CHAPTER VI: HOMOLOGICAL ALGEBRA I

Throughout this chapter: R is a commutative ring with identity, $M, N, \dots$ are R-modules, maps are R-linear. $\mathcal{M}(R)$ denotes the category of R-modules and R-linear maps.

§1: THE HOM FUNCTORS

(6.1) Definition: The set of R-linear maps from M to N is denoted by

$$\mathrm{Hom}_R(M,N) = \{ f : M \longrightarrow N \mid f \ R\text{-linear} \}.$$

(6.2) Remark: $\mathrm{Hom}_R(M,N)$ is an R-module under the operations: For $f, g \in \mathrm{Hom}_R(M,N)$, $a \in R$, $m \in M$ set $(f+g)(m) = f(m) + g(m)$ and $(af)(m) = a f(m)$. If R is not commutative, $\mathrm{Hom}_R(M,N)$ is an abelian group, but, in general, not an R-module.

(6.3) Definition: Let $\alpha : N \longrightarrow N'$ and $\beta : M \longrightarrow M'$ be R-linear maps of R-modules.

(a) α induces an R-linear map $\mathrm{Hom}(M,\alpha) = \alpha_* : \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N')$ defined by $\alpha_*(f) = \alpha f$ for all $f \in \mathrm{Hom}_R(M,N)$.

(b) β induces an R-linear map $\mathrm{Hom}(\beta,N) = \beta^* : \mathrm{Hom}_R(M',N) \longrightarrow \mathrm{Hom}_R(M,N)$ defined by $\beta^*(g) = g\beta$ for all $g \in \mathrm{Hom}_R(M',N)$.

(6.4) Remark: (a) Obviously, $\mathrm{Hom}_R(M, \mathrm{id}_N) = (\mathrm{id}_N)_* = \mathrm{id}_{\mathrm{Hom}(M,N)}$ and for R-linear maps $N_1 \xrightarrow{\alpha_1} N_2 \xrightarrow{\alpha_2} N_3$, $(\alpha_2\alpha_1)_* = \alpha_{2*}\alpha_{1*}$. Thus for a fixed R-module M $\mathrm{Hom}_R(M,-)$ is a covariant functor from $\mathcal{M}(R)$ to $\mathcal{M}(R)$.

(b) Similarly, for a fixed R-module N, $\mathrm{Hom}(\mathrm{id}_M, N) = (\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{Hom}(M,N)}$ and for R-linear maps $M_1 \xrightarrow{\beta_1} M_2 \xrightarrow{\beta_2} M$, $(\beta_2\beta_1)^* = \beta_1^*\beta_2^*$. $\mathrm{Hom}_R(-,N)$ is a contravariant functor from $\mathcal{M}(R)$ to $\mathcal{M}(R)$.

(6.5) Definition: Let $F : \mathcal{M}(R) \longrightarrow \mathcal{M}(R)$ be a functor (contravariant functor).

(a) F is called an <u>additive</u> functor if for any two R-modules M, M' the induced map $\operatorname{Hom}_R(M,M') \longrightarrow \operatorname{Hom}_R(F(M),F(M'))$ ($\operatorname{Hom}_R(M,M') \longrightarrow \operatorname{Hom}_R(F(M'),F(M))$, resp.) is a homomorphism of abelian groups.

(b) If F is additive, F is called <u>left exact</u> if whenever $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ ($M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$) is an exact sequence, then $0 \longrightarrow F(M') \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(M'')$ ($0 \longrightarrow F(M'') \xrightarrow{F(\beta)} F(M) \xrightarrow{F(\alpha)} F(M')$) is exact.

(c) If F is additive, F is called <u>right exact</u> if whenever $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ ($0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$) is an exact sequence, then $F(M') \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(M'') \longrightarrow 0$ ($F(M'') \xrightarrow{F(\beta)} F(M) \xrightarrow{F(\alpha)} F(M') \longrightarrow 0$) is exact.

(d) F is <u>exact</u> if F is left and right exact.

(6.6) <u>Proposition</u>: $\operatorname{Hom}_R(M,-)$ and $\operatorname{Hom}_R(-,N)$ are additive functors.

<u>Proof</u>: Homework

(6.7) <u>Theorem</u>: $\operatorname{Hom}_R(M,-)$ and $\operatorname{Hom}_R(-,N)$ are left exact functors.

<u>Proof</u>: (a) $\operatorname{Hom}_R(M,-)$ is left exact. Let $0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ be an exact sequence.
α_* is injective: Let $f \in \operatorname{Hom}_R(M,N')$ with $\alpha_*(f) = \alpha f = 0$. Since α is injective, $f = 0$.
$\operatorname{im}\alpha_* \subseteq \ker\beta_*$: By assumption $\beta\alpha = 0$. Thus $\beta_*\alpha_* = (\beta\alpha)_* = 0_* = 0$.
$\ker\beta_* \subseteq \operatorname{im}\alpha_*$: Let $f \in \ker\beta_*$. Then $\beta_*(f) = \beta f = 0$ and $f(M) \subseteq \ker\beta = \operatorname{im}\alpha \cong N'$. There is an R-linear map $f' : M \longrightarrow N'$ with $\alpha f' = f$ and $f = \alpha_*(f') \in \operatorname{im}(\alpha_*)$.
(b) $\operatorname{Hom}_R(-,N)$ is left exact. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence.
β^* is injective: Let $f \in \operatorname{Hom}_R(M'',N)$ with $\beta^*(f) = f\beta = 0$. Since β is surjective, $f = 0$.
$\operatorname{im}\beta^* \subseteq \ker\alpha^*$: Since $\beta\alpha = 0$, $(\beta\alpha)^* = \alpha^*\beta^* = 0^* = 0$.
$\ker\alpha^* \subseteq \operatorname{im}\beta^*$: Let $f \in \ker\alpha^*$, i.e. $f : M \longrightarrow N$ with $\alpha^*(f) = f\alpha = 0$. Then $f(\operatorname{im}\alpha) = 0$ and since $\operatorname{im}\alpha = \ker\beta$, $f(\ker\beta) = 0$. Thus f factors and there is an R-linear map $f'' : M'' \longrightarrow N$ with $f''\beta = f$.

$$\begin{tikzcd}
 & M \arrow[r, "f"] \arrow[dl, "\beta"'] \arrow[d] & N \\
M'' \cong & M/\ker\beta \arrow[ur, "\bar{f}"'] &
\arrow[from=2-1, to=1-3, bend right=60, "f''"']
\end{tikzcd}$$

(6.8) Remark: In general, neither $\mathrm{Hom}_R(M,-)$ nor $\mathrm{Hom}_R(-,N)$ are right exact. Consider the exact sequence of $\mathbb{Z}$-modules $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/(2) \longrightarrow 0$ where $\alpha(n) = 2n$ for all $n \in \mathbb{Z}$ and β is the natural map.

(a) Let $M = \mathbb{Z}/(2)$. The sequence $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \longrightarrow 0$ is not exact since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}) = 0$ but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \neq 0$

(b) Let $N = \mathbb{Z}$. The sequence $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$ is not exact. For all $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ $\alpha^*(f) = f\alpha$ and $f(\alpha(n)) = f(2n) = 2f(n)$. Thus $\mathrm{im}(\alpha^*(f)) \subseteq (2)$.

(6.9) Definition: An R-module P is called projective if for every surjective R-linear map $\beta : M \longrightarrow N$ and every R-linear map $\alpha : P \longrightarrow N$ there is an R-linear map $\gamma : P \longrightarrow M$ so that $\alpha = \beta\gamma$.

$$\begin{tikzcd}
 & P \arrow[dl, dotted, "\exists\gamma"'] \arrow[d, "\alpha"] & \\
M \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

(6.10) Theorem: An R-module P is projective if and only if the functor $\mathrm{Hom}_R(P,-)$ is exact, that is, for every exact sequence $0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$ of R-modules the sequence $0 \longrightarrow \mathrm{Hom}_R(P,N') \xrightarrow{\alpha_*} \mathrm{Hom}_R(P,N) \xrightarrow{\beta_*} \mathrm{Hom}_R(P,N'') \longrightarrow 0$ is exact.

Proof: "$\longrightarrow$": Since $\mathrm{Hom}_R(P,-)$ is left exact, the sequence $0 \longrightarrow \mathrm{Hom}_R(P,N') \xrightarrow{\alpha_*} \mathrm{Hom}_R(P,N) \xrightarrow{\beta_*} \mathrm{Hom}_R(P,N'')$ is exact. Let $f \in \mathrm{Hom}_R(P,N'')$. Since P is projective, there is a $g \in \mathrm{Hom}_R(P,N)$ with $\beta g = f$ and β_* is surjective.

"$\longleftarrow$" Suppose that there are given R-linear maps

$$\begin{tikzcd}
 & P \arrow[d, "f"] & \\
M \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

with β surjective. Consider the short exact sequence $0 \longrightarrow \ker\beta \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$. By assumption $\mathrm{Hom}_R(P,M) \xrightarrow{\beta_*} \mathrm{Hom}_R(P,N)$ is surjective. P is projective.

(6.11) Proposition: Every free R-module is projective.

Proof: Let F be a free R-module with basis $\{e_i\}_{i \in I}$. Consider

$$\begin{tikzcd}
 & F \arrow[d, "f"] & \\
M \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

where f, β are R-linear and β surjective. For every $i \in I$ choose an element $m_i \in M$ with $\beta(m_i) = f(e_i)$. Since F is free there is an R-linear map $g: F \longrightarrow M$ with $g(e_i) = m_i$ for all $i \in I$. Then $\beta g = f$.

(6.12) <u>Proposition</u>: Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of R-modules and R-linear maps. The following are equivalent:

(a) There is an R-linear map $\gamma: M \longrightarrow M'$ with $\gamma\alpha = id_{M'}$.

(b) There is an R-linear map $\delta: M'' \longrightarrow M$ with $\beta\delta = id_{M''}$.

(c) $M = im(\alpha) \oplus N$ for a submodule $N \subseteq M$.

The submodule N of (c) is isomorphic to M''.

<u>Proof</u>: (a) $\Longrightarrow$ (c): Let $\gamma: M \longrightarrow M'$ with $\gamma\alpha = id_{M'}$. Set $N = ker(\gamma)$. For $m \in M$, $m - \alpha\gamma(m) \in ker(\gamma)$ and $M = im(\alpha) + N$. If $n \in im(\alpha) \cap N$ then $n = \alpha(t)$ for some $t \in M'$ and $\gamma(n) = 0 = \gamma\alpha(t) = t$. Thus $M = im(\alpha) \oplus N$.

(c) $\Longrightarrow$ (a): Define $\gamma: M \longrightarrow M'$ by $\gamma = \tilde{\alpha}p$ where $p: im(\alpha) \oplus N \longrightarrow im(\alpha)$ is the projection and $\tilde{\alpha}: im(\alpha) \longrightarrow M'$ is defined by $\tilde{\alpha}(\alpha(m)) = m$ (since $M' \cong im(\alpha)$). Then $\gamma\alpha = id_{M'}$.

(b) $\Longrightarrow$ (c): Let $\delta: M'' \longrightarrow M$ be such that $\beta\delta = id_{M'}$. Set $N = im(\delta) \cong M''$. For $m \in M$, $m - \delta\beta(m) \in ker(\beta) = im(\alpha)$ and $M = im(\alpha) + N$. If $n \in im(\alpha) \cap N$ then $n = \delta(t)$ for some $t \in M''$ and $\beta(n) = 0 = \beta\delta(t) = t$. Hence $M = im(\alpha) \oplus N$.

(c) $\Longrightarrow$ (b): Define $\delta: M'' \longrightarrow M$ as follows: since $im(\alpha) = ker(\beta)$ and $M'' \cong M/ker(\beta)$ the R-linear map $\beta|_N : N \longrightarrow M''$ is an isomorphism. Let $\delta = i\,(\beta|_N)^{-1}$ where $i: N \longrightarrow M$ is the embedding. Then $\beta \circ \delta = id_{M''}$.

