)}\mathrm{Tor}_i(M,N)\xrightarrow{\mathrm{Tor}_i(\psi,N)}\mathrm{Tor}_i(M'',N)\xrightarrow{\Delta_i}\mathrm{Tor}_{i-1}(M',N)\longrightarrow\cdots$$
$$\cdots\longrightarrow\mathrm{Tor}_1(M'',N)\longrightarrow M'\otimes N\longrightarrow M\otimes N\longrightarrow M''\otimes N\longrightarrow 0.$$

Furthermore this sequence is natural.

$\mathrm{Ext}_R^i(N,-)$ is an additive functor, $\mathrm{Ext}_R^0(N,-)\simeq\mathrm{Hom}_R(N,-)$ (since $\mathrm{Hom}_R(N,-)$ is left exact), and $\mathrm{Ext}_R^i(N,I)=0$ if I is injective and $i>0$.

$\mathrm{Ext}_R^i(-,N)$ is an additive contravariant functor, $\mathrm{Ext}_R^0(-,N)\simeq\mathrm{Hom}_R(-,N)$ (since $\mathrm{Hom}_R(-,N)$ is left exact), and $\mathrm{Ext}_R^i(P,N)=0$ if P is projective and $i>0$.

(7.40) Theorem: If $0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$ is an exact sequence, then there are long exact sequences:

(a) $0\longrightarrow\mathrm{Hom}(N,M')\longrightarrow\mathrm{Hom}(N,M)\longrightarrow\mathrm{Hom}(N,M'')\longrightarrow\mathrm{Ext}^1(N,M')\longrightarrow\cdots$

(b) $0\longrightarrow\mathrm{Hom}(M'',N)\longrightarrow\mathrm{Hom}(M,N)\longrightarrow\mathrm{Hom}(M',N)\longrightarrow\mathrm{Ext}^1(M'',N)\longrightarrow\cdots$

Furthermore these sequences are natural.

The symbols $\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$ are well defined due to the following fact:

(7.41) <u>Theorem</u>: (a) $\mathrm{Tor}_i^R(-,N)(M) \cong \mathrm{Tor}_i^R(M,-)(N)$

(b) $\mathrm{Ext}_R^i(-,N)(M) \cong \mathrm{Ext}_R^i(M,-)(N)$

<u>Proof</u>: We only prove (b), the proof of (a) is similar. Consider a projective resolution $P_\cdot$ of M and an injective resolution $I^\cdot$ of N and write:

$$\begin{tikzcd}[column sep=small]
\cdots & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & K_2 \arrow[ur] \arrow[dr] & & K_1 \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
0 \arrow[r] & N \arrow[r] & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[rr] \arrow[dr] & & I^2 & \cdots \\
& & & L^1 \arrow[ur] \arrow[dr] & & L^2 \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

The sequences $0 \to K_1 \to P_0 \to M \to 0$ and $0 \to N \to I^0 \to L^1 \to 0$ together with (7.40) yield a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}(M,N) \arrow[r] \arrow[d] & \mathrm{Hom}(M,I^0) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}(M,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1(M,-)(N) \arrow[r] & \mathrm{Ext}^1(M,-)(I^0)=0 \\
0 \arrow[r] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,L^1) \arrow[r] \arrow[d, "\gamma"] & 0 & \\
0 \arrow[r] & \mathrm{Hom}(K_1,N) \arrow[r] \arrow[d] & \mathrm{Hom}(K_1,I^0) \arrow[r, "\beta"] \arrow[d] & \mathrm{Hom}(K_1,L^1) \arrow[r] \arrow[d] & \mathrm{Ext}^1(K_1,-)(N) \arrow[r] & 0 \\
& \mathrm{Ext}^1(-,N)(M) \arrow[d] & 0 & \mathrm{Ext}^1(-,L^1)(M) \arrow[d] & & \\
& \mathrm{Ext}^1(-,N)(P_0)=0 & & 0 & &
\end{tikzcd}$$

By the Snake Lemma (7.5), $\mathrm{coker}\,\alpha \cong \mathrm{Ext}^1(-,N)(M)$, whereas by the first row of the diagram $\mathrm{coker}\,\alpha \cong \mathrm{Ext}^1(M,-)(N)$. Thus $\mathrm{Ext}^1(-,N)(M) \cong \mathrm{Ext}^1(M,-)(N)$ and we may write $\mathrm{Ext}^1(M,N)$ for this module.

In the above diagram we also have $\mathrm{im}\,\beta = \mathrm{im}\,\gamma$ and therefore $\mathrm{coker}\,\gamma \cong \mathrm{coker}\,\beta$ which gives: $\quad (*) \quad \mathrm{Ext}^1(M,L^1) \cong \mathrm{Ext}^1(K_1,N)$

Let $i > 1$. Then

$$\begin{aligned}
\mathrm{Ext}^i(-,N)(M) &\cong \mathrm{Ext}^1(-,N)(K_{i-1}) && \text{by } (7.37)(c)\\
&\cong \mathrm{Ext}^1(K_{i-1},N)\\
&\cong \mathrm{Ext}^1(K_{i-2},L^1) \cong \ldots \cong \mathrm{Ext}^1(M,L^{i-1}) && \text{by } (*)\\
&\cong \mathrm{Ext}^1(M,-)(L^{i-1})\\
&\cong \mathrm{Ext}^i(M,-)(N) && \text{by } (7.35)(c).
\end{aligned}$$

(7.42) Proposition: Let M be an R-module and $n>0$ a positive integer. The following are equivalent:

(a) $\mathrm{projdim}\, M \leq n$

(b) Every n-th syzygy module of M is projective.

(c) $\mathrm{Ext}^i_R(M,N)=0$ for all $i>n$ and every R-module N.

(d) $\mathrm{Ext}^{n+1}_R(M,N)=0$ for every R-module N.

Proof: (b) $\Rightarrow$ (a): clear

(a) $\Rightarrow$ (c): (7.37)(b)

(c) $\Rightarrow$ (d): clear

(d) $\Rightarrow$ (b): Let K_n be an nth syzygy of M. By (7.37)(c) $\mathrm{Ext}^1_R(K_n,N) \cong \mathrm{Ext}^{n+1}_R(M,N)=0$. Since $\mathrm{Ext}^1_R(K_n,N)=0$ for every R-module N, the long exact sequence (7.40)(a) shows that the functor $\mathrm{Hom}_R(K_n,-)$ is exact. Thus K_n is projective.

(7.43) Proposition: Let M be an R-module and $n>0$ a positive integer. The following are equivalent:

(a) $\mathrm{injdim}\, M \leq n$

(b) If $0 \to M \to I^0 \to \ldots \to I^{n-1} \to L^n \to 0$ is an exact sequence with I^j injective then L^n is injective.

(c) $\mathrm{Ext}^i_R(N,M)=0$ for all $i>n$ and every R-module N.

(d) $\mathrm{Ext}^{n+1}_R(N,M)=0$ for every R-module N.

(e) $\mathrm{Ext}^{n+1}_R(R/I,M)=0$ for every R-ideal I.

Proof: (e) $\Rightarrow$ (b): By (7.35) $\mathrm{Ext}^1_R(R/I,L^n) \cong \mathrm{Ext}^{n+1}_R(R/I,M)=0$ for every ideal I. Thus by (7.40)(b) the sequence $\mathrm{Hom}_R(R, L^n) \longrightarrow \mathrm{Hom}_R(I, L^n) \longrightarrow \mathrm{Ext}^1_R(R/I, L^n) = 0$ is exact for every ideal $I \subseteq R$. By (6.61) L^n is injective.

(7.44) <u>Corollary</u>: Let R be a ring, then
$\sup\{\mathrm{projdim}\, M \mid M \text{ an } R\text{-module}\} = \sup\{\mathrm{projdim}\, R/I \mid I \text{ an } R\text{-ideal}\} =$
$\sup\{\mathrm{injdim}\, M \mid M \text{ an } R\text{-module}\} = \sup\{n \mid \mathrm{Ext}^n_R(M,N) \neq 0 \text{ for some } R\text{-modules } M, N\}$.
This (not necessarily finite) number is called the <u>global dimension</u> of R, denoted <u>gl dim</u> R.

(7.45) <u>Examples</u>: (a) If R is a field then $\mathrm{gldim}\, R = 0$.
(b) If R is a Dedekind domain then $\mathrm{gldim}\, R = 1$ (since every ideal is principal).
(c) $\mathrm{gldim}(\mathbb{Z}/(4)) = \infty$ (Homework)

(7.46) <u>Definition</u>: (a) A <u>flat resolution</u> of a module M is an acyclic complex $F.$ with flat modules F_i for all i together with an isomorphism $H_0(F.) \cong M$.
(b) The <u>flat dimension</u> of M, $\mathrm{fldim}_R M = \mathrm{fldim}\, M$, is the minimal length of a flat resolution of M.

(7.47) <u>Proposition</u>: (a) If F is a flat R-module, then $\mathrm{Tor}^R_i(F, N) = 0$ for all $i > 0$ and all R-modules N.
(b) If $F.$ is a flat resolution of M, then $\mathrm{Tor}^R_i(M,N) \cong H_i(F. \otimes_R N)$ for all i.

<u>Proof</u>: (a) If F is R-flat then the functor $F \otimes_R -$ is exact. Thus $\mathrm{Tor}^R_i(F, -) = L_i(F \otimes_R -) = 0$ whenever $i > 0$ by (7.27)(b).
(b) By induction on i: If $i = 0$ then the claim holds since $- \otimes_R N$ is right exact. Write $0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ and $E.: \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow 0$ which is a flat resolution of K_1. Let $i = 1$. By the long exact sequence (7.39) one has an exact sequence $\mathrm{Tor}^R_1(F_0, N) = 0 \longrightarrow \mathrm{Tor}^R_1(M,N) \xrightarrow{\varphi} K_1 \otimes_R N \longrightarrow F_0 \otimes_R N$. Hence $\mathrm{Tor}^R_1(M,N) \cong \ker \varphi \cong \ker(F_1 \otimes N / \mathrm{im}(F_2 \otimes N) \longrightarrow F_0 \otimes N) = H_1(F. \otimes_R N)$. If $i > 1$, then by (7.39) $\mathrm{Tor}^R_i(M,N) \cong \mathrm{Tor}^R_{i-1}(K_1, N) \cong H_{i-1}(E. \otimes_R N) \cong H_i(F. \otimes_R N)$.

(7.48) Proposition: The following are equivalent for an integer $n \geq 0$:

(a) $\operatorname{fldim} M \leq n$

(b) If $0 \to K_n \to F_{n-1} \to \dots \to F_0 \to M \to 0$ is an exact sequence with F_j flat, then K_n is flat.

(c) $\operatorname{Tor}_i^R(M,N) = 0$ for all $i > n$ and every R-module N.

(d) $\operatorname{Tor}_{n+1}^R(M,N) = 0$ for every R-module N.

(e) $\operatorname{Tor}_{n+1}^R(M, R/I) = 0$ for every R-ideal I.

Proof: (a) $\Rightarrow$ (c): follows from (7.47)

(e) $\Rightarrow$ (b): By (7.47) $\operatorname{Tor}_1^R(K_n, R/I) \cong \operatorname{Tor}_{n+1}^R(M, R/I)$ for every R-ideal I. Then $0 \to I \to R \to R/I \to 0$ yields an exact sequence $0 \to I \otimes_R K_n \to R \otimes_R K_n \cong K_n$. Thus for every ideal I: $I \otimes_R K_n \xrightarrow{\cong} IK_n$ via the natural map. K_n is flat by a homework problem.

(7.49) Theorem: Let R be a ring, $S \subseteq R$ a multiplicative subset, and M, N R-modules. Then:

(a) $\operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N) \cong S^{-1}\operatorname{Tor}_i^R(M,N)$

(b) If R is Noetherian and M a finite R-module: $\operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N) \cong S^{-1}\operatorname{Ext}^i_R(M,N)$.

Proof: (b) By induction on i: If $i=0$ then by (6.46) $S^{-1}\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. For $i > 0$, consider the exact sequence $0 \to K \to F_{i-1} \xrightarrow{d} F_{i-2} \to \dots \to F_0 \to M \to 0$ where the F_j are finite free R-modules and $K = \ker d$. With $L = \operatorname{im} d$ we have exact sequences $0 \to K \to F_{i-1} \to L \to 0$ and $0 \to L \to F_{i-2} \to \dots \to F_0 \to M \to 0$. Since R is Noetherian, K and L are finitely generated. The long exact sequence $0 \to \operatorname{Hom}_R(L,N) \to \operatorname{Hom}_R(F_{i-1},N) \to \operatorname{Hom}_R(K,N) \to \operatorname{Ext}^1_R(L,N) \to 0$ yields that $\operatorname{Ext}^1_R(L,N) = \operatorname{coker}(\operatorname{Hom}_R(F_{i-1},N) \to \operatorname{Hom}_R(K,N))$. Since localization is exact:

$$\begin{aligned} S^{-1}(\operatorname{Ext}^1_R(L,N)) &\cong S^{-1}\operatorname{coker}(\operatorname{Hom}_R(F_{i-1},N) \to \operatorname{Hom}_R(K,N)) \\ &\cong \operatorname{coker}(S^{-1}\operatorname{Hom}_R(F_{i-1},N) \to S^{-1}\operatorname{Hom}_R(K,N)) \\ &\cong \operatorname{coker}(\operatorname{Hom}_{S^{-1}R}(S^{-1}F_{i-1}, S^{-1}N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}K, S^{-1}N)) \quad \text{by (6.46).} \end{aligned}$$

Using the exact sequence $0 \to S^{-1}K \to S^{-1}F_{i-1} \to S^{-1}L \to 0$ we see that the last module

is isomorphic to $\mathrm{Ext}^1_{S^{-1}R}(S^{-1}L, S^{-1}N)$. Thus $S^{-1}\mathrm{Ext}^1_R(L,N) \cong \mathrm{Ext}^1_{S^{-1}R}(S^{-1}L, S^{-1}N)$. By (7.27)
$\mathrm{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N) \cong \mathrm{Ext}^1_{S^{-1}R}(S^{-1}L, S^{-1}N) \cong S^{-1}\mathrm{Ext}^1_R(L,N) \cong S^{-1}\mathrm{Ext}^i_R(M,N)$.
(a) follows by a similar argument.

(7.50) <u>Corollary</u>: Let R be a Noetherian ring, M an R-module.
(a) $\mathrm{fldim}_R M = \sup\{\mathrm{fldim}_{R_m} M_m \mid m \in m\mathrm{Spec}\, R\}$
(b) If M is a finite R-module, $\mathrm{projdim}_R M = \sup\{\mathrm{projdim}_{R_m} M_m \mid m \in m\mathrm{Spec}\, R\}$
(c) $\mathrm{injdim}_R M = \sup\{\mathrm{injdim}_{R_m} M_m \mid m \in m\mathrm{Spec}\, R\}$
(d) $\mathrm{gldim}\, R = \sup\{\mathrm{gldim}\, R_m \mid m \in m\mathrm{Spec}\, R\}$

<u>Proof</u>: Use (7.49), (7.48), (7.44), (7.43), (7.42).

§4: MINIMAL RESOLUTIONS

A <u>free resolution</u> of a module M is a projective resolution $F_\cdot$ of M with F_i free for all i. Let (R,m) be a Noetherian local ring and M a finite R-module. A <u>minimal free resolution</u> of M is a free resolution $(F_\cdot, \partial_\cdot)$ of M with F_i finite and $\operatorname{im}\partial_{i+1} \subseteq mF_i$ for all i.

(7.51) <u>Remark and Definition</u>: Let (R,m) be a Noetherian local ring and M a finite R-module. The cardinality of every minimal generating set of M is the same, denoted by $\mu(M)$, and is called the <u>minimal number of generators</u> of M. By Nakayama, $\mu(M) = \dim_{R/m}(M/mM)$.

(7.52) <u>Proposition</u>: Let (R,m) be a Noetherian local ring and M a finite R-module. Then:

(a) M has a minimal free resolution $F_\cdot$

(b) $F_\cdot$ is unique up to isomorphism

(c) If $P_\cdot$ is a projective resolution of M, then $F_\cdot$ is isomorphic to a direct summand of $P_\cdot$

<u>Proof</u>: see (7.18) and (7.19)

(7.53) <u>Definition</u>: Let (R,m) be a Noetherian local ring with $k = R/m$ and M a finite R-module. The ith <u>Betti number</u> of M is defined by: $b_i(M) = \dim_k \operatorname{Tor}_i^R(k,M)$.