(6.13) <u>Definition</u>: An exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ is called <u>split</u> (or <u>split exact</u>) if there is an R-linear map $\delta: M'' \longrightarrow M$ with $\beta\delta = id_{M''}$ (or equivalently, if there is an R-linear map $\gamma: M \longrightarrow M''$ with $\gamma\alpha = id_{M'}$).

(6.14) <u>Proposition</u>: Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be a split exact sequence of

R-modules and R-linear maps, $F : \mathcal{M}(R) \longrightarrow \mathcal{M}(R)$ a left (right) exact functor (contravariant functor). Then $0 \longrightarrow F(M') \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(M'') \longrightarrow 0$ (or if F is contravariant $0 \longrightarrow F(M'') \xrightarrow{F(\beta)} F(M) \xrightarrow{F(\alpha)} F(M') \longrightarrow 0$) is split exact.

Proof: Let $\gamma : M \longrightarrow M'$ with $\gamma\alpha = id_{M'}$ and $\delta : M'' \longrightarrow M$ with $\beta\delta = id_{M''}$. Then $F(\gamma\alpha) = F(\gamma)F(\alpha) = F(id_{M'}) = id_{F(M')}$ and $F(\beta\delta) = F(\beta)F(\delta) = F(id_{M''}) = id_{F(M'')}$ (or in the contravariant case $F(\gamma\alpha) = F(\alpha)F(\gamma) = id_{F(M')}$ and $F(\beta\delta) = F(\delta)F(\beta) = id_{F(M'')}$.)

(6.15) Proposition: Every direct summand of a projective module is projective.

Proof: Let P be a projective module, $N, Q \subseteq P$ submodules, and $P = N \oplus Q$. A diagram of R-linear maps

$$\begin{tikzcd} & Q \arrow[d, "f"] & \\ M \arrow[r, "\beta"] & M' \arrow[r] & 0 \end{tikzcd}$$

can be extended to

$$\begin{tikzcd} & P \arrow[d, "\pi"] \arrow[dl, "g"'] & \\ & Q \arrow[d, "f"] \arrow[u, bend right, "i"'] & \\ M \arrow[r, "\beta"] & M' \arrow[r] & 0 \end{tikzcd}$$

where $\pi : P \longrightarrow Q$ is the projection and $i : Q \longrightarrow P$ is the embedding. Note that $\pi i = id_Q$. Since P is projective, there is an R-linear map $g : P \longrightarrow M$ with $\beta g = f\pi$. Then $\beta(gi) = f(\pi i) = f$ and Q is projective.

(6.16) Proposition: Let P be an R-module. The following are equivalent:

(a) P is projective.

(b) P is (isomorphic to) a direct summand of a free module.

(c) Every exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is split exact.

Proof: (c) $\Longrightarrow$ (b): Every module is a homomorphic image of a free module. Consider an exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow P \longrightarrow 0$ with F a free R-module. By (6.12) P is isomorphic to a direct summand of F.

(b) $\Longrightarrow$ (a): By (6.11) and (6.16).

(a) $\Longrightarrow$ (c): If $0 \longrightarrow N \longrightarrow M \xrightarrow{\beta} P \longrightarrow 0$ is exact, consider

$$\begin{tikzcd} & P \arrow[d, "id_P"] \arrow[dl, dotted, "\delta"'] & \\ M \arrow[r, "\beta"] & P \arrow[r] & 0 \end{tikzcd}$$

Since P is projective, there is an R-linear map $\delta : P \longrightarrow M$ with $\beta\delta = \mathrm{id}_P$.

(6.17) Examples: (a) Let $R = \mathbb{Z}/(6)$. By the Chinese remainder theorem $R \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$. $P = \mathbb{Z}/(2)$ is a projective R-module, but P is not a free R-module.
(b) Let R be a Dedekind domain which is not factorial (for example, $\mathbb{Z}[\sqrt{-5}]$). We will show later that every nonzero ideal of R is projective. An ideal I of a domain R is free if and only if $I \neq (0)$ and I is principal. Moreover, every PID is factorial. Thus every nonprincipal ideal of R is projective but not free.

(6.18) Definition: An R-module E is called injective if for every R-linear map $\alpha : N \longrightarrow M$ and every R-linear map $\delta : N \longrightarrow E$ there is an R-linear map $\sigma : M \rightarrow E$ which extends δ, that is, $\delta = \sigma\alpha$.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M \arrow[dl, "\sigma"] \\ & E & \end{tikzcd}$$

(6.19) Theorem: An R-module E is injective if and only if $\mathrm{Hom}_R(-, E)$ is exact, that is, for every exact sequence of R-modules $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ the sequence $0 \longrightarrow \mathrm{Hom}_R(M'', E) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, E) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M', E) \longrightarrow 0$ is exact.

Proof: If E is injective then $\mathrm{Hom}_R(M, E) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M', E)$ is surjective by the definition of injective modules. Conversely, suppose that $\mathrm{Hom}_R(-, E)$ is exact and consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\delta"'] & M \\ & E & \end{tikzcd}$$

with α injective. Set $N' = \mathrm{coker}(\alpha)$. Then the sequence $0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} N' \longrightarrow 0$ is exact and so is the sequence $0 \rightarrow \mathrm{Hom}_R(N', E) \xrightarrow{\beta^*} \mathrm{Hom}_R(M, E) \xrightarrow{\alpha^*} \mathrm{Hom}_R(N, E) \longrightarrow 0$. Thus there is a $\sigma \in \mathrm{Hom}_R(M, E)$ with $\alpha^*(\sigma) = \sigma\alpha = \delta$.

(6.20) Remark: It is easy to show that the direct product of injective modules is injective. The direct sum of injective modules may not be injective (but finite direct sums are). There is a theorem which states that a commutative ring R is Noetherian if and only if every direct sum of injective R-modules is injective.

(6.21) <u>Proposition</u>: Every direct summand D of an injective R-module E is injective.

<u>Proof</u>: Similar to the proof of (6.15).

(6.22) <u>Theorem</u>: For an R-module E the following conditions are equivalent:

(a) E is injective.

(b) Every exact sequence $0 \longrightarrow E \longrightarrow M \longrightarrow N \longrightarrow 0$ is split exact.

<u>Proof</u>: (a) $\Rightarrow$ (b): Let $0 \to E \xrightarrow{i} M \to N \to 0$ be exact. Since E is injective, there is a map $g: M \longrightarrow E$ such that

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "i"] \arrow[d, "id"'] & M \arrow[dl, "g"] \\ & E & \end{tikzcd}$$

commutes. The sequence splits.

(b) $\to$ (a): Consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \\ & E & (*) \end{tikzcd}$$

with α injective. Set $T = E \oplus M/W$ where $W = \{(f(m), -\alpha(m)) \mid m \in K\}$. W is a submodule of $E \oplus M$. Consider the R-linear map $\alpha': E \longrightarrow T$ defined by $\alpha'(e) = (e,0) + W$ and $f': M \longrightarrow T$ defined by $f'(m) = (0,m) + W$. Then diagram $(*)$ extends to the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \arrow[d, "f'"] \\ & E \arrow[r, "\alpha'"] & T. & (**) \end{tikzcd}$$

$(**)$ is commutative: For all $m \in K$: $\alpha' f(m) = (f(m), 0) + W = (f(m),0) - (f(m), -\alpha(m)) + W =$
$= (0, \alpha(m)) + W = f'\alpha(m)$.

α' is injective: Let $e \in E$ with $\alpha'(e) = (e,0) + W = (0,0)$. Then there is an element $m \in K$ with $(e,0) = (f(m), -\alpha(m))$ in $E \oplus M$. Thus $\alpha(m) = 0$ and $m = 0$ since α is injective. Hence $e = 0$.

By assumption (b) there is a map $\beta: T \longrightarrow E$ with $\beta\alpha' = id_E$. Let $g = \beta f'$. Then $g\alpha = \beta f'\alpha = \beta\alpha' f = f$. E is an injective R-module.

(6.23) <u>Remark</u>: The module T together with the maps α' and f' (in the proof of (6.22)) is called the <u>pushout</u> of the diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "f"'] & M \\ E & \end{tikzcd}$$

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§2: THE TENSOR FUNCTORS

For a fixed R-module M there is a covariant functor $M\otimes_R -$ from the category of R-modules $\mathcal{M}(R)$ into itself. $M\otimes_R -$ is defined as follows: for every R-module N set $(M\otimes_R -)(N) = M\otimes_R N$ and for every R-linear map $\varphi: N \longrightarrow N'$, $(M\otimes_R -)(\varphi) = id_M \otimes \varphi$. For a fixed R-module N the covariant functor $-\otimes_R N$ is defined accordingly.

(6.24) Theorem: (adjoint isomorphism) Let M, N, P be R-modules. Then there is an isomorphism $\quad Hom_R(M\otimes_R N, P) \cong Hom_R(M, Hom_R(N,P))$.

Proof: Set $Bilin_R(M\times N, P) = \{\varphi: M\times N \longrightarrow P \mid \varphi \text{ R-bilinear}\}$. $Bilin_R(M\times N, P)$ is naturally an R-module which is isomorphic to $Hom_R(M\otimes_R N, P)$. Thus it suffices to show that the R-modules $Bilin_R(M\times N, P)$ and $Hom_R(M, Hom_R(N,P))$ are isomorphic. Define:

$$\begin{aligned}
\Phi: Bilin_R(M\times N, P) &\longrightarrow Hom_R(M, Hom_R(N,P))\\
\varphi &\longmapsto \Phi(\varphi): M \longrightarrow Hom_R(N,P)\\
&\qquad\qquad m \longmapsto \Phi(\varphi)(m) = \varphi(m,-): N \longrightarrow P\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \longmapsto \varphi(m,n).
\end{aligned}$$

Since φ is R-bilinear, $\phi(\varphi)(m)$ is R-linear. Moreover, $\Phi(\varphi)$ and Φ are R-linear. Conversely, define:

$$\begin{aligned}
\Psi: Hom_R(M, Hom_R(N,P)) &\longrightarrow Bilin_R(M\times N, P)\\
\gamma &\longmapsto \Psi(\gamma): M\times N \longrightarrow P\\
&\qquad\qquad (m,n) \longmapsto \gamma(m)(n).
\end{aligned}$$

$\Psi(\gamma)$ is R-bilinear and Ψ is R-linear. Φ and Ψ are inverse to each other, that is, $\Psi\circ\Phi = id_{Bilin}$ and $\Phi\Psi = id_{Hom}$.

(6.25) Remark: The functors $-\otimes_R N$ and $Hom_R(N,-)$ are adjoint pairs. This means, for all R-linear maps $\alpha: M' \longrightarrow N$ and $\beta: P' \longrightarrow P$ the diagrams

$$\begin{tikzcd}
Hom_R(M\otimes_R N, P) \arrow[r, "\cong"] \arrow[d, "(\alpha\otimes id)^*"'] & Hom_R(M, Hom_R(N,P)) \arrow[d, "\alpha^*"] \\
Hom_R(M'\otimes_R N, P) \arrow[r, "\cong"] & Hom_R(M', Hom_R(N,P))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_R(M\otimes_R N, P') \arrow[r, "\cong"] \arrow[d, "\beta_*"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P')) \arrow[d, "\beta_{**}"] \\
\mathrm{Hom}_R(M\otimes_R N, P) \arrow[r, "\cong"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))
\end{tikzcd}$$

commute where the horizontal isomorphisms are the isomorphisms of (6.24).

Proof: Homework

(6.26) Lemma: Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R-modules. If for all R-modules N the sequence $\mathrm{Hom}_R(M'',N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M,N) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M',N)$ is exact, the sequence $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is exact.

Proof: Suppose that $\mathrm{Hom}_R(M'',N) \xrightarrow{\beta^*} \mathrm{Hom}_R(M,N) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M',N)$ is exact for all N.

(a) Let $N = M''$. Then $0 = \alpha^*\beta^*(\mathrm{id}_{M''}) = \mathrm{id}_{M''}\beta\alpha = \beta\alpha$ and $\mathrm{im}(\alpha) \subseteq \ker(\beta)$.

(b) In order to show $\ker(\beta) \subseteq \mathrm{im}(\alpha)$ set $N = M/\mathrm{im}(\alpha)$ and let $\nu : M \longrightarrow M/\mathrm{im}(\alpha)$ be the natural map. Since $\alpha^*(\nu) = \nu\alpha = 0$, there is an R-linear map $\sigma : M'' \longrightarrow M/\mathrm{im}(\alpha)$ with $\beta^*(\sigma) = \sigma\beta = \alpha$:

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "\nu"'] & M'' \arrow[dl, "\sigma"] \\
M/\mathrm{im}(\alpha) &
\end{tikzcd}$$

. Thus $\mathrm{im}(\alpha) = \ker(\nu) = \ker(\sigma\beta) \supseteq \ker(\beta)$.

(6.27) Theorem: The functor $-\otimes_R N$ is right exact, that is, if $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules, then the sequence $M'\otimes_R N \longrightarrow M\otimes_R N \longrightarrow M''\otimes_R N \longrightarrow 0$ is exact.

Proof: For every R-module P the sequence:
$0 \longrightarrow \mathrm{Hom}_R(M'', \mathrm{Hom}_R(N,P)) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) \longrightarrow \mathrm{Hom}_R(M', \mathrm{Hom}_R(N,P))$ is exact. Thus by (6.24) and (6.25) the sequence $0 \longrightarrow \mathrm{Hom}_R(M''\otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M\otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M'\otimes_R N, P)$ is exact for every R-module P. By (6.26) $M'\otimes_R N \longrightarrow M\otimes_R N \longrightarrow M''\otimes_R N \longrightarrow 0$ is exact.

(6.28) Corollary: Let $I \subseteq R$ be an ideal and M an R-module. Then $(R/I)\otimes_R M \cong M/IM$.

Proof: Consider the exact sequence $0 \to I \xrightarrow{i} R \to R/I \to 0$. Tensoring with M yields an exact sequence $I \otimes_R M \xrightarrow{i\otimes M} R \otimes_R M \to (R/I) \otimes_R M \to 0$ which can be extended to a diagram with exact rows:

$$\begin{tikzcd}
 & I\otimes_R M \arrow[r, "i\otimes M"] \arrow[d, "\delta"'] \arrow[dr, phantom, "①"] & R\otimes_R M \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "②"] & (R/I)\otimes_R M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & IM \arrow[r] & M \arrow[r] & M/IM \arrow[r] & 0
\end{tikzcd}$$

where δ is defined by $\delta(a\otimes m) = am$ and φ is induced by the R-bilinear map $\tau: (R/I) \times M \to M/IM$ with $\tau(\bar{a}, m) = \overline{am}$. Thus $\varphi(\bar{a} \otimes m) = \overline{am}$ and φ is surjective. δ is also surjective and squares ① and ② commute. By diagram chasing φ is an isomorphism.

(6.29) Remark: If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules and N an R-module, in general, the sequence $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ is not exact. For example, consider the sequence (*) $0 \to \mathbb{Z} \xrightarrow{\sigma} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$ where $\sigma(n) = 2n$ for all $n \in \mathbb{Z}$. (*) is exact, but the sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \xrightarrow{\sigma\otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \to (\mathbb{Z}/(2)) \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \to 0$$

is not exact since $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \cong \mathbb{Z}/(2) \neq 0$ and $\sigma \otimes \mathrm{id}$ is the zero map.

(6.30) Definition: An R-module N is called flat over R if for all exact sequences of R-modules $0 \to M' \to M$ the sequence $0 \to M' \otimes_R N \to M \otimes_R N$ is exact.

(6.31) Proposition: Let $\{N_i\}_{i\in I}$ be a set of R-modules. The following are equivalent:

(a) $\bigoplus_{i\in I} N_i$ is flat over R.

(b) For all $i \in I$, N_i is flat over R.

Proof: If $\{M_i'\}_{i\in I}$ and $\{M_i\}_{i\in I}$ are sets of R-modules and $\{f_i : M_i' \to M_i\}_{i\in I}$ R-linear maps, then there is a unique R-linear map $\bigoplus f_i : \bigoplus M_i' \to \bigoplus M_i$ with $(\bigoplus f_i)((m_j)) = (f_j(m_j))$. Moreover, $\bigoplus f_i$ is injective if and only if f_i is injective for all $i \in I$. For an R-linear map $\tau : M' \longrightarrow M$ by (0.46) there is a commutative diagram:

$$\begin{tikzcd}
\left(\bigoplus_{i\in I} N_i\right) \otimes_R M' \arrow[r, "id \otimes \tau"] \arrow[d, "\cong"'] & \left(\bigoplus_{i\in I} N_i\right) \otimes_R M \arrow[d, "\cong"] \\
\bigoplus_{i\in I} (N_i \otimes_R M') \arrow[r, "\bigoplus (id\otimes\tau)"] & \bigoplus_{i\in I} (N_i \otimes_R M)
\end{tikzcd}$$

where the vertical arrows are isomorphisms. The top row is injective if and only if the bottom row is.

(6.32) <u>Corollary</u>: Every projective module is flat. In particular, every free module is flat.

<u>Proof</u>: Since R is a flat R-module, by (6.31) every free R-module is flat. Projective modules are direct summands of free modules.

(6.33) <u>Theorem</u>: Let R be a ring, $S \subseteq R$ a multiplicative set, and M an R-module. Then

$$(S^{-1}R) \otimes_R M \cong S^{-1}M .$$

The isomorphism is natural in the following sense: if $\tau : M' \longrightarrow M$ is an R-linear map then the diagram:

$$\begin{tikzcd}
(S^{-1}R) \otimes_R M' \arrow[r, "S^{-1}R\otimes\tau"] \arrow[d, "\cong"'] & (S^{-1}R) \otimes_R M \arrow[d, "\cong"] \\
S\ M' \arrow[r, "S^{-1}\tau"] & S^{-1}M
\end{tikzcd}$$

commutes.

<u>Proof</u>: The R-bilinear map $\alpha : S^{-1}R \times M \longrightarrow S^{-1}M$ with $\alpha(a/s, m) = am/s$ induces an R-linear map $\varphi : (S^{-1}R) \otimes_R M \longrightarrow S^{-1}M$ with $\varphi((a/s)\otimes m) = am/s$. Conversely, define $\psi : S^{-1}M \longrightarrow (S^{-1}R) \otimes_R M$ by $\psi(m/s) = (1/s) \otimes m$. ψ is well defined. If $m/s = m'/s'$ in $S^{-1}M$ then there is an $t \in S$ with $tsm' = ts'm$. Thus $(1/s) \otimes m = (1/tss') \otimes ts'm = (1/tss') \otimes tsm' = (1/s') \otimes m'$. ψ is R-linear and $\varphi \circ \psi = id_{S^{-1}M}$. For an element $(a/s) \otimes m \in (S^{-1}R) \otimes_R M$, $\psi\varphi((a/s)\otimes m) = \psi(am/s) = (a/s) \otimes m$. Since $\psi\varphi$ is the identity on the generators of $(S^{-1}R)\otimes_R M$, also $\psi\varphi = id_{S^{-1}R\otimes M}$. It is easy to check that the isomorphism is natural.

(6.34) <u>Remark</u>: The isomorphism $(S^{-1}R) \otimes_R M \cong S^{-1}M$ of (6.33) is also $S^{-1}R$-linear.

(6.35) Corollary: Let R be a ring and $S \subseteq R$ a multiplicative set. Then $S^{-1}R$ is a flat R-module.

Proof: Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. By (1.51) the sequence $0 \longrightarrow S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M'' \longrightarrow 0$ is exact. This sequence is by (6.33) naturally isomorphic to $0 \longrightarrow (S^{-1}R)\otimes_R M' \longrightarrow (S^{-1}R)\otimes_R M \longrightarrow (S^{-1}R)\otimes_R M'' \longrightarrow 0$ which has to be exact too. Thus $S^{-1}R$ is R-flat.

(6.36) Proposition: If N and L are flat R-modules then $N \otimes_R L$ is a flat R-module.

Proof: Let $\tau : M' \longrightarrow M$ be an injective R-linear map. Then $\tau \otimes N : M' \otimes_R N \longrightarrow M \otimes_R N$ is injective since N is R-flat. Thus $(\tau \otimes N) \otimes L : (M' \otimes_R N) \otimes_R L \longrightarrow (M \otimes_R N) \otimes_R L$ is injective since L is R-flat. By (0.48) $\tau \otimes (N \otimes L) : M' \otimes_R (N \otimes_R L) \longrightarrow M \otimes_R (N \otimes_R L)$ is injective.

(6.37) Proposition: Let R be a ring and $S \subseteq R$ a multiplicative set. For R-modules M and N there are natural isomorphisms:

$$S^{-1}(M \otimes_R N) \cong (S^{-1}M) \otimes_R N \cong M \otimes_R (S^{-1}N) \cong (S^{-1}M) \otimes_R (S^{-1}N) \cong (S^{-1}M) \otimes_{S^{-1}R} (S^{-1}N).$$

Proof:
$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \cong (S^{-1}R \otimes_R M) \otimes_R N \cong (S^{-1}M) \otimes_R N \\ &\cong (M \otimes_R N) \otimes_R S^{-1}R \cong M \otimes_R (N \otimes_R S^{-1}R) \cong M \otimes_R (S^{-1}N) \\ &\cong S^{-1}R \otimes_R (S^{-1}R \otimes_R (M \otimes_R N)) \cong S^{-1}R \otimes_R (M \otimes_R S^{-1}N) \cong (S^{-1}M) \otimes_R (S^{-1}N) \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \cong (S^{-1}M) \otimes_{S^{-1}R} (S^{-1}N). \end{aligned}$$

(6.38) Theorem: Let N be an R-module. The following conditions are equivalent:

(a) A sequence of R-modules $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact if and only if the sequence $N \otimes_R M' \xrightarrow{N \otimes f} N \otimes_R M \xrightarrow{N \otimes g} N \otimes_R M''$ is exact.

(b) N is flat over R and for all $m \in m\mathrm{Spec}(R)$, $N/mN \neq 0$.

(c) N is flat over R and whenever $N \otimes_R M = 0$ for an R-module M then $M = 0$.

(d) N is flat over R and whenever $N \otimes h = 0$ for an R-linear map $h : M' \longrightarrow M$ then $h = 0$.

Proof: (a) $\Longrightarrow$ (b): Obviously, N is flat over R. For $m \in m\operatorname{Spec}(R)$ consider the exact sequence $0 \longrightarrow m \longrightarrow R \longrightarrow R/m \longrightarrow 0$. Then $0 \longrightarrow N \otimes_R m \longrightarrow N \otimes_R R \longrightarrow N \otimes_R (R/m) \longrightarrow 0$ is exact. If $N/mN \cong N \otimes_R (R/m) = 0$ then $0 \longrightarrow N \otimes_R m \longrightarrow N \otimes_R R \longrightarrow 0$ is exact. By (a) $0 \longrightarrow m \longrightarrow R \longrightarrow 0$ is exact, a contradiction.