(7.54) <u>Theorem</u>: Let (R,m,k) be a Noetherian local ring and M a finite R-module. Then $b_i(M) = \dim_k \operatorname{Ext}_R^i(M,k)$ and for the minimal free R-resolution $F_\cdot$ of M one has that $\operatorname{rank} F_i = b_i(M)$.

<u>Proof</u>: Let $(F_\cdot, \partial_\cdot)$ be the minimal free R-resolution of M. Then $\operatorname{im}\partial_i \subseteq mF_{i-1}$ and hence $k\otimes_R \partial_i = 0$ and $\operatorname{Hom}_R(\partial_i, k) = 0$. Therefore $H_\cdot(k\otimes_R F_\cdot) = k\otimes_R F_\cdot$ and $H^\cdot(\operatorname{Hom}_R(F_\cdot,k)) = \operatorname{Hom}_R(F_\cdot,k)$. Write $F_i = R^{n_i}$. Then $b_i(M) = \dim_k(\operatorname{Tor}_i^R(k,M)) = \dim_k H_i(k\otimes_R F_\cdot) = \dim_k k\otimes_R F_i = n_i$ and $\dim_k \operatorname{Ext}_R^i(M,k) = \dim_k H^i(\operatorname{Hom}_R(F_\cdot,k)) = \dim_k \operatorname{Hom}_R(F_i,k) = n_i$.

(7.55) Corollary: Let R be a Noetherian ring and M a finite R-module. Then $\operatorname{projdim} M = \operatorname{fldim} M$.

Proof: By (7.50) we may assume that R is local with residue field k. Obviously, $\operatorname{projdim} M \geq \operatorname{fldim} M$. By (7.48) $b_i(M) = \operatorname{Tor}_i^R(k,M) = 0$ for $i > \operatorname{fldim} M$. Thus by (7.54) the minimal free resolution of M has length $\leq \operatorname{fldim} M$.

(7.56) Corollary: Let (R,m) be a Noetherian local ring with residue field k. Then $\operatorname{gldim} R = \operatorname{projdim}_R k = \operatorname{fldim}_R k = \operatorname{injdim}_R k$.

Proof: By (7.55) and (7.44) it suffices to show that for every finite R-module M, $\operatorname{projdim}_R M \leq \operatorname{projdim}_R k$ and $\operatorname{projdim}_R M \leq \operatorname{injdim}_R k$. However, $\operatorname{Tor}_i^R(k,M) = 0$ for $i > \operatorname{projdim}_R k$ and $\operatorname{Ext}_R^i(M,k) = 0$ for $i > \operatorname{injdim}_R k$. Now use (7.54).

(7.57) Definition: An R-module M is called indecomposable if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$. Otherwise it is called decomposable.

In the following the injective hull of an R-module M is denoted by $E(M)$ or $E_R(M)$.

(7.58) Remark: Let R be a ring, M an R-module, and $E \subseteq M$ an injective submodule. Then $M = E \oplus F$ for some submodule $F \subseteq M$.

Proof: Consider the diagram
$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, hook, "i"] \arrow[d, "\mathrm{id}"'] & M \arrow[dl, dashed, "f"] \\ & E & \end{tikzcd}$$
where i is the embedding. Since E is injective there is an R-linear map $f: M \longrightarrow E$ with $fi = \mathrm{id}_E$. Then $M = E \oplus \ker f$.

(7.59) Proposition: Let R be a Noetherian ring.

(a) For every prime ideal $P \subseteq R$ the injective hull $E_R(R/P)$ is indecomposable.

(b) Any indecomposable injective R-module is of the form $E_R(R/Q)$ for some $Q \in \operatorname{Spec} R$.

<u>Proof</u>: (a) Let $N_1, N_2 \subseteq E(R/P)$ be nonzero submodules. Since $E(R/P)$ is an essential extension of R/P, $N_1 \cap R/P = K_1 \neq 0$ and $N_2 \cap R/P = K_2 \neq 0$. K_1 and K_2 are nonzero ideals of the domain R/P, thus $0 \neq K_1K_2 \subseteq K_1 \cap K_2 \subseteq N_1 \cap N_2$.

(b) Let N be an indecomposable injective R-module. Since R is Noetherian, $\mathrm{Ass}_R(N) \neq \emptyset$. Let $Q \in \mathrm{Ass}_R(N)$, then $R/Q \subseteq N$. Since N is injective, there is an R-linear map $\varphi: E(R/Q) \longrightarrow N$ which extends the embedding $R/Q \hookrightarrow N$. Moreover, $\ker(\varphi) = 0$ since $E(R/Q)$ is an essential extension of R/Q and $R/Q \cap \ker(\varphi) = (0)$. $E(R/Q)$ is isomorphic to an injective submodule of N. By (7.58): $N \cong E(R/Q)$.

(7.60) <u>Proposition</u>: Let R be a Noetherian ring and $P \subseteq R$ a prime ideal.

(a) For every $a \in R-P$ multiplication by a induces an automorphism on $E(R/P)$.

(b) If $Q \in \mathrm{Spec}\, R$ with $P \neq Q$, then $E(R/P) \neq E(R/Q)$.

(c) For every $\xi \in E(R/P)$ there is an $n \in \mathbb{N}$ with $P^n \xi = 0$.

<u>Proof</u>: (a) Let $\varphi: E(R/P) \longrightarrow E(R/P)$ with $\varphi(\xi) = a\xi$ be the multiplication by a. Since $\ker\varphi \cap R/P = 0$, it follows that $\ker\varphi = 0$ and therefore $E(R/P) \cong \mathrm{im}\,\varphi$. $\mathrm{im}\,\varphi$ is an injective submodule of $E(R/P)$ with $R/P \subseteq \mathrm{im}\,\varphi$. Thus $E(R/P) = \mathrm{im}\,\varphi$.

(b) If $P \nsubseteq Q$, every element $a \in P-Q$ is regular on $E(R/Q)$ but not on $E(R/P)$.

(c) Since $R/P \subseteq E(R/P)$, $\mathrm{Ass}_R(R/P) = \{P\} \subseteq \mathrm{Ass}_R(E(R/P))$. Let $Q \in \mathrm{Ass}_R(E(R/P))$. Then $N = R/Q \subseteq E(R/P)$ and $N \cap R/P \neq 0$. Therefore $Q \in \mathrm{Ass}_R(R/P)$ and $P = Q$. This shows that $\mathrm{Ass}_R(E(R/P)) = \{P\}$. If $\xi \in E(R/P)$, then $R\xi \cong R/\mathrm{ann}(\xi)$ is a submodule of $E(R/P)$. Hence $\mathrm{Ass}_R(R/\mathrm{ann}(\xi)) = \{P\}$ and $\mathrm{ann}(\xi)$ is P-primary.

(7.61) <u>Proposition</u>: Let R be a Noetherian ring and $Q \subseteq P \subseteq R$ prime ideals. Then:

(a) $E_R(R/Q)$ is an R_P-module.

(b) $E_R(R/Q) = E_{R_P}(R_P/QR_P)$.

<u>Proof</u>: (a) By (7.60) for every $a \in R-P \subseteq R-Q$ multiplication by a is an isomorphism of

$E_R(R/Q)$. Thus $E_R(R/Q)$ is an R_P-module.

(b) By (a) $R/Q \subseteq (R/Q)_P \subseteq E_R(R/Q)$ and the R_P-module $E_R(R/Q)$ is an essential extension of the R_P-module $(R/Q)_P$. It remains to show that $E_R(R/Q)$ is injective as an R_P-module. Consider the diagram of R_P-modules and R_P-linear maps:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dotted, "h"] \\ & E(R/Q) & \end{tikzcd}$$

Since f and g are R-linear there is an R-linear map $h: M \longrightarrow E(R/Q)$ with $hf = g$. h is also R_P-linear and $E(R/Q)$ is an injective R_P-module.

(7.62) <u>Example</u>: Let R be a DVR with maximal ideal $m = (p)$, field of quotients $K = Q(R)$, and residue class field $k = R/m$. Then $E_R(R) = K$ and $E_R(k) = K/R$.

<u>Proof</u>: K is an essential extension of R and by (6.61) K is an injective R-module. Let $I = (p^r)$ be an ideal of R and $f: I \longrightarrow K/R$ an R-linear map. We need to extend f to an R-linear map $g: R \longrightarrow K/R$. Let $f(p^r) = [\alpha]$ for some $\alpha \in K$. Define $g: R \longrightarrow K/R$ by $g(1) = [\alpha/p^r]$. Then g extends f and K/R is an injective R-module. Moreover, $k = R/pR \cong p^{-1}R/R \subseteq K/R$. If $\beta \in K$ with $[\beta] \neq 0$ in K/R, then $\beta = u/p^n$ for some $u \in R^*$ and $n > 0$. Then $p^{n-1}[\beta] = [p^{n-1}\beta] = [p^{-1}u] \in k$ and K/R is an essential extension of k.

(7.63) <u>Lemma</u>: Let R be a Noetherian ring, $P \in \operatorname{Spec} R$, and M an R-module. Then:

(a) $\operatorname{Ass}_R(E(M)) = \operatorname{Ass}_R(M)$

(b) $\operatorname{Hom}_{R_P}(k(P), E(R/P)_P) \cong k(P)$

<u>Proof</u>: (a) Since $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(E(M))$, it suffices to show that $\operatorname{Ass}(E(M)) \subseteq \operatorname{Ass}(M)$. Let $Q \in \operatorname{Ass}(E(M))$. Then there is a submodule $N \subseteq E(M)$ with $N \cong R/Q$. Since $E(M)$ is an essential extension of M, $N \cap M \neq 0$. Thus $\emptyset \neq \operatorname{Ass}(N \cap M) \subseteq \operatorname{Ass}(N) = \{Q\}$ and $\{Q\} = \operatorname{Ass}(N \cap M) \subseteq \operatorname{Ass}(M)$.

(b) By (7.61) $E_R(R/P)_P = E_R(R/P) = E_{R_P}(k(P))$. Thus we may replace R by R_P to assume that

R is local with maximal ideal $P = m$ and residue field $k = R/m$. We have to show that $\mathrm{Hom}_R(k, E(k)) \cong k$. $\mathrm{Hom}_R(k, E(k))$ can be identified with $0 :_{E(k)} m \subseteq E(k)$. Obviously, $k \subseteq 0 :_{E(k)} m$. Suppose $k \subsetneq 0 :_{E(k)} m$. Then the k-vector space $0 :_{E(k)} m$ contains a nontrivial subspace N with $N \cap k = 0$. But this is impossible, since $k \subseteq E(k)$ is an essential extension.

(7.64) <u>Theorem</u>: Let R be a Noetherian ring and E an injective R-module. Then

(a) E is a direct sum of indecomposable injective R-modules.

(b) For $P \in \mathrm{Spec}\, R$, $E(R/P)$ appears in this decomposition if and only if $P \in \mathrm{Ass}(E)$. The multiplicity with which $E(R/P)$ appears is $\dim_{k(P)} \mathrm{Hom}_{R_P}(k(P), E_P)$. In particular, the direct sum decomposition of E is unique.

<u>Proof</u>: (a) Let $\Gamma = \{ S \mid S$ a set of indecomposable injective submodules of E with $\sum_{I \in S} I = \bigoplus_{I \in S} I \}$ be partially ordered by inclusion. If $P \in \mathrm{Ass}(E)$, then $E(R/P) \subseteq E$ and $\Gamma \neq \emptyset$. By Zorn's Lemma Γ has a maximal element S. Set $E' = \bigoplus_{I \in S} I$. Since R is Noetherian, E' is injective. (Homework). Thus $E = E' \oplus E''$ by (7.58). If $E'' = 0$, we are done. If $E'' \neq 0$, there is a $P \in \mathrm{Ass}(E'')$ and $E(R/P) \subseteq E''$ since E'' is injective (6.21). Thus $E' \cap E(R/P) = 0$. By (7.59) $E(R/P)$ is an indecomposable injective submodule and $S \subsetneq S \cup \{E(R/P)\} \in \Gamma$, contradiction.

(b) Let $E = \bigoplus_{I \in S} I$, where $I \neq 0$ are indecomposable injective submodules of E. Then each I is of the form $E(R/P)$ for some $P \in \mathrm{Spec}(R)$ and $\mathrm{Ass}(E(R/P)) = \{P\}$ by (7.63). Finally, $\mathrm{Ass}(E) = \bigcup_{I \in S} \mathrm{Ass}(I)$. This shows (b).

In order to show the last statement let $P \in \mathrm{Ass}(E)$. Then $\mathrm{Hom}_{R_P}(k(P), E_P) \cong \mathrm{Hom}_{R_P}(k(P), \bigoplus_{I \in S} I_P) \cong \bigoplus_{I \in S} \mathrm{Hom}_{R_P}(k(P), I_P)$, since $k(P)$ is a finite R_P-module. (Homework) By (7.63) $k(P) \cong \mathrm{Hom}_{R_P}(k(P), E(R/P)_P)$. It remains to show that $\mathrm{Hom}_{R_P}(k(P), E(R/Q)_P) = 0$ for $P \neq Q \in \mathrm{Spec}\, R$. If $Q \not\subseteq P$, then $Q \cap (R - P) \neq \emptyset$ and $E(R/Q)_P = 0$ by (7.60)(c). If $Q \subsetneq P$ by (7.60)(a) every element $a \in P - Q$ is a NZD on $E(R/Q)$. Thus no nonzero element of $E(R/Q) = E(R/Q)_P$ is annihilated by P. Hence if $P \neq Q$, $\mathrm{Hom}_{R_P}(k(P), E(R/Q)_P) = 0$.

(7.65) Definition: A minimal injective resolution of a module M is an injective resolution $(E^\bullet, \partial^\bullet)$ so that $E^i = E_R(Z^i(E^\bullet))$ for all i.

(7.66) Remark: Every R-module has a minimal injective resolution $E^\bullet$. Furthermore by (7.22):

(a) $E^\bullet$ is unique up to isomorphisms.

(b) If $I^\bullet$ is an injective resolution of M, then $E^\bullet$ is isomorphic to a direct summand of $I^\bullet$.

(7.67) Definition: Let R be a Noetherian ring and M a finite R-module. For $P \in \operatorname{Spec} R$, $\mu_i(P, M) = \dim_{k(P)} \operatorname{Ext}^i_{R_P}(k(P), M_P)$ is called the ith Bass number of M with respect to P.

(7.68) Remark: The Bass numbers $\mu_i(P,M)$ are finite as can be seen by taking a free R_P-resolution $F_\bullet$ of $k(P)$ where all F_j are finite R_P-modules.

(7.69) Theorem: Let R be a Noetherian ring and M a finite R-module. If $E^\bullet$ is a minimal injective R-resolution of M, then $E^i \cong \bigoplus_{P \in \operatorname{Spec}(R)} E(R/P)^{\mu_i(P,M)}$.

Proof: By (7.64) we have to show that $\dim_{k(P)} \operatorname{Hom}_{R_P}(k(P), E^i_P) = \mu_i(P,M)$ for every prime ideal $P \in \operatorname{Spec}(R)$. Fix $P \in \operatorname{Spec}(R)$. Since $E^\bullet_P$ is a minimal injective R_P-resolution of M_P (Homework), we may replace R by R_P. Write m for the maximal ideal of R and k for R/m. It suffices to show $\operatorname{Hom}_R(k, E^i) \cong \operatorname{Ext}^i_R(k, M)$. Since $\operatorname{Ext}^i_R(k,M) = H^i(\operatorname{Hom}_R(k, E^\bullet))$, this will follow once we have shown that the differential on $\operatorname{Hom}_R(k, E^\bullet)$ is trivial. Note that $\operatorname{Hom}_R(k, E^\bullet) \cong C^\bullet$ where $C^\bullet$ is the subcomplex of $E^\bullet$ with $C^i = 0 :_{E^i} m$.
If $0 \longrightarrow M \xrightarrow{\partial^{-1}} E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \longrightarrow \ldots$ then $\partial^i(C^i) = 0$ for all $i \geq 0$ if $C^i \subseteq \operatorname{im} \partial^{i-1}$.
Let $x \in C^i$. Since the extension $\operatorname{im} \partial^{i-1} \subseteq E^i$ is essential, there is an $a \in R$ with $0 \neq ax \in \operatorname{im} \partial^{i-1}$. As $mx = 0$ it follows that $a \in R - m = R^*$ and $x \in \operatorname{im} \partial^{i-1}$.