(b) $\Longrightarrow$ (c): Let M be a nonzero R-module and $x \in M - (0)$. Then $\operatorname{ann}(x) \neq R$ and the submodule $Rx \subseteq M$ is isomorphic to $R/\operatorname{ann}(x)$. Let $\operatorname{ann}(x) \subseteq m$ for a maximal ideal $m \subseteq R$ and consider the exact sequence $Rx \longrightarrow R/m \longrightarrow 0$. Then $N \otimes_R Rx \longrightarrow N \otimes_R (R/m) \longrightarrow 0$ is exact. By assumption $N \otimes_R (R/m) \cong N/mN \neq 0$ implying that $N \otimes_R Rx \neq 0$. By flatness of N the sequence $0 \longrightarrow N \otimes_R Rx \longrightarrow N \otimes_R M$ is exact and $N \otimes_R M \neq 0$.

(c) $\Longrightarrow$ (d): Let $h: M' \longrightarrow M$ be an R-linear map so that $N \otimes h : N \otimes_R M' \longrightarrow N \otimes_R M$ is the zero map. With $U = \operatorname{im}(h)$ the map h factors into $M' \xrightarrow{\alpha} U \xrightarrow{\beta} M$ where $\alpha(m) = h(m)$ for all $m \in M$ and β is the embedding of U into M. α is surjective while β is injective and $h = \beta\alpha$. Then $N \otimes h = (N \otimes \beta)(N \otimes \alpha)$, $N \otimes \alpha$ is surjective, and by the flatness of N the map $N \otimes \beta$ is injective. Since $N \otimes h = 0$, $N \otimes \alpha = 0$ and therefore $N \otimes U = 0$. Thus $U = 0$.

(d) $\Longrightarrow$ (a): The flatness of N implies the forward direction. For the backward direction let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence with $N \otimes_R M' \xrightarrow{N \otimes f} N \otimes_R M \xrightarrow{N \otimes g} N \otimes_R M''$ exact. Set $U = \operatorname{im}(f)$ and $V = \ker(g)$. Then $N \otimes (gf) = (N \otimes g)(N \otimes f) = 0$ and by assumption (d) $gf = 0$. Therefore $U \subseteq V$.

Consider the exact sequence $0 \longrightarrow U \longrightarrow V \longrightarrow V/U \longrightarrow 0$ which induces an exact sequence $0 \longrightarrow N \otimes_R U \longrightarrow N \otimes_R V \longrightarrow N \otimes_R (V/U) \longrightarrow 0$. By the flatness of N $N \otimes_R U = N \otimes_R \operatorname{im}(f) = \operatorname{im}(N \otimes f)$ and $N \otimes_R V = N \otimes_R \ker(g) = \ker(N \otimes g)$. Thus $N \otimes_R U = N \otimes_R V$ and the map $N \otimes_R V \longrightarrow N \otimes_R (V/U)$ is the zero map. Thus $V \longrightarrow V/U$ is the zero map and $U = V$.

(6.39) Definition: A R-module N is called faithfully flat over R if N satisfies one of the equivalent condition of Theorem (6.38).

(6.40) Examples: (a) Every free R-module is faithfully flat over R.

(b) The R-module $\bigoplus_{m \in \mathrm{mSpec}(R)} R_m$ is faithfully flat.

(c) $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, but $\mathbb{Q}$ is not faithfully flat over $\mathbb{Z}$.

(6.41) Remark: Let $f: R \longrightarrow S$ and $g: R \longrightarrow T$ be homomorphisms of rings. The R-module $D = S \otimes_R T$ is a commutative ring under the multiplication

$$\left(\sum_i{}' \, b_i \otimes c_i\right)\left(\sum_j{}' \, \tilde{b}_j \otimes \tilde{c}_j\right) = \sum_{i,j}{}' \, b_i \tilde{b}_j \otimes c_i \tilde{c}_j.$$

The natural maps $S \longrightarrow S \otimes_R T$ with $b \mapsto b \otimes 1_T$ and $T \longrightarrow S \otimes_R T$ with $c \mapsto 1_S \otimes c$ are homomorphisms of rings and $S \otimes_R T$ is an $(S\text{-}T)$-bialgebra.

§3: FLATNESS AND PROJECTIVITY

(6.42) Remark: Let R be a commutative diagram with identity, $S \subseteq R$ a multiplicative subset, and M, N R-modules. There is a commutative diagram of R-modules:

$$\begin{tikzcd}
\mathrm{Hom}_R(M,N) \arrow[r, "\sigma"] \arrow[d, "i"'] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \\
S^{-1}\mathrm{Hom}_R(M,N) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where $i = i_{\mathrm{Hom},S}$, $\sigma: \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ defined by $\sigma(f) = S^{-1}f$, and $\varphi: S^{-1}\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ defined by $\varphi(f/s) = \frac{1}{s} S^{-1}f$.

(6.43) Example: Let $R = \mathbb{Z}$ and $S = R - (0) = \mathbb{Z} - (0)$.

(a) If $M = \mathbb{Q}$ and $N = \mathbb{Z}$, then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, but $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q} \neq 0$. This shows that φ may not be surjective.

(b) If $M = N = \mathbb{Q}/\mathbb{Z}$, then $\mathrm{id}_{\mathbb{Q}/\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ and for all $n \in \mathbb{Z} - (0)$, $n\,\mathrm{id}_{\mathbb{Q}/\mathbb{Z}} \neq 0$. Thus $S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \neq 0$. Since $S^{-1}(\mathbb{Q}/\mathbb{Z}) = 0$, $\mathrm{Hom}_{\mathbb{Q}}(S^{-1}(\mathbb{Q}/\mathbb{Z}), S^{-1}(\mathbb{Q}/\mathbb{Z})) = 0$ and φ may not be injective.

(6.44) Definition: Let R be a ring and M an R-module. M is called an R-module of finite presentation if there is an exact sequence $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ for some $n, m \in \mathbb{N}$.

(6.45) Remark: (a) If M is an R-module of finite presentation then M is a finite R-module.

(b) If R is a Noetherian ring every finite R-module is of finite presentation.

(c) If R is a non Noetherian ring, $I \subseteq R$ an ideal which is not finitely generated, then the R-module R/I is finite but not of finite presentation.

(d) If P is a finite projective R-module then P is of finite presentation.

Proof: (d) There is an $n \in \mathbb{N}$ and a surjective R-linear map $\varphi: R^n \longrightarrow P$. Since P is projective, $R^n \cong P \oplus \ker(\varphi)$ and $\ker(\varphi) \cong R^n/P$. Thus there is a surjective map

$\psi: R^n \to \ker(\varphi)$ and an exact sequence $R^n \xrightarrow{\psi} R^n \xrightarrow{\varphi} P \to 0$.

(6.46) <u>Theorem</u>: Let M be an R-module of finite presentation, $S \subseteq R$ a multiplicative subset, and N an R-module. The map $\varphi: S^{-1}\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ defined by $\varphi(f/t) = 1/t\, S^{-1}f$ is an isomorphism of $S^{-1}R$-modules.

<u>Proof</u>: If $M = R^n$ is a finite free R-module then $\mathrm{Hom}_R(R^n, N) \cong N^n$ and $S^{-1}\mathrm{Hom}_R(R^n, N) \cong S^{-1}(N^n) \cong (S^{-1}N)^n \cong \mathrm{Hom}_{S^{-1}R}((S^{-1}R)^n, S^{-1}N) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.
For an arbitrary R-module M of finite presentation let $R^m \to R^n \to M \to 0$ be exact.
The following diagram with exact rows is commutative:

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\mathrm{Hom}_R(M,N) \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\swarrow"] & S^{-1}\mathrm{Hom}_R(R^n,N) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\swarrow"] & S^{-1}\mathrm{Hom}_R(R^m,N) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R^n, S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R^m, S^{-1}N)
\end{tikzcd}$$

φ is an isomorphism.

(6.47) <u>Remark</u>: If M is a finite R-module then φ is injective but not necessarily surjective.

(6.48) <u>Remark</u>: The isomorphism φ of (6.46) is natural in the following sense: If $\alpha: M' \to M$ and $\beta: N' \to N$ are R-linear maps, then the diagrams

$$\begin{tikzcd}
S^{-1}\mathrm{Hom}_R(M,N) \arrow[r, "S^{-1}\alpha^*"] \arrow[d, "\varphi"', "\cong"] & S^{-1}\mathrm{Hom}_R(M',N) \arrow[d, "\cong"', "\varphi'"] \\
\mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \arrow[r, "(S^{-1}\alpha)^*"] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M', S^{-1}N)
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{-1}\mathrm{Hom}_R(M,N') \arrow[r, "S^{-1}\beta_*"] \arrow[d, "\varphi'"', "\cong"] & S^{-1}\mathrm{Hom}_R(M,N) \arrow[d, "\cong"', "\varphi"] \\
\mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N') \arrow[r, "(S^{-1}\beta)_*"] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)
\end{tikzcd}$$

commute.

(6.49) <u>Corollary</u>: Let P be an R-module of finite presentation. P is projective if and only if $P_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \mathrm{mSpec}(R)$.

<u>Proof</u>: "$\Rightarrow$": There is an R-module Q so that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. Since the

tensor product commutes with direct sums, for all $m \in m\mathrm{Spec}(R)$, $P_m \oplus Q_m \cong (R_m)^n$. P_m is a projective R_m-module.

"$\Longleftarrow$": Let $M \xrightarrow{\varphi} M' \longrightarrow 0$ be an exact sequence of R-modules. The sequence $\mathrm{Hom}_R(P, M) \xrightarrow{\varphi_*} \mathrm{Hom}_R(P, M') \longrightarrow K \longrightarrow 0$ is exact where $K = \mathrm{coker}(\varphi_*)$. We have to show that $K = 0$. For all $m \in m\mathrm{Spec}(R)$ P_m is a projective R_m-module and the sequence $\mathrm{Hom}_{R_m}(P_m, M_m) \xrightarrow{\varphi_{m*}} \mathrm{Hom}_{R_m}(P_m, M'_m)$ is exact. By (6.46)

$$\begin{tikzcd}
\mathrm{Hom}_{R_m}(P_m, M_m) \arrow[r] & \mathrm{Hom}_{R_m}(P_m, M'_m) \arrow[r] & 0 \\
\mathrm{Hom}_R(P, M)_m \arrow[u, "\cong"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Hom}_R(P, M')_m \arrow[u, "\cong"'] &
\end{tikzcd}$$

Thus $K_m = 0$ for all $m \in m\mathrm{Spec}(R)$ and by the local-global principle $K = 0$.

(6.50) <u>Proposition</u>: Let $\varphi : R \longrightarrow S$ be a homomorphism of rings and M a flat R-module. Then $M \otimes_R S$ is a flat S-module.

<u>Proof</u>: If $0 \longrightarrow N' \longrightarrow N$ is an exact sequence of S-modules, then $0 \longrightarrow (M \otimes_R S) \otimes_S N' \cong M \otimes_R N' \longrightarrow (M \otimes_R S) \otimes_S N \cong M \otimes_R N$ is exact.

(6.51) <u>Corollary</u>: Let M be an R-module. M is flat over R if and only if M_m is flat over R_m for all $m \in m\mathrm{Spec}(R)$.

<u>Proof</u>: "$\Longrightarrow$": By (6.50) $M_m \cong M \otimes_R R_m$ is flat over R_m.

"$\Longleftarrow$": Let $0 \longrightarrow N' \xrightarrow{\varphi} N$ be an exact sequence of R-modules. With $K = \ker(\varphi \otimes M)$ the sequence $0 \longrightarrow K \longrightarrow N' \otimes_R M \xrightarrow{\varphi \otimes M} N \otimes_R M$ is exact. Thus for all $m \in m\mathrm{Spec}(R)$

$$\begin{tikzcd}
0 \arrow[r] & K_m \arrow[r] & (N' \otimes_R M)_m \arrow[r, "(\varphi \otimes M)_m"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & (N \otimes_R M)_m \arrow[d, "\cong"] \\
& & N'_m \otimes_{R_m} M_m \arrow[r, "\varphi_m \otimes M_m"] & N_m \otimes_{R_m} M_m
\end{tikzcd}$$

is exact

Since M_m is R_m-flat, the map $\varphi_m \otimes M_m$ is injective. Since the square is commutative, $(\varphi \otimes M)_m$ is injective and $K_m = 0$ for all $m \in m\mathrm{Spec}(R)$.