§5: MORE ON FLATNESS

A homomorphism of rings $\varphi: R \to S$ is called flat (faithfully flat) if S as an R-module is flat (faithfully flat). Equivalently one says that S is a flat (faithfully flat) R-algebra.

(7.70) Proposition: Let S be an R-algebra and M an S-module.
(a) If S is flat (faithfully flat) over R and M is flat (faithfully flat) over S, then M is flat (faithfully flat) over R.
(b) Let M be faithfully flat over S. Then S is flat (faithfully flat) over R if and only if M is flat (faithfully flat) over R.

Proof: Let $\mathcal{L}_\bullet: \ldots \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow \ldots$ be a sequence of R-modules and R-linear maps. Then $(\mathcal{L}_\bullet \otimes_R S) \otimes_S M \cong \mathcal{L}_\bullet \otimes_R (S \otimes_S M) \cong \mathcal{L}_\bullet \otimes_R M$.

(7.71) Lemma: Let $\varphi: R \to S$ be a homomorphism of rings, $Q \in \operatorname{Spec} S$, $P = \varphi^{-1}(Q)$, M an S-module, and N an R-module. Then $\operatorname{Tor}_i^R(M,N)_Q \cong \operatorname{Tor}_i^{R_P}(M_Q, N_P)$.

Proof: Let $F_\bullet$ be a free R-resolution of N. Since R_P is R-flat, $R_P \otimes_R F_\bullet$ is a free R_P-resolution of $R_P \otimes_R N \cong N_P$. Then

$$\begin{aligned}
\operatorname{Tor}_i^R(M,N)_Q &\cong S_Q \otimes_S \operatorname{Tor}_i^R(M,N) \\
&= S_Q \otimes_S H_i(M \otimes_R F_\bullet) \\
&\cong H_i(S_Q \otimes_S (M \otimes_R F_\bullet)) \qquad \text{since } S_Q \text{ is } S\text{-flat} \\
&\cong H_i(M_Q \otimes_R F_\bullet) \\
&\cong H_i((M_Q \otimes_R R_P) \otimes_R F_\bullet) \\
&\cong H_i(M_Q \otimes_R (R_P \otimes_R F_\bullet)) \\
&\cong \operatorname{Tor}_i^{R_P}(M_Q, N_P).
\end{aligned}$$

By (6.51) an R-module M is flat if and only if M_m is R_m-flat for all $m \in m\operatorname{Spec} R$.

(7.72) <u>Proposition</u>: Let $\varphi: R \longrightarrow S$ be a homomorphism of rings and M an S-module. For $Q \in \operatorname{Spec} S$ set $P = \varphi^{-1}(Q) \in \operatorname{Spec} R$. Then the following are equivalent:

(a) M is R-flat

(b) M_Q is R_P-flat for all $Q \in \operatorname{Spec} S$

(c) M_Q is R_P-flat for all $Q \in m\operatorname{Spec} S$.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (7.48) we have to show that for every R_P-ideal J, $\operatorname{Tor}_1^{R_P}(M_Q, R_P/J) = 0$. Since $J = I_P$ for some R-ideal I, by (7.71) $\operatorname{Tor}_1^{R_P}(M_Q, R_P/J) \cong \operatorname{Tor}_1^{R_P}(M_Q, (R/I)_P) \cong \operatorname{Tor}_1^{R}(M, R/I)_Q$. The latter module vanishes by (7.48).

(c) $\Rightarrow$ (a): By (7.48) we have to show that for every R-ideal I, $\operatorname{Tor}_1^R(M, R/I) = 0$. By (7.71) for every $Q \in m\operatorname{Spec} S$, $\operatorname{Tor}_1^R(M, R/I)_Q \cong \operatorname{Tor}_1^{R_P}(M_Q, R_P/I_P) = 0$ by (7.48). Since the S-module $\operatorname{Tor}_1^R(M, R/I)$ vanishes at every maximal ideal of S, by the Local-global principle $\operatorname{Tor}_1^R(M, R/I) = 0$.

Let $\varphi: R \longrightarrow S$ be a homomorphism of rings and let ${}^a\varphi: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ denote the induced map given by ${}^a\varphi(Q) = \varphi^{-1}(Q)$. Note that ${}^a\varphi$ is a continuous map. For $P \in \operatorname{Spec} R$, write $k(P) = R_P/PR_P$. Then $k(P) \otimes_R S \cong W^{-1}S/PW^{-1}S$ where $W = R - P$ and:

$$\begin{aligned}\operatorname{Spec}(k(P) \otimes_R S) &= \{Q(k(P) \otimes_R S) \mid Q \in \operatorname{Spec} S,\ Q \cap \varphi(W) = \emptyset,\ Q \supseteq \varphi(P)\} \\ &= \{Q(k(P) \otimes_R S) \mid Q \in \operatorname{Spec} S,\ \varphi^{-1}(Q) = P\} \\ &\underset{\text{homeo}}{\cong} \{Q \mid Q \in \operatorname{Spec} S,\ \varphi^{-1}(Q) = P\} = ({}^a\varphi)^{-1}(P).\end{aligned}$$

$\operatorname{Spec}(k(P) \otimes_R S)$ is called the <u>fiber</u> over P.

(7.73) <u>Theorem</u>: Let $\varphi: R \longrightarrow S$ be a homomorphism of rings and let M be an S-module.

(a) If M is faithfully flat over R then ${}^a\varphi(\operatorname{Supp}_S(M)) = \operatorname{Spec}(R)$.

(b) Let M be a finite S-module. M is faithfully flat over R if and only if M is R-flat and $m\operatorname{Spec} R \subseteq {}^a\varphi(\operatorname{Supp}(M))$.

<u>Proof</u>: (a) Let $P \in \operatorname{Spec} R$ and $T = k(P) \otimes_R S$. Then $T \otimes_S M = k(P) \otimes_R S \otimes_S M \cong k(P) \otimes_R M$ and

$k(P)\otimes_R M \neq 0$, since M is faithfully flat over R. Thus for the T-module $T\otimes_S M$: $\mathrm{Supp}_T(k(P)\otimes_R M)\neq\emptyset$. Pick $Q_0\in\mathrm{Supp}_T(k(P)\otimes_R M)$ and let Q be the preimage of Q_0 in S. Then $\varphi^{-1}(Q)=P$. Since $(k(P)\otimes_R M)_{Q_0}$ is a homomorphic image of M_Q, we have that $M_Q\neq 0$ and $Q\in\mathrm{Supp}_S(M)$.

(b) We have to show that if M is flat over R and $m\mathrm{Spec}\,R\subseteq {}^a\varphi(\mathrm{Supp}_S(M))$ then M is faithfully flat over R. By (6.38) it suffices to show that for every $m\in m\mathrm{Spec}\,R$, $M\neq mM$. By assumption there is a $Q\in\mathrm{Spec}\,S$ with $\varphi^{-1}(Q)=m$ and $M_Q\neq 0$. Suppose that $M=mM$. Then $M=QM$, since $\varphi(m)\subseteq Q$, and therefore $M_Q=QM_Q$. M_Q is a finite module over the local ring S_Q and by Nakayama's lemma $M_Q=0$, contradiction.

(7.74) <u>Corollary</u>: Let $\varphi: R\longrightarrow S$ be a homomorphism of rings. The following are equivalent:

(a) S is faithfully flat over R.

(b) S is flat over R and φ satisfies lying over (i.e. ${}^a\varphi$ is surjective).

(c) S is flat over R and for all $m\in m\mathrm{Spec}(R)$ there is a $Q\in\mathrm{Spec}(S)$ lying over m.

Let (R,m) and (S,n) be local rings. A homomorphism of rings $\varphi: R\longrightarrow S$ is called <u>local</u> if $\varphi(m)\subseteq n$, i.e. $\varphi^{-1}(n)=m$. By (7.74) a local homomorphism is faithfully flat if and only if it is flat. A homomorphism of rings $\varphi: R\longrightarrow S$ is said to satisfy <u>going down</u> if for every chain of prime ideals $P_0\supset P_1\supset\ldots\supset P_n$ and every $Q_0\in\mathrm{Spec}(S)$ with $\varphi^{-1}(Q_0)=P_0$ there exists a chain $Q_0\supset Q_1\supset\ldots\supset Q_n$ with $Q_i\in\mathrm{Spec}(S)$ and $\varphi^{-1}(Q_i)=P_i$. If φ satisfies going down and lying over then for every R-ideal I, $\mathrm{ht}\,IB\geq\mathrm{ht}\,I$.

(7.75) <u>Theorem:</u> Let $\varphi: R\longrightarrow S$ be a flat homomorphism of rings. Then φ satisfies going down.

<u>Proof</u>: By induction it suffices to consider a chain of length one, $P_0\supset P_1$, $P_i\in\mathrm{Spec}(R)$. Let $Q_0\in\mathrm{Spec}(S)$ with $\varphi^{-1}(Q_0)=P_0$. By (7.72) S_{Q_0} is flat over R_{P_0}. Since the homomorphism $R_{P_0}\longrightarrow S_{Q_0}$ is local, it is faithfully flat and satisfies lying over. Thus there is a $Q_1'\in\mathrm{Spec}(S_{Q_0})$ lying over $P_1R_{P_0}$. With Q_1 the preimage of Q_1' in S, Q_1 lies over P_1 and $Q_1\subset Q_0$.

(7.76) Theorem: (a) Let R be a ring, M a flat R-module, N an R-module and N_1, N_2 submodules of N. Then as submodules of $N \otimes_R M$ one has that $(N_1 \cap N_2) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M)$.
(b) Let $\varphi: R \longrightarrow S$ be a flat homomorphism of rings and $I_1, I_2 \subseteq R$ ideals. Then $(I_1 \cap I_2) S = I_1 S \cap I_2 S$.
(c) If in addition I_2 is finitely generated, then $(I_1 : I_2) S = I_1 S : I_2 S$.

Proof: (a) Consider the R-linear map $f: N \longrightarrow N/N_1 \oplus N/N_2$ defined by $f(x) = (x+N_1, x+N_2)$. Then $\ker(f) = N_1 \cap N_2$ and $0 \longrightarrow N_1 \cap N_2 \longrightarrow N \longrightarrow N/N_1 \oplus N/N_2$ is exact. Since M is flat, the sequence $0 \longrightarrow (N_1 \cap N_2) \otimes M \longrightarrow N \otimes M \xrightarrow{f \otimes M} N\otimes M/N_1 \otimes M \oplus N \otimes M / N_2 \otimes M$ is exact. Thus $(N_1 \cap N_2) \otimes M = (N_1 \otimes M) \cap (N_2 \otimes M)$.
(b) If S is flat over R, for every R-ideal I the natural map $I \otimes_R S \longrightarrow IS$ is an isomorphism. The statement follows from (a).
(c) First consider the case where $I_2 = (a)$ is principal. Define $f: R \xrightarrow{a} R/I_1$ by $f(x) = ax + I_1$. Then $\ker(f) = I_1 : (a)$ and $0 \longrightarrow I_1 : (a) \longrightarrow R \xrightarrow{a} R/I_1$ is exact. Tensoring with S and using the R-flatness of S yields an exact sequence $0 \longrightarrow (I_1 : (a)) S \longrightarrow S \xrightarrow{a} S/I_1 S$. Thus $(I_1 : (a)) S = I_1 S : (a) S$. If $I_2 = (a_1, \text{—}, a_n)$ is a finitely generated R-ideal, then $I_1 : I_2 = \bigcap_{i=1}^{n} (I_1 \cap (a_i))$. By the above case: $(I_1 : (a_i)) S = I_1 S : (a_i) S$ and therefore with (b): $(\bigcap_{i=1}^{n} (I_1 : (a_i))) S = (I_1 : I_2) S = \bigcap_{i=1}^{n} (I_1 : (a_i)) S = \bigcap_{i=1}^{n} (I_1 S : (a_i) S) = I_1 S : I_2 S$.

(7.77) Theorem: Let $\varphi: R \longrightarrow S$ be a faithfully flat homomorphism of rings.
(a) For every R-module M, the map $M \cong M \otimes_R R \xrightarrow{M \otimes \varphi} M \otimes_R S$ is injective.
(b) φ is injective
(c) $\varphi^{-1}(IS) = I$ for every R-ideal I.

Proof: (a) Let $\psi: M \longrightarrow M \otimes_R S$ be given by $\psi(m) = m \otimes 1$ and let $U = \ker \psi$. Since S is flat over R, $U \otimes_R S \subseteq M \otimes_R S$ and $U \otimes_R S = 0$ by definition of U. Since S is faithfully flat over R, $U = 0$ and ψ is injective.
(b) Apply (a) with $M = R$.

(b) Apply (a) with $M = R/I$. Then $\psi: R/I \longrightarrow S/IS$ is injective and $0 = \ker\psi = \varphi^{-1}(IS)/I$.

(7.78) <u>Theorem</u>: Let M be an R-module. The following are equivalent:

(a) M is flat

(b) $\operatorname{Tor}_1^R(M, R/I) = 0$ for every finitely generated R-ideal I.

(c) $I \otimes_R M \xrightarrow{\sim} IM$ via the natural map for every finitely generated R-ideal I.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (7.48)

(b) $\Rightarrow$ (c): Follows from the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$.

(c) $\Rightarrow$ (a): Let $I \subseteq R$ be any ideal. By Homework, it suffices to show that $I \otimes_R M \cong IM$ via the natural map. $I = \bigcup_{\lambda\in\Lambda} I_\lambda$ with $\{I_\lambda \mid \lambda \in \Lambda\}$ the set of all finitely generated R-ideals contained in I. $\{I_\lambda \mid \text{incl.}\}$ is a direct system with $I \cong \varinjlim I_\lambda$. Also, $\{R_\lambda = R; \text{id}\}$ is a direct system with $R \cong \varinjlim R_\lambda$. By assumption (c), the natural maps $I_\lambda \otimes_R M \longrightarrow R_\lambda \otimes_R M \cong M$ are injective. Thus the induced map $\varinjlim (I_\lambda \otimes_R M) \longrightarrow \varinjlim (R_\lambda \otimes_R M)$ is injective by (6.92). By (6.93) $\varinjlim (I_\lambda \otimes_R M) \cong (\varinjlim I_\lambda) \otimes_R M \cong I \otimes_R M$ and $\varinjlim (R_\lambda \otimes_R M) \cong (\varinjlim R_\lambda) \otimes_R M \cong R \otimes_R M \cong M$. Thus the natural map $I \otimes_R M \longrightarrow M$ is injective and $I \otimes_R M \cong M$ via the natural map.

(7.79) <u>Theorem</u>: (Equitorial criterion for flatness) Let M be an R-module. Consider the following condition $(*)$: Given an $r \times n$ system of linear equations:

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \quad a_{ij} \in R, \ x_j \in M, \ 1 \le i \le r,$$

there exists an integer s, elements $y_k \in M$ with $1 \le k \le s$, elements $b_{jk} \in R$ with $1 \le j \le n$, $1 \le k \le s$ so that

$$\sum_{j=1}^{n} a_{ij} b_{jk} = 0 \text{ for every } i,k \quad \text{and} \quad x_j = \sum_{k=1}^{s} b_{jk} y_k$$

i.e. every solution in M is a linear combination of solutions in R with coefficients in M.

Then:

(a) If M is flat then $(*)$ holds.

(b) If $(*)$ holds for $r=1$, then M is flat.

Proof: Consider $\varphi: R^n \longrightarrow R^r$ given by the $r \times n$ matrix (a_{ij}) and let $K = \ker \varphi \subseteq R^n$. The vector $(b_1, -, b_n) \in R^n$ is in K if and only if $\sum_i a_{ij} b_j = 0$ for $1 \leq i \leq r$. Consider $\varphi \otimes M : R^n \otimes_R M = M^n \longrightarrow R^r \otimes_R M = M^r$. Then $(x_1, -, x_n) \in M^n$ is in $\ker(\varphi \otimes M)$ if and only if $\sum_i a_{ij} x_j = 0$ for $1 \leq i \leq r$. $(*)$ means that $\ker(\varphi \otimes M)$ is contained in the image of $K \otimes M$ in $R^n \otimes_R M = M^n$. Thus $(*)$ is equivalent to $(**)$: If $0 \to K \longrightarrow R^n \longrightarrow R^r$ is exact, then $K \otimes M \longrightarrow R^n \otimes M \longrightarrow R^r \otimes M$ is exact.

(a) If M is flat $(**)$ holds.