(6.52) <u>Proposition</u>: Let (R, m, k) be a local ring and M a finite R-module. Then there is an $n \in \mathbb{N}$ and a surjective R-linear map $\varphi : R^n \longrightarrow M$ with $\ker(\varphi) \subseteq mR^n$.

<u>Proof</u>: Let $\dim_k(M/mM) = n$ and let $x_1, \_\_, x_n \in M$ so that $x_1 + mM, \_\_, x_n + mM$ is a basis of the k-vector space M/mM. By Nakayama $M = Rx_1 + \_\_ + Rx_n$. Define $\varphi : R^n \longrightarrow M$ by $\varphi(e_i) = x_i$ where $e_1, \_\_, e_n$ is the standard basis of R^n. The induced map $\varphi \otimes k : R^n \otimes_R k \longrightarrow M \otimes_R k$ is surjective with $\dim_k(R^n \otimes_R k) = \dim_k(M \otimes_R k) = n$. Thus $\varphi \otimes k$ is an isomorphism and $\ker(\varphi) \subseteq mR^n$.

(6.53) <u>Corollary</u>: Let (R, m) be a local ring. Every finite projective R-module P is free.

<u>Proof</u>: Consider an exact sequence $(*)$ $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} P \longrightarrow 0$ with $N = \ker(\varphi) \subseteq mR^n$. Since P is projective $R^n \cong N \oplus P$ and $(*)$ is split exact. Thus for every R-module M, the sequence $0 \longrightarrow N \otimes_R M \longrightarrow R^n \otimes_R M \xrightarrow{\varphi \otimes M} P \otimes_R M \longrightarrow 0$ is exact. In particular, the sequence $0 \longrightarrow N/mN \longrightarrow R^n/mR^n \xrightarrow{\overline{\varphi}} P/mP \longrightarrow 0$ is exact. Since $N \subseteq mR^n$, $\overline{\varphi}$ is an isomorphism and $N/mN = 0$. N is a finite R-module and by Nakayama $N = 0$.

(6.54) <u>Remark</u>: Kaplansky showed that every projective module over a local ring is free.

Consider the commutative diagram of R-modules and R-maps with exact rows and columns:

$$\begin{tikzcd}
& & & 0 \arrow[d] \\
& K' \arrow[r, "\alpha'"] \arrow[d, "f'"] & K \arrow[r, "\beta'"] \arrow[d, "f"] & K'' \arrow[d, "f''"] \\
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "g'"] & M \arrow[r, "\beta"] \arrow[d, "g"] & M'' \\
& N' \arrow[r, "\alpha''"] \arrow[d, dash] & N & \\
& 0 & &
\end{tikzcd}$$

(6.55) <u>Lemma</u>: The R-linear map α'' in the above diagram is injective.

Proof: by diagram chasing: Let $x \in N'$ with $\alpha''(x)=0$. Then there is a $y \in M$ with $g'(y)=x$. Thus $g\alpha(y) = \alpha'' g'(y) = \alpha''(x) = 0$ and $\alpha(y) \in \ker(g) = \mathrm{im}(f)$. There is an element $z \in K$ with $f(z) = \alpha(y)$ and $f''\beta'(z) = \beta f(z) = \beta\alpha(y) = 0$. Since f'' is injective $\beta'(z)=0$ and there is an element $w \in K'$ with $\alpha'(w) = z$. Then $\alpha f'(w) = f\alpha'(w) = f(z) = \alpha(y)$. Since α is injective, $f'(w) = y$ and hence $x = g'(y) = g'f'(w) = 0$.

(6.56) Theorem: Let F be a flat R-module and $0 \longrightarrow M' \xrightarrow{\alpha} M \longrightarrow F \longrightarrow 0$ an exact sequence of R-modules. For all R-modules N the sequence

$0 \longrightarrow M' \otimes_R N \xrightarrow{\alpha \otimes N} M \otimes_R N \longrightarrow F \otimes_R N \longrightarrow 0$ is exact.

Proof: We need to show that $\alpha \otimes N : M' \otimes_R N \longrightarrow M \otimes_R N$ is injective. Consider an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P a projective R-module. In particular, P is R-flat. This yields a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & M' \otimes K \arrow[r] \arrow[d, dash] & M \otimes K \arrow[r] \arrow[d, dash] & F \otimes K \arrow[r] \arrow[d, dash] & 0 \\
0 \arrow[r] & M' \otimes P \arrow[r] \arrow[d, dash] & M \otimes P \arrow[r] \arrow[d, dash] & F \otimes P \arrow[r] \arrow[d, dash] & 0 \\
 & M' \otimes N \arrow[r, "\alpha \otimes N"] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & F \otimes N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By (6.55) $\alpha \otimes N$ is injective.

(6.57) Proposition: Let (R, m, k) be a local ring, M an R-module of finite presentation, and N a finite R-module. If $f : N \longrightarrow M$ is an R-linear map with $f \otimes k : N/mN \longrightarrow M/mM$ an isomorphism, then $\ker(f)$ is a finite R-module.

Proof: Since $f \otimes k$ is an isomorphism, $M = \mathrm{im}(f) + mM$ and by Nakayama $M = \mathrm{im}(f)$, i.e. f is surjective. By assumption there are integers $m, n \in \mathbb{N}$ and an exact sequence

$R^m \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \longrightarrow 0$. Consider the diagram:

$$\begin{tikzcd} & R^n \arrow[dl, dotted, "g"'] \arrow[d, "\beta"] & \\ N \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

Since R^n is projective, there is an R-linear map $g: R^n \longrightarrow N$ with $\beta = fg$. We claim that $\ker(f) = \operatorname{im}(g\alpha)$ (which implies that $\ker(f)$ is finitely generated.) Tensoring with k yields maps:

$$\begin{tikzcd} R^n \otimes_R k \arrow[r, "g\otimes k"] \arrow[rr, bend right, "\beta\otimes k"] & N \otimes_R k \arrow[r, "f\otimes k \ \cong"] & M \otimes_R k \end{tikzcd}$$

$\beta\otimes k$ is surjective and $f \otimes k$ is an isomorphism. Thus $g \otimes k$ is surjective and g is surjective by Nakayama. Since $0 = \beta\alpha = fg\alpha$, $\operatorname{im}(g\alpha) \subseteq \ker(f)$. Let $x \in N$ with $f(x) = 0$ and let $y \in R^n$ with $g(y) = x$. Then $\beta(y) = fg(y) = f(x) = 0$ and there is a $z \in R^m$ with $\alpha(z) = y$. Then $x = g(y) = (g\alpha)(z) \in \operatorname{im}(g\alpha)$ and $\ker(f) \subseteq \operatorname{im}(g\alpha)$.

(6.58) Corollary: Let (R, m, k) be a local ring and M a flat R-module of finite presentation. Then M is free.

Proof: By (6.52) there is an exact sequence $0 \longrightarrow N \xrightarrow{g} R^n \xrightarrow{f} M \longrightarrow 0$ with $N = \ker(f) \subseteq mR^n$. The k-linear map $f\otimes k : R^n \otimes k \longrightarrow M \otimes k$ is an isomorphism of k-vector spaces and by (6.56) the sequence $0 \longrightarrow N\otimes k \xrightarrow{g\otimes k} R^n \otimes k \xrightarrow{f\otimes k} M \otimes k \longrightarrow 0$ is exact. Thus $N \otimes k \cong N/mN = 0$. By (6.57) $N = \ker(f)$ is finite and by Nakayama $N = 0$. M is free.

(6.59) Corollary: Let (R, m, k) be a Noetherian local ring and M a finite R-module. The following are equivalent:

(a) M is free.

(b) M is projective.

(c) M is flat.

(6.60) Theorem: Let R be a ring and P an R-module. The following are equivalent:

(a) P is finite and projective.

(b) P is of finite presentation and flat.

(c) P is of finite presentation and for all $m \in m\operatorname{Spec}(R)$ the R_m-module P_m is free.

__Proof:__ (a) $\Rightarrow$ (b): By (6.32) and (6.45).

(b) $\Rightarrow$ (c): By base change $P_m = P \otimes_R R_m$ is a flat R_m-module. Since P_m is an R_m-module of finite presentation, by (6.58) P_m is a free R_m-module.

(c) $\Rightarrow$ (a): By (6.49).

§4: MORE ON INJECTIVE MODULES

(6.61) Theorem: For an R-module E the following conditions are equivalent:

(a) E is injective

(b) Every R-linear map $f: I \longrightarrow E$, where $I \subseteq R$ is an ideal, extends to an R-linear map $g: R \longrightarrow E$.

Proof: We have to show that (b) $\Rightarrow$ (a). Suppose there is given a diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "f"'] & M \\ & E & \end{tikzcd}$$

where α is injective and we may assume that $N \subseteq M$ is a submodule. Consider the set $\mathfrak{M} = \{(N', g') \mid N \subseteq N' \subseteq M$ a submodule, $g': N' \longrightarrow E$ an R-linear map with $g'|_N = f\}$.
Since $(N, f) \in \mathfrak{M}$, $\mathfrak{M} \neq \emptyset$. $\mathfrak{M}$ is partially ordered by:

$$(N', g') \leq (N'', g'') \Longleftrightarrow N' \subseteq N'' \text{ and } g''|_{N'} = g'.$$

In order to show that $\mathfrak{M}$ is inductively ordered let $\mathcal{K} = \{(N_i, g_i)\}_{i \in I}$ be a chain in $\mathfrak{M}$. Then $\tilde{N} = \bigcup_{i \in I} N_i$ is a submodule of M and the map $\tilde{g}: \tilde{N} \longrightarrow E$ with $\tilde{g}(n) = g_i(n)$ if $n \in N_i$ is well defined. Thus $(\tilde{N}, \tilde{g}) \in \mathfrak{M}$ is an upper bound of $\mathcal{K}$. By Zorn's Lemma there is a maximal element $(N_0, g_0) \in \mathfrak{M}$. If $N_0 = M$ we are done. If $N_0 \neq M$ let $m \in M - N_0$ and consider $I = \{a \in R \mid am \in N_0\}$. Obviously, I is an ideal of R. Define $h: I \longrightarrow E$ by $h(a) = g_0(am)$ and note that h is R-linear. By assumption (b) h extends to an R-linear map $h': R \longrightarrow E$. Set $N_1 = N_0 + Rm$ and define $g_1: N_1 \longrightarrow E$ by $g_1(n_0 + am) = g_0(n_0) + a\,h'(1)$ for all $n_0 \in N_0$ and $a \in R$. We claim that g_1 is well defined. Suppose $n_0 + am = n_0' + a'm$ for some $n_0, n_0' \in N_0$ and $a, a' \in R$. Then $n_0 - n_0' = (a' - a)m \in N_0$ and $a' - a \in I$. Thus $g_0(n_0 - n_0') = g_0((a'-a)m) = h(a'-a) = (a'-a)\,h'(1)$ and therefore $g_0(n_0) - g_0(n_0') = a'h'(1) - a\,h'(1)$ and $g_0(n_0) + a\,h'(1) = g_0(n_0') + a'h'(1)$. g_1 is a well defined R-linear map which extends g_0. Thus $(N_1, g_1) \in \mathfrak{M}$ with $(N_0, g_0) \lneq (N_1, g_1)$, a contradiction. This implies that $N_0 = M$.

(6.62) Example: By (6.61) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Next we will show that every R-module M is isomorphic to a submodule of an injective R-module E. We first use (6.61) to characterize injective modules over PID's.

(6.63) Definition: Let M be an R-module and $m \in M$, $a \in R$. m is divisible by a if $m = am'$ for some $m' \in M$. The R-module M is called divisible if every $m \in M$ is divisible by every nonzero divisor $a \in R$ of R.

(6.64) Example: $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module.

(6.65) Proposition: Every injective R-module E is divisible.

Proof: Let $m \in E$ and $a \in R$ a nonzero divisor of R. Define $f : (a) \longrightarrow E$ by $f(ax) = xm$. Since a is a NZD of A, f is a well defined R-linear map with $f(a) = m$. By (6.61) f extends to an R-linear map $g : R \longrightarrow E$. If $g(1) = n$ then $g(a) = an = f(a) = m$.

(6.66) Proposition: Let R be a PID and M an R-module. The following are equivalent:

(a) M is injective.

(b) M is divisible.

Proof: (a) $\Rightarrow$ (b): By (6.65).

(b) $\Rightarrow$ (a): Let $I \subseteq R$ be an ideal and $f : I \longrightarrow M$ an R-linear map. By (6.61) we have to show that f extends to an R-linear map $g : R \longrightarrow M$. If $I = (0)$ there is nothing to show. If $I = (a) \neq (0)$ let $f(a) = m$. Since M is divisible and $a \in R$ is a NZD of R there is an $n \in M$ with $m = an$. The R-linear map $g : R \longrightarrow M$ with $g(1) = n$ extends f.

(6.67) Lemma: (a) Every quotient of a divisible module is divisible.

(b) Every direct summand of a divisible module is divisible.

(c) Every direct product of divisible modules is divisible.

(d) Every direct sum of divisible modules is divisible.

Proof: Homework

(6.68) Proposition: Every injective $\mathbb{Z}$-module M can be embedded into an injective $\mathbb{Z}$-module.

Proof: Write $M = F/S$ where $F = \bigoplus_{i\in I} \mathbb{Z}$ is a free $\mathbb{Z}$-module. Then $F = \bigoplus_{i\in I} \mathbb{Z} \subseteq \bigoplus_{i\in I} \mathbb{Q} = E$ and E is a divisible $\mathbb{Z}$-module. S is a $\mathbb{Z}$-submodule of E with $M = F/S \subseteq E/S$. E/S is divisible and thus injective by (6.66).

(6.69) Lemma: Let D be a divisible $\mathbb{Z}$-module and R a ring. Then $\operatorname{Hom}_{\mathbb{Z}}(R,D)$ is an injective R-module.

Proof: $\operatorname{Hom}_{\mathbb{Z}}(R,D)$ is an R-module under the scaler multiplication $(af)(x) = f(ax)$ for all $a,x \in R$ and $f \in \operatorname{Hom}_{\mathbb{Z}}(R,D)$. We have to show: For every ideal $I \subseteq R$ and every R-linear map $\varphi : I \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R,D)$ φ extends to an R-linear map $\rho : R \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R,D)$. First note that φ induces a map $g : I \longrightarrow D$ defined by $g(a) = [\varphi(a)](1_R)$. g is a $\mathbb{Z}$-linear map. Hence g extends to a $\mathbb{Z}$-linear map $\bar{g} : R \longrightarrow D$. Define $\rho : R \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(R,D)$ by $\rho(a) : R \longrightarrow D$ is the map given by $[\rho(a)](x) = \bar{g}(ax)$ for all $a,x \in R$. For every $a \in R$ the map $\rho(a)$ is a homomorphism of the additive groups and ρ is well defined. It is easy to see that ρ is $\mathbb{Z}$-linear.

Claim 1: ρ is R-linear.

Pf of Cl.1: Let $a,b,x \in R$. Then $\rho(ab)(x) = \bar{g}((ab)x) = \bar{g}(b(ax)) = \rho(b)(ax)$ and by the definition of the R-module structure on $\operatorname{Hom}_{\mathbb{Z}}(R,D)$ $\rho(b)(ax) = [a\rho(b)](x)$. Thus $\rho(ab) = a\rho(b)$.

Claim 2: ρ extends φ.

Pf of Cl.2: Let $a \in I$ and $x \in R$. Then $\rho(a)(x) = \bar{g}(ax) = g(ax) = [\varphi(ax)](1_R)$ since $ax \in I$. By the definition of the R-module structure on $\operatorname{Hom}_{\mathbb{Z}}(R,D)$ $[\varphi(ax)](1_R) = [x\varphi(a)](1_R) = \varphi(a)(x)$. Thus $\rho(a) = \varphi(a)$ for all $a \in I$.

(6.70) Theorem: Let R be a ring and M an R-module. M can be embedded into an injective R-module.

Proof: Consider M as a $\mathbb{Z}$-module. By (6.68) there is a divisible $\mathbb{Z}$-module D and an injective $\mathbb{Z}$-linear map $\varepsilon: M \longrightarrow D$. Since $\mathrm{Hom}_{\mathbb{Z}}(R,-)$ is left exact, there is an injective $\mathbb{Z}$-linear map $\varepsilon_*: \mathrm{Hom}_{\mathbb{Z}}(R,M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,D)$ with $\varepsilon_*(f) = \varepsilon f$. By definition of the R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(R,-)$ for all $a,x \in R$: $\varepsilon_*(af)(x) = (\varepsilon(af))(x) = \varepsilon f(ax) = (\varepsilon f)(ax) = \varepsilon_*(f)(ax) = a\,\varepsilon_*(f)(x)$. Thus $\varepsilon_*(af) = a\,\varepsilon_*(f)$ and ε_* is R-linear. Every R-linear map $g: R \longrightarrow M$ is also $\mathbb{Z}$-linear. Hence $\mathrm{Hom}_R(R,M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R,M)$. By (6.69) $\mathrm{Hom}_{\mathbb{Z}}(R,D)$ is an injective R-module. Since $M \cong \mathrm{Hom}_R(R,M)$ we obtain the desired embedding: $M \cong \mathrm{Hom}_R(R,M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,M) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{\mathbb{Z}}(R,D)$.

In the following we want to show that for every R-module M there is a smallest injective R-module which contains M.

(6.71) Definition: An essential extension of an R-module M is an R-module E which contains M such that for every nonzero submodule $N \subseteq E$ the intersection $N \cap M \neq 0$. If, in addition, $M \neq E$ then E is called a proper essential extension of M.

(6.72) Example: $\mathbb{Q}$ is a proper essential extension of $\mathbb{Z}$.

(6.73) Theorem: An R-module M is injective if and only if M has no proper essential extensions.

Proof: "$\Longrightarrow$": Suppose that M is injective and let $M \subseteq E$ be an essential extension of M. By (6.22) there is an R-module $N \subseteq E$ with $E = M \oplus N$. Thus $E = M$.
"$\Longleftarrow$": Suppose that M has no proper essential extensions. Let E be an injective R-module with $M \subseteq E$. Consider the set $\mathcal{M} = \{N \subseteq E \mid N \text{ a submodule and } N \cap M = 0\}$.

Since $(0)\in\mathcal{M}$, $\mathcal{M}\neq\emptyset$. $\mathcal{M}$ is partially ordered by inclusion and every chain in $\mathcal{M}$ has an upper bound in $\mathcal{M}$. Thus Zorn's Lemma applies and there is a maximal submodule $N_0\subseteq E$ with $M\cap N_0=0$. The composition of R-linear maps $\varphi=\pi i: M\xhookrightarrow{i} E\xrightarrow{\pi} E/N_0$ is injective and we may consider M as an R-submodule of E/N_0. We claim that $M\subseteq E/N_0$ is an essential extension. Every nonzero submodule $\overline{N}\subseteq E/N_0$ is image of a submodule $N\subseteq E$ with $N_0\subsetneq N$. By the maximality of N_0 $N\cap M\neq 0$ and hence $M\cap\overline{N}\neq 0$. Since M has no proper essential extensions, φ is surjective and $M\cong E/N_0$. This implies that $E=M+N_0$ and, since $M\cap N_0=0$, $E=M\oplus N_0$. By (6.21) every direct summand of an injective module is injective.

(6.74) <u>Theorem</u>: Let E be an R-module, $M\subseteq E$ a submodule. The following conditions are equivalent:

(a) E is a maximal essential extension of M, that is, no proper extension of E is an essential extension of M.

(b) E is injective and E is an essential extension of M.

(c) E is injective and there is no proper injective submodule E' of E with $M\subseteq E'\subsetneq E$.

Moreover, for every R-module M an R-module E exists so that conditions (a), (b), (c) are satisfied.

<u>Proof</u>: (a) $\Rightarrow$ (b): By (6.73) it suffices to show that E has no proper essential extensions. Suppose that $E\subseteq F$ is an essential extension of E and let $N\subseteq F$ be a nonzero submodule. $N_0=N\cap E$ is a nonzero submodule of E. Thus $N_0\cap M\neq 0$ and therefore $N\cap M\neq 0$. F is an essential extension of M. By the maximality of E, $E=F$.

(b) $\Rightarrow$ (c): Suppose that $E'\subseteq E$ is an injective submodule with $M\subseteq E'$. By (6.22) there is a submodule $E''\subseteq E$ with $E=E'\oplus E''$. Then $E''\cap M=0$ and therefore $E''=0$ and $E=E'$.

(c) $\Rightarrow$ (a): Consider $\mathcal{M}=\{N\subseteq E \mid N \text{ a submodule}, M\subseteq N, \text{ and } N \text{ an essential extension of } M\}$. Since $M\in\mathcal{M}$, $\mathcal{M}\neq\emptyset$, and $\mathcal{M}$ is partially ordered by inclusion. Let $\mathcal{H}=\{N_i\}_{i\in I}\subseteq\mathcal{M}$ be a chain in $\mathcal{M}$ and set $N_0=\bigcup_{i\in I}N_i$. N_0 is a submodule of E with $M\subseteq N_0$.

Let $U \subseteq N_0$ be a nonzero submodule. Then there is an $i \in I$ with $U \cap N_i \neq 0$. Since N_i is an essential extension of M, $(U\cap N_i)\cap M \neq 0$ and N_0 is an essential extension of M. By Zorn's Lemma $\mathcal{M}$ has a maximal element E'. We claim that E' is a maximal essential extension of M. Let N be an R-module with $M \subseteq E' \subseteq N$ so that N is an essential extension of M. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "j"] \arrow[d, "i"'] & N \arrow[dl, dotted, "\varphi"] \\ & E & \end{tikzcd}$$

where i and j are the inclusions. Since E is injective, there is an R-linear map $\varphi: N \to E$ with $\varphi|_{E'} = i$. Since N is an essential extension of M, N is an essential extension of E'. Thus if $\ker(\varphi) \neq 0$, then $\ker(\varphi)\cap E' \neq 0$ and i fails to be injective. Thus $\ker(\varphi) = 0$ and $N \cong \varphi(N) \subseteq E$ is an essential extension of M. By the maximality of E', $\varphi(N) = E'$ and E' is a maximal essential extension of M. By (a) $\Rightarrow$ (b) E' is an injective R-module and by (c) $E = E'$.

In order to show the existence of an R-module E with properties (a), (b), (c), let $\tilde{E}$ be an injective R-module with $M \subseteq \tilde{E}$. The proof (c) $\Rightarrow$ (a) shows that $\tilde{E}$ contains a maximal essential extension E of M.

(6.75) Definition: Let M be an R-module. An R-module E that satisfies one of the equivalent conditions of (6.74) is called the injective hull of M.

The following proposition shows that the injective hull is unique up to isomorphism

(6.76) Proposition: Let M be an R-module and E an injective hull of M.

(a) If D is an injective R-module with $M \subseteq D$, $i: M \to D$ the inclusion, then there is an injective R-linear map $\varphi: E \to D$ with $\varphi|_M = i$.

(b) If E' is another injective hull of M then there is an isomorphism $\varphi: E \to E'$ with $\varphi|_M = id_M$.

Proof: (a) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, hook, "i"'] & E \arrow[dl, dotted, "\varphi"] \\
& D &
\end{tikzcd}$$

By the injectivity of D there is an R-linear map $\varphi : E \longrightarrow D$ with $\varphi|_M = i$. If $\ker(\varphi) \neq 0$ then $\ker(\varphi) \cap M \neq 0$ since E is an essential extension of M. Thus $\ker(\varphi) = 0$ and φ is injective.