(b) By (7.78) we have to show that $I \otimes M \xrightarrow{\sim} IM$ via the natural map for every finitely generated R-ideal I. Let I be generated by n elements. Then there is an exact sequence $0 \to K \longrightarrow R^n \longrightarrow R \longrightarrow R/I \longrightarrow 0$. By $(**)$ for $r=1$ the sequence $K \otimes M \longrightarrow M^n \longrightarrow M$ is exact. Thus $K \otimes M \xrightarrow{f} M^n \longrightarrow IM \longrightarrow 0$ is exact. From $K \longrightarrow R^n \longrightarrow I \longrightarrow 0$ and the right exactness of $\otimes_R M$ we obtain an exact sequence $K \otimes M \xrightarrow{f} M^n \longrightarrow I \otimes M \longrightarrow 0$. Thus $I \otimes M \cong \operatorname{coker} f \cong IM$.

The local flatness criterion

(7.80) Definition: Let R be a ring, $I \subseteq R$ an ideal, and M an R-module. M is called separated in the I-adic topology if $\bigcap_{n \in \mathbb{N}} I^n M = (0)$.

(7.81) Remark: (a) If R is a Noetherian ring, $I \subseteq \operatorname{Jac}(R)$, and M a finite R-module then by (4.21) $\bigcap_{n \in \mathbb{N}} I^n M = (0)$.

(b) Every ideal $I \subseteq R$ defines a topology, the so-called I-adic topology, on an R-module M. This topology is separated (Hausdorff) if and only if $\bigcap_{n \in \mathbb{N}} I^n M = (0)$. We discuss the I-adic topology in Chapter 9.

(7.82) Theorem: Let R be a ring, S a Noetherian R-algebra, M a finite S-module, and $J \subseteq \operatorname{Jac}(S)$ an ideal. For $n \geq 0$ set $M_n = M/J^{n+1}M$. If M_n is flat over R for all $n \geq 0$, then M is flat over R.

Proof: We have to show that for every finitely generated ideal I of R the natural map $u: I\otimes_R M \longrightarrow M$ is injective. Then $I\otimes_R M = M'$ is a finite S-module and hence separated in the J-adic topology. Let $x\in \ker(u)$. We want to show that $x\in \bigcap_{n\in\mathbb{N}} J^n M' = 0$. For all $n\geq 0$: $M'_n = M'/J^{n+1}M' = (I\otimes_R M)\otimes_S S/J^{n+1} = I\otimes_R M_n$ and the map $u_n: M'_n \longrightarrow M_n$ is injective, since M_n is R-flat. From the commutative diagram:

$$\begin{array}{ccc} M' & \xrightarrow{u} & M \\ \downarrow & & \downarrow \\ M'_n & \xrightarrow{u_n} & M_n \end{array}$$

we obtain that $x\in J^{n+1}M'$.

(7.83) Theorem: Let R be a ring, S a Noetherian R-algebra, and M a finite S-module. Let $b\in \operatorname{Jac}(S)$ be an M-regular element. If M/bM is flat over R, so is M.

Proof: For all $i>0$, the sequence $0\longrightarrow M/b^iM \xrightarrow{b} M/b^{i+1}M \longrightarrow M/bM \longrightarrow 0$ is exact. Using Theorem (7.78) it follows by induction on i that M/b^iM is R-flat for all $i>0$. Apply (7.82).

(7.84) Definition: Let R be a ring and $I\subseteq R$ an ideal. An R-module M is called I-adically ideal separated if for every finitely generated ideal $K\subseteq R$ the R-module $K\otimes_R M$ is separated in the I-adic topology.

(7.85) Example: If S is a Noetherian R-algebra and $IS\subseteq \operatorname{Jac}(S)$, then a finite S-module M is I-adically ideal separated as an R-module. (Since $K\otimes_R M$ is a finite S-module for every finitely generated ideal $K\subseteq R$.)

Let R be a ring, $I\subseteq R$ an ideal, and M an R-module. For all $n\geq 0$ set $R_n = R/I^{n+1}$ and $M_n = M/I^{n+1}M$. Consider $gr_I(R) = gr(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and $gr_I(M) = gr(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$. For all $n\geq 0$ there are natural maps

$$\gamma_n: (I^n/I^{n+1})\otimes_{R_0} M_0 \longrightarrow I^nM/I^{n+1}M$$

which induce a morphism of the $gr(R)$-modules:

$$\gamma : gr(R) \otimes_{R_0} M_0 \longrightarrow gr(M).$$

(7.86) Theorem: With the above notation suppose that R is a Noetherian ring and that M is I-adically ideal separated. Then the following are equivalent:

(a) M is flat over R.

(b) $Tor_1^R(N,M) = 0$ for every R_0-module N.

(c) M_0 is flat over R_0 and $I \otimes_R M \cong IM$.

(c') M_0 is flat over R_0 and $Tor_1^R(R_0, M) = 0$.

(d) M_0 is flat over R_0 and γ_n is an isomorphism for all $n \geq 0$.

(d') M_0 is flat over R_0 and γ is an isomorphism.

(e) M_n is flat over R_n for all $n \geq 0$.

Proof: (a) $\Rightarrow$ (b): clear

(b) $\Rightarrow$ (c): Let N be an R_0-module. Then $N \otimes_R M \cong (N \otimes_{R_0} R_0) \otimes_R M \cong N \otimes_{R_0} M_0$. For every exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of R_0-modules, the sequence $0 = Tor_1^R(N_3, M) \to N_1 \otimes_{R_0} M_0 \to N_2 \otimes_{R_0} M_0 \to N_3 \otimes_{R_0} M_0 \to 0$ is exact and M_0 is flat over R_0. From the exact sequence $0 \to I \to R \to R_0 \to 0$ we get the exact sequence $0 = Tor_1^R(R_0, M) \to I \otimes_R M \to M \to M_0 \to 0$ and $I \otimes_R M \cong IM$.

(c) $\Leftrightarrow$ (c'): Consider the exact sequence $0 \to I \to R \to R_0 \to 0$ and the induced long exact sequence: $0 = Tor_1^R(R,M) \to Tor_1^R(R_0, M) \to I \otimes_R M \to M \to M_0 \to 0$.

(c') $\Rightarrow$ (b): Let N be an R_0-module and $0 \to K \xrightarrow{\varphi} F_0 \to N \to 0$ an exact sequence with F_0 a free R_0-module. Then $0 = Tor_1^R(F_0, M) \to Tor_1^R(N,M) \to K \otimes_{R_0} M_0 \xrightarrow{\varphi \otimes M_0} F_0 \otimes_{R_0} M_0 \to N \otimes_{R_0} M_0 \to 0$ is exact. Since M_0 is R_0-flat, $\varphi \otimes M_0$ is injective and $Tor_1^R(N,M) = 0$.

(b) $\Rightarrow$ (d): By (b) $Tor_1^R(I/I^2, M) = 0$. Thus the exact sequence $0 \to I^2 \to I \to I/I^2 \to 0$ induces an exact sequence $0 \to I^2 \otimes M \to I \otimes M \to (I/I^2) \otimes M \to 0$. Since $IM \cong I \otimes M$ we obtain that $I^2 \otimes M \cong I^2 M$. Proceeding by induction on n we obtain from the exact sequence $0 \to I^{n+1} \to I^n \to I^n/I^{n+1} \to 0$ that $I^{n+1} \otimes M \cong I^{n+1} M$. Thus

$(I^n/I^{n+1}) \otimes M \cong I^nM/I^{n+1}M$.

(d) $\Leftrightarrow$ (d'): clear

(d) $\Rightarrow$ (e): Fix $n>0$. For $i \le n$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} & (I^{i+1}/I^{n+1})\otimes M & \longrightarrow & (I^i/I^{n+1})\otimes M & \longrightarrow & (I^i/I^{i+1})\otimes M & \longrightarrow 0 \\ & \downarrow \alpha_{i+1} & & \downarrow \alpha_i & & \downarrow \gamma_i & \\ 0 \longrightarrow & I^{i+1}M/I^{n+1}M \cong I^{i+1}M_n & \longrightarrow & I^iM/I^{n+1}M \cong I^iM_n & \longrightarrow & I^iM/I^{i+1}M & \longrightarrow 0 \end{array}$$

By assumption γ_i is an isomorphism and α_{n+1} is an isomorphism. Thus, by descending induction, α_i is an isomorphism for all $1 \le i \le n+1$. In particular, $\alpha_1 : (I/I^{n+1}) \otimes_R M = (IR_n) \otimes_{R_n} M_n \cong IM_n$. Thus R_n, M_n, and $I/I^{n+1} \subseteq R_n$ satisfy the conditions of (c). By (c)$\Rightarrow$(b) $\mathrm{Tor}_1^{R_n}(N, M_n) = 0$ for all R_0-modules N. If N is an R_1-module, then $0 \to IN \to N \to N/IN \to 0$ is exact with R_0-modules IN and N/IN. Thus $\mathrm{Tor}_1^{R_n}(N, M_n) = 0$. Proceeding by induction on i we see that if N is an R_i-module, then IN and N/IN are R_{i-1}-modules. Assuming that $\mathrm{Tor}_1^{R_n}(K, M_n) = 0$ for all R_{i-1}-modules, the exact sequence $0 \to IN \to N \to N/IN \to 0$ yields that $\mathrm{Tor}_1^{R_n}(N, M_n) = 0$. Thus M_n is R_n-flat.