(b) If E' is another injective hull of M, by (a) there is an injective R-linear map $\varphi : E \longrightarrow E'$ with $\varphi|_M = \mathrm{id}_M$. Then $\mathrm{im}(\varphi)$ is an injective submodule of E' with $M \leq \mathrm{im}(\varphi)$. By (6.74) $E' = \mathrm{im}(\varphi)$ and φ is an isomorphism.

§5: THE TORSION FUNCTOR Γ_I

(6.77) Definition: Let R be a ring, $I \subseteq R$ an ideal, and M an R-module. The set $\Gamma_I(M) = \bigcup_{n\in\mathbb{N}} (0 :_M I^n)$ is called the I-torsion of M.

(6.78) Remark: $\Gamma_I(M)$ is the set of all elements of M which are annihilated by some power of I. Obviously, $(0:_M I) \subseteq (0:_M I^2) \subseteq \dots \subseteq (0:_M I^n) \subseteq \dots$ is an increasing chain of submodules of M which is stationary if M is Noetherian. In any case, $\Gamma_I(M)$ is a submodule of M.

(6.79) Definition: A covariant functor $F: \mathcal{M}(R) \longrightarrow \mathcal{M}(R)$ from the category of R-modules into itself is called R-linear if the map $\mathrm{Hom}_R(M,N) \xrightarrow{\phi_{M,N}} \mathrm{Hom}_R(FM,FN)$ defined by $\phi_{M,N}(f) = Ff$ is R-linear for all R-modules M and N. That is, for all $f,g: M \longrightarrow N$ and all $a \in R$, $F(f+g) = Ff + Fg$ and $F(af) = aFf$.

(6.80) Proposition: $\Gamma_I(-)$ is a covariant R-linear functor from $\mathcal{M}(R)$ into itself.

Proof: If $f: M \longrightarrow N$ is an R-linear map of R-modules then $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$. Let $\Gamma_I(f)$ be the restriction of f: $\Gamma_I(f) = f|_{\Gamma_I(M)}: \Gamma_I(M) \longrightarrow \Gamma_I(N)$. If $g: N \longrightarrow L$ is another R-linear map then obviously, $\Gamma_I(gf) = \Gamma_I(g)\Gamma_I(f)$. Moreover, $\Gamma_I(\mathrm{id}_M) = \mathrm{id}_{\Gamma_I(M)}$. $\Gamma_I(-)$ is a covariant functor from $\mathcal{M}(R)$ into itself. It is easy to verify that $\Gamma_I(-)$ is R-linear.

(6.81) Proposition: Let R be a Noetherian ring and $I, J \subseteq R$ ideals. Then:

(a) For all R-modules M, $\Gamma_I(\Gamma_J(M)) = \Gamma_{I+J}(M)$

(b) $\Gamma_I(-) = \Gamma_J(-)$ if and only if $\mathrm{rad}(I) = \mathrm{rad}(J)$.

Proof: (a) Let $x \in \Gamma_I(\Gamma_J(M))$. Then there are $r,s \in \mathbb{N}$ with $I^r x = 0$ and $J^s x = 0$ and therefore $(I+J)^{r+s} x = 0$ and $x \in \Gamma_{I+J}(M)$. For the other inclusion let $x \in \Gamma_{I+J}(M)$. Then $(I+J)^r x = 0$ for some $r \in \mathbb{N}$. In particular, $I^r x = 0$ and $J^r x = 0$ and $x \in \Gamma_I(\Gamma_J(M))$.

(b) "$\Rightarrow$": $\Gamma_I(R/\mathrm{rad}(J)) = \Gamma_J(R/\mathrm{rad}(J)) = R/\mathrm{rad}(J)$ and $I \subseteq \mathrm{rad}(J)$. Similarly, $J \subseteq \mathrm{rad}(I)$.
"$\Leftarrow$": If $\mathrm{rad}(I) = \mathrm{rad}(J)$ then $I^n \subseteq J$ for some $n \in \mathbb{N}$ since R is Noetherian. Thus $\Gamma_J(M) \subseteq \Gamma_I(M)$ for every R-module M. Similarly, $\Gamma_I(M) \subseteq \Gamma_J(M)$.

(6.82) Proposition: The I-torsion functor $\Gamma_I(-)$ is left exact.

Proof: Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence of R-modules. Obviously, $\Gamma_I(f)$ is injective and $\Gamma_I(g)\Gamma_I(f) = \Gamma_I(gf) = 0$. It remains to show that $\ker(\Gamma_I(g)) \subseteq \mathrm{im}(\Gamma_I(f))$. Let $x \in \ker(\Gamma_I(g))$. Then $g(x) = 0$ and $I^r x = 0$ for some $r \in \mathbb{N}$. Let $y \in L$ with $f(y) = x$. For all $a \in I^r$, $f(ay) = ax = 0$ and thus $ay = 0$ since f is injective. Hence $y \in \Gamma_I(L)$ and $x \in \mathrm{im}(\Gamma_I(f))$.

(6.83) Proposition: Let R be a Noetherian ring and M an R-module. Then:
(a) $\Gamma_I(\Gamma_I(M)) = \Gamma_I(M)$
(b) $\Gamma_I(M/\Gamma_I(M)) = 0$
(c) If M is a finite R-module then $\Gamma_I(M) = 0$ if and only if I contains a NZD of M.

Proof: (a), (b) trivial
(c) "$\Rightarrow$": If $\Gamma_I(M) = 0$ then $I \nsubseteq P$ for all $P \in \mathrm{Ass}_R(M)$. Since $\mathrm{Ass}_R(M)$ is finite, $I \nsubseteq \bigcup_{P \in \mathrm{Ass}_R(M)} P$. Every element $a \in I - \bigcup_{P \in \mathrm{Ass}_R(M)} P$ is a NZD of M.
"$\Leftarrow$": If $a \in I$ is a NZD of M then $a^n m \neq 0$ for all $n \in \mathbb{N}$ and all $m \in M - (0)$.

(6.84) Remark: Together with the Hom-functors $\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,N)$, and the tensor functor $M \otimes_R -$, the torsion functor $\Gamma_I(-)$ is an important tool in commutative algebra. Its derived functors (Chapter VII) are the local cohomology functors $H^i_I(-)$. For a deeper understanding of these functors more investigations into $\Gamma_I(-)$ are needed. For example, a basic result states that $\Gamma_I(-)$ is a direct limit of certain Hom-functors. This yields that its derived functors $H^i_I(-)$ are direct limits of derived Hom-functors.

§6: DIRECT LIMITS

(6.85) *Definition*: (a) A *directed set* is a partially ordered set Λ such that for all $\lambda, \mu \in \Lambda$ there is an element $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$.

(b) Let R be a ring, Λ a directed set, and $\{M_\lambda \mid \lambda \in \Lambda\}$ a family of R-modules. Suppose that for each pair $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ there is given an R-linear map $f_{\mu\lambda}: M_\lambda \longrightarrow M_\mu$ so that (i) $f_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ for all $\lambda \in \Lambda$ and (ii) $f_{\nu\mu} f_{\mu\lambda} = f_{\nu\lambda}$ whenever $\lambda \leq \mu \leq \nu$. Then $\{M_\lambda, f_{\mu\lambda}\}$ is called a *direct system* of R-modules over the directed set Λ.

Note that one can define direct systems of sets, groups, rings etc. accordingly.

(6.86) *Definition*: (a) Let $\{M_\lambda, f_{\mu\lambda}\}$ and $\{M'_\lambda, f'_{\mu\lambda}\}$ be direct systems of R-modules over the directed set Λ. A *morphism* (or an R-*linear map*) $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M'_\lambda, f'_{\mu\lambda}\}$ is a family of R-linear maps $\varphi = \{\varphi_\lambda : M_\lambda \longrightarrow M'_\lambda \mid \lambda \in \Lambda$ and φ_λ R-linear$\}$ so that for all $\lambda \leq \mu$, $f'_{\mu\lambda} \varphi_\lambda = \varphi_\mu f_{\mu\lambda}$, that is, the diagram

$$\begin{tikzcd}
M_\lambda \arrow[r, "f_{\mu\lambda}"] \arrow[d, "\varphi_\lambda"'] & M_\mu \arrow[d, "\varphi_\mu"] \\
M'_\lambda \arrow[r, "f'_{\mu\lambda}"] & M'_\mu
\end{tikzcd}$$

commutes.

(b) Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of R-modules over Λ and X an R-module. A *morphism* (or an R-*linear map*) $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ is a family of R-linear maps $\varphi = \{\varphi_\lambda : M_\lambda \longrightarrow X \mid \lambda \in \Lambda$ and φ_λ R-linear$\}$ so that for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, $\varphi_\lambda = \varphi_\mu f_{\mu\lambda}$, that is, the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "\varphi_\lambda"] \arrow[d, "f_{\mu\lambda}"'] & X \\
M_\mu \arrow[ur, "\varphi_\mu"'] &
\end{tikzcd}$$

commutes.

(c) Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of R-modules over Λ. An R-module M_∞ together with an R-linear map $\psi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow M_\infty$ is called a *direct limit* of $\{M_\lambda, f_{\mu\lambda}\}$ if for every R-linear map $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ there is a unique R-linear map $h: M_\infty \longrightarrow X$ so that $\varphi_\lambda = h \psi_\lambda$ for all $\lambda \in \Lambda$, that is, the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[dr, "\psi_\lambda"] \arrow[drr, bend left, "\varphi_\lambda"] \arrow[d, "f_{\mu\lambda}"'] & & \\
& M_\infty \arrow[r, "h"] & X \\
M_\mu \arrow[ur, "\psi_\mu"] \arrow[urr, bend right, "\varphi_\mu"'] & &
\end{tikzcd}$$

commutes. Notation: $M_\infty = \varinjlim M_\lambda = \varinjlim_{\Lambda} M_\lambda$.

(6.87) Theorem: Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of R-modules over Λ. Then $\varinjlim M_\lambda$ exists.

Proof: Let $C = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be the direct sum of the M_λ and $i_\lambda : M_\lambda \longrightarrow \bigoplus_{\mu\in\Lambda} M_\mu$ the natural embedding. Let $D \subseteq C$ be the submodule which is generated by

$$\{\, i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda,\ \lambda \leq \mu \,\}.$$

Set $M_\infty = C/D$ and define $\psi : \{M_\lambda, f_{\mu\lambda}\} \longrightarrow M_\infty$ by $\psi_\lambda(a) = i_\lambda(a) + D$ for all $a \in M_\lambda$. Obviously, $\psi_\lambda = \psi_\mu f_{\mu\lambda}$ for $\lambda \leq \mu$. Let $\varphi : \{M_\lambda, f_{\mu\lambda}\} \longrightarrow X$ be an R-linear map. By the universal property of the direct sum there is a unique R-linear map $g : C = \bigoplus M_\lambda \longrightarrow X$ so that $g i_\lambda = \varphi_\lambda$ for all $\lambda \in \Lambda$. Then for all $\lambda\in\Lambda$ and all $a \in M_\lambda$:

$$g(i_\lambda(a) - i_\mu(f_{\mu\lambda}(a))) = (g i_\lambda)(a) - (g i_\mu)(f_{\mu\lambda}(a)) = \varphi_\lambda(a) - \varphi_\mu(f_{\mu\lambda}(a)) = 0.$$

Hence g factors through M_∞:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "v"'] & X \\
M_\infty \arrow[ur, "h"'] &
\end{tikzcd}$$

where v is the natural map.

Moreover, for all $\lambda \in \Lambda$ the diagram:

$$\begin{tikzcd}
M_\lambda \arrow[r, "i_\lambda"'] \arrow[rr, bend left, "\psi_\lambda"] \arrow[dr, "\varphi_\lambda"'] & C \arrow[r, "v"'] \arrow[d, "g"] & M_\infty \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

commutes.