(e) $\Rightarrow$ (a): We have to show that the natural map $\psi : J \otimes_R M \longrightarrow M$ is injective for every ideal $J \subseteq R$. Since M is I-adically ideal separated, $\bigcap_n I^n(J \otimes M) = 0$ and it suffices to show that $\ker(\psi) \subseteq I^n(J \otimes M)$ for all $n > 0$. Fix $n > 0$. By Artin–Rees $I^k \cap J \subseteq I^n J$ for sufficiently large $k > n$. Consider the natural map:

$$J \otimes M \xrightarrow{f} (J/J \cap I^k) \otimes M \xrightarrow{g} (J/I^nJ) \otimes_R M \cong J \otimes M / I^n(J \otimes M).$$

Since M_{k-1} is flat over $R_{k-1} = R/I^k$ the map

$$(J/J \cap I^k) \otimes_R M \cong (J/J \cap I^k) \otimes_{R_{k-1}} M_{k-1} \longrightarrow M_{k-1}$$

is injective. From the commutative diagram

$$\begin{array}{ccc} J \otimes M & \xrightarrow{f} & (J/J \cap I^k) \otimes M \\ \psi \downarrow & & \downarrow \mathrm{inj} \\ M & \longrightarrow & M_{k-1} \end{array}$$

we get that $\ker(\psi) \subseteq \ker(f) \subseteq \ker(gf) = I^n(J \otimes M)$.

§6: SOME IDENTITIES

In this section we list some homological identities. The proofs, if not provided, can be found in most books on homological algebra, for example in J. Rotman's book: An introduction to homological algebra.

(7.87) Remark: Let $\{M_i\}_{i\in I}$ be a family of R-modules and N an R-module. Then

(a) $\mathrm{Hom}_R(\oplus M_i, N) \cong \prod \mathrm{Hom}_R(M_i, N)$

(b) $\mathrm{Hom}_R(N, \prod M_i) \cong \prod \mathrm{Hom}_R(N, M_i)$

We know from (0.46) that the tensor product commutes with direct sums:

(c) $(\oplus M_i) \otimes_R N \cong \oplus (M_i \otimes_R N)$.

Since $\oplus$ and $\prod$ commute with the formation of homology, the identities of (7.87) extend to the derived functors:

(7.88) Proposition: Let $\{M_i\}_{i\in I}$ be a family of R-modules and N an R-module.

(a) $\mathrm{Ext}^i_R(\oplus_{j\in I} M_j, N) \cong \prod_{j\in I} \mathrm{Ext}^i_R(M_j, N)$

(b) $\mathrm{Ext}^i_R(N, \prod_{j\in I} M_j) \cong \prod_{j\in I} \mathrm{Ext}^i_R(N, M_j)$

(c) $\mathrm{Tor}^R_i(\oplus_{j\in I} M_j, N) \cong \oplus_{j\in I} \mathrm{Tor}^R_i(M_j, N)$.

(7.89) Theorem: Let $R \longrightarrow S$ be a homomorphism of rings so that S is a flat R-module and let M and N be R-modules. Then

(a) $\mathrm{Tor}^S_i(S\otimes_R M, S\otimes_R N) \cong S\otimes_R \mathrm{Tor}^R_i(M, N)$

(b) If R is Noetherian and M is finite, then

$$\mathrm{Ext}^i_S(S\otimes_R M, S\otimes_R N) \cong S\otimes_R \mathrm{Ext}^i_R(M, N).$$

Proof: We only prove (b). Since M is finite and R is Noetherian, M has a free resolution $F_\bullet$ with F_j finite, say $F_j \cong R^{n_j}$. Then by (7.88):

$$S\otimes_R \mathrm{Hom}_R(F_j,N) \cong S\otimes_R \mathrm{Hom}_R(R^{n_j},N) \cong S\otimes_R \oplus^{n_j}\mathrm{Hom}_R(R,N) \cong S\otimes_R \oplus^{n_j} N \cong$$
$$\cong \oplus^{n_j} S\otimes_R N \cong \oplus^{n_j}\mathrm{Hom}_S(S, S\otimes_R N) \cong \mathrm{Hom}_S(\oplus^{n_j}S, S\otimes_R N) \cong$$
$$\cong \mathrm{Hom}_S(S\otimes_R F_j, S\otimes_R N),$$

and this isomorphism is natural. Hence there is an isomorphism of complexes: $S\otimes_R \mathrm{Hom}_R(F_\bullet,N) \cong \mathrm{Hom}_S(S\otimes_R F_\bullet, S\otimes_R N)$. Since S is R-flat, $S\otimes_R F_\bullet$ is a free S-resolution of $S\otimes_R M$. Hence:

$$\begin{aligned} \mathrm{Ext}^i_S(S\otimes_R M, S\otimes_R N) &= H^i(\mathrm{Hom}_S(S\otimes_R F_\bullet, S\otimes_R N)) \\ &\cong H^i(S\otimes_R \mathrm{Hom}_R(F_\bullet,N)) \\ &\cong S\otimes_R H^i(\mathrm{Hom}_R(F_\bullet,N)) \qquad \text{since } S \text{ is } R\text{-flat} \\ &= S\otimes_R \mathrm{Ext}^i_R(M,N). \end{aligned}$$

From the universal property of inverse limits one can derive the following properties:

(17.90) Proposition: Let $\{M_i\}$ be a direct (inverse) system of R-modules, N an R-module.

(a) $\mathrm{Hom}_R(\varinjlim M_i, N) \cong \varprojlim \mathrm{Hom}_R(M_i,N)$

(b) $\mathrm{Hom}_R(N, \varprojlim M_i) \cong \varprojlim \mathrm{Hom}_R(N,M_i)$.

Let $I\subseteq R$ be an ideal and $\Gamma_I(-)$ the torsion functor. Notice that $\Gamma_I(-)$ is an additive functor. For every R-module M:

$$\begin{aligned} \Gamma_I(M) &= \{m\in M \mid I^n m = 0 \text{ for some } n\in\mathbb{N}\} \\ &= \varinjlim (0:_M I^n) \\ &\cong \varinjlim \mathrm{Hom}_R(R/I^n, M). \end{aligned}$$

The right derived functors of $\Gamma_I(-)$ are the local cohomology functors. Notation: $H^i_I(-) = R^i\Gamma_I(-)$. $H^i_I(-)$ are additive functors with $H^0_I(-) = \Gamma_I(-)$.

(17.91) Proposition: $H^i_I(M) \cong \varinjlim \mathrm{Ext}^i_R(R/I^n, M)$ and this isomorphism is natural.

Proof: $H^i_I(M) \cong H^i_I(\Gamma_I(I^\bullet_M)) \cong H^i(\varinjlim \mathrm{Hom}_R(R/I^n, I^\bullet_M))$

$$\cong \varinjlim H^i(\mathrm{Hom}_R(R/I^n, I^\bullet_M)) \qquad \text{since } \varinjlim \text{ preserves exactness}$$
$$\cong \varinjlim \mathrm{Ext}^i_R(R/I^n, M).$$

(7.92) <u>Theorem</u>: If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then there is a long exact sequence:

$$0 \to H^0_I(M') = \Gamma_I(M') \longrightarrow H^0_I(M) = \Gamma_I(M) \longrightarrow H^0_I(M'') = \Gamma_I(M'') \longrightarrow H^1_I(M') \longrightarrow H^1_I(M) \longrightarrow \cdots$$

Furthermore this exact sequence is natural.