The uniqueness of h follows since M_∞ is generated by $\bigcup_{\lambda\in\Lambda} \psi_\lambda(M_\lambda)$. Thus $M_\infty = \varinjlim M_\lambda$ together with ψ is a direct limit of $\{M_\lambda, f_{\mu\lambda}\}$.

(6.88) Proposition: The direct limit $\varinjlim M_\lambda$ of a direct system $\{M_\lambda, f_{\mu\lambda}\}$ is unique up to isomorphism.

Proof: Homework

(6.89) Examples: (a) Let M be an R-module, $M_\lambda \subseteq M$ submodules, $\lambda \leq \mu$ if and only if

$M_\lambda \subseteq M_\mu$, and $f_{\mu\lambda}: M_\lambda \longrightarrow M_\mu$ the inclusion. If $\{M_\lambda, f_{\mu\lambda}\}$ is a direct system, then $\varinjlim M_\lambda \cong \bigcup_{\lambda\in\Lambda} M_\lambda$.

(b) Let M be an R-module and $\{M_\lambda, \text{incl.}\}$ the direct system of all finite submodules of M. Then $M \cong \varinjlim M_\lambda$ is a direct limit of finite R-modules.

(c) Let R be a ring and $\{R_\lambda, \text{incl.}\}$ the direct system of all finitely generated $\mathbb{Z}$-subalgebras of R. Then $R \cong \varinjlim R_\lambda$ is a direct limit of Noetherian rings.

(6.90) Remark: Let $\{M_\lambda, f_{\mu\lambda}\}$ and $\{M'_\lambda, f'_{\mu\lambda}\}$ be direct systems of R-modules over the directed set Λ and let $\varphi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M'_\lambda, f'_{\mu\lambda}\}$ be an R-linear map. Then there is a unique R-linear map $\varinjlim \varphi_\lambda : \varinjlim M_\lambda \longrightarrow \varinjlim M'_\lambda$ with $\psi'_\lambda \varphi_\lambda = (\varinjlim \varphi_\lambda)\psi_\lambda$ (where $\psi: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \varinjlim M_\lambda$ and $\psi': \{M'_\lambda, f'_{\lambda\mu}\} \longrightarrow \varinjlim M'_\lambda$ are the natural maps.)

Proof: By the definition of the direct limit (6.88)(c).

(6.91) Proposition: Let $\{M_\lambda, f_{\mu\lambda}\}$ be a direct system of R-modules with direct limit $\varinjlim M_\lambda$ and natural maps $\psi_\lambda : M_\lambda \longrightarrow \varinjlim M_\lambda$.

(a) For all $m \in \varinjlim M_\lambda$ there are a $\lambda \in \Lambda$ and an $m_\lambda \in M_\lambda$ with $\psi_\lambda(m_\lambda) = m$.

(b) If $m_\lambda \in M_\lambda$ with $\psi_\lambda(m_\lambda) = 0$ then there is a $\mu \geq \lambda$ with $f_{\mu\lambda}(m_\lambda) = 0$.

Proof: (a) Let $m \in \varinjlim M_\lambda$. Since $\varinjlim M_\lambda$ is generated by $\{\psi_\lambda(M_\lambda)\}_{\lambda\in\Lambda}$, $m = \sum_{i=1}^{r} \psi_{\tau_i}(m_{\tau_i})$. Let $\lambda \in \Lambda$ with $\tau_i \leq \lambda$ for all $1 \leq i \leq r$. Then $\psi_{\tau_i}(m_{\tau_i}) = \psi_\lambda(f_{\lambda\tau_i}(m_{\tau_i}))$ and with $m_\lambda = \sum_i f_{\lambda\tau_i}(m_{\tau_i})$ we have that $m = \psi_\lambda(m_\lambda)$.

(b) Let $i_\lambda : M_\lambda \longrightarrow C = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be the natural injections and $D \subseteq C$ the submodule generated by $\{ i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda,\ \lambda \leq \mu\}$. By the construction of $\varinjlim M_\lambda = C/D$ $\psi_\lambda(m_\lambda) = 0$ implies that

$$(*) \qquad i_\lambda(m_\lambda) = \sum\nolimits'_\tau \left(i_{\mu(\tau)} f_{\mu(\tau)\tau}(m_\tau) - i_\tau(m_\tau) \right).$$

Let $\sigma \in \Lambda$ be larger than any index which occurs in $(*)$. In the following we write μ instead of $\mu(\tau)$ with μ dependent on τ. Then:

$$i_\sigma f_{\sigma\lambda}(m_\lambda) = (i_\sigma f_{\sigma\lambda}(m_\lambda) - i_\lambda(m_\lambda)) + i_\lambda(m_\lambda)$$
$$= (i_\sigma f_{\sigma\lambda}(m_\lambda) - i_\lambda(m_\lambda)) + \sum_\tau (i_\mu f_{\mu\tau}(m_\tau) - i_\tau(m_\tau))$$

and for each τ:

$$i_\mu f_{\mu\tau}(m_\tau) - i_\tau(m_\tau) = (i_\sigma f_{\sigma\tau}(m_\tau) - i_\tau(m_\tau)) +$$
$$+ (i_\sigma f_{\sigma\mu}(-f_{\mu\tau}(m_\tau)) - i_\mu(-f_{\mu\tau}(m_\tau))).$$

Thus we can write

$$i_\sigma f_{\sigma\lambda}(m_\lambda) = \sum_\rho (i_\sigma f_{\sigma\rho}(n_\rho) - i_\rho(n_\rho)) \in \oplus M_\lambda.$$

If $\rho \neq \sigma$, then $i_\rho(n_\rho) = 0$ and $n_\rho = 0$. This implies:

$$i_\sigma f_{\sigma\lambda}(m_\lambda) = i_\sigma f_{\sigma\sigma}(n_\sigma) - i_\sigma(n_\sigma) = 0$$

since $f_{\sigma\sigma} = \mathrm{id}$. Therefore $f_{\sigma\lambda}(m_\lambda) = 0$.

(6.92) <u>Theorem</u>: Let $\{M'_\lambda, f'_{\mu\lambda}\}$, $\{M_\lambda, f_{\mu\lambda}\}$, $\{M''_\lambda, f''_{\mu\lambda}\}$ be direct systems of R-modules over the directed set Λ and let $\alpha: \{M'_\lambda, f'_{\mu\lambda}\} \longrightarrow \{M_\lambda, f_{\mu\lambda}\}$ and $\beta: \{M_\lambda, f_{\mu\lambda}\} \longrightarrow \{M''_\lambda, f''_{\mu\lambda}\}$ be R-linear maps so that for all $\lambda \in \Lambda$ the sequence $M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda$ is exact. Then

$$\varinjlim M'_\lambda \xrightarrow{\varinjlim \alpha_\lambda} \varinjlim M_\lambda \xrightarrow{\varinjlim \beta_\lambda} \varinjlim M''_\lambda \qquad \text{is exact.}$$

<u>Proof</u>: Let $\psi_\lambda: M_\lambda \longrightarrow \varinjlim M_\lambda$ and $\psi'_\lambda, \psi''_\lambda$, respectively, denote the natural maps into the direct limits. Set $\alpha_\infty = \varinjlim \alpha_\lambda$ and $\beta_\infty = \varinjlim \beta_\lambda$. For all $\lambda \in \Lambda$ there are commutative diagrams:

$$\begin{tikzcd} M'_\lambda \arrow[r, "\alpha_\lambda"] \arrow[d, "\psi'_\lambda"'] & M_\lambda \arrow[d, "\psi_\lambda"] \\ \varinjlim M'_\lambda \arrow[r, "\alpha_\infty"] & \varinjlim M_\lambda \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_\lambda \arrow[r, "\beta_\lambda"] \arrow[d, "\psi_\lambda"'] & M''_\lambda \arrow[d, "\psi''_\lambda"] \\ \varinjlim M_\lambda \arrow[r, "\beta_\infty"] & \varinjlim M''_\lambda \end{tikzcd},$$

(1) $\beta_\infty \alpha_\infty = 0$.

<u>Pf</u> (1): Let $z' \in \varinjlim M'_\lambda$. Then there are a $\lambda \in \Lambda$ and an $m'_\lambda \in M'_\lambda$ with $\psi'_\lambda(m'_\lambda) = z'$. Thus $\alpha_\infty(z') = \alpha_\infty \psi'_\lambda(m'_\lambda) = \psi_\lambda \alpha_\lambda(m'_\lambda)$ and $\beta_\infty \alpha_\infty(z') = \beta_\infty \psi_\lambda \alpha_\lambda(m'_\lambda) = \psi''_\lambda \beta_\lambda \alpha_\lambda(m'_\lambda) = 0$.

(2) $\ker(\beta_\infty) \subseteq \mathrm{im}(\alpha_\infty)$.

<u>Pf</u> (2): Let $z \in \varinjlim M_\lambda$ with $\beta_\infty(z) = 0$ and let $m_\lambda \in M_\lambda$ with $\psi_\lambda(m_\lambda) = z$. Then $\beta_\infty \psi_\lambda(m_\lambda) = \psi''_\lambda \beta_\lambda(m_\lambda) = 0$. By (6.91) there is a $\mu \in \Lambda$ with $\lambda \leq \mu$ so that $f''_{\mu\lambda} \beta_\lambda(m_\lambda) = 0$. Since $f''_{\mu\lambda} \beta_\lambda = \beta_\mu f_{\mu\lambda}$, $\beta_\mu(f_{\mu\lambda}(m_\lambda)) = 0$ and therefore $f_{\mu\lambda}(m_\lambda) \in \ker(\beta_\mu) = \mathrm{im}(\alpha_\mu)$. Let

$m'_\mu \in M'_\mu$ with $\alpha_\mu(m'_\mu) = f_{\mu\lambda}(m_\lambda)$. Then $\alpha_\infty \psi'_\mu(m'_\mu) = \psi_\mu \alpha_\mu(m_\mu) = \psi_\mu f_{\mu\lambda}(m_\lambda) = \psi_\lambda(m_\lambda) = z$.

(6.93) <u>Theorem</u>: Let R be a ring, N an R-module, and $\{M_\lambda, f_{\mu\lambda}\}$ a direct system of R-modules over Λ. Then $\{M_\lambda \otimes_R N, f_{\mu\lambda} \otimes N\}$ is a direct system of R-modules over Λ and

$$\varinjlim (M_\lambda \otimes_R N) \cong (\varinjlim M_\lambda) \otimes_R N$$

i.e. the tensor product commutes with direct limits.

<u>Proof</u>: Obviously, $\{M_\lambda \otimes_R N, f_{\mu\lambda} \otimes N\}$ is a direct system of R-modules. There is an isomorphism $(*)$ $\oplus (M_\lambda \otimes_R N) \cong (\oplus M_\lambda) \otimes_R N$ which maps $i_\lambda(a \otimes b)$ to $i_\lambda(a) \otimes b$. Under this map $i_\lambda(a\otimes b) - i_\mu(f_{\mu\lambda} \otimes N)(a \otimes b)$ is mapped to $(i_\lambda(a) - i_\mu(f_{\mu\lambda}(a))) \otimes b$. Let D be the submodule of $\oplus M_\lambda$ generated by $\{i_\lambda(a) - i_\mu(f_{\mu\lambda}(a)) \mid a \in M_\lambda, \lambda \le \mu\}$. The sequence

$$D \otimes_R N \xrightarrow{h} (\oplus M_\lambda) \otimes_R N \longrightarrow (\oplus M_\lambda / D) \otimes_R N \longrightarrow 0$$

is exact and under the isomorphism $(*)$ the submodule $\tilde{D}$ of $\oplus (M_\lambda \otimes_R N)$, which is generated by $\{i_\lambda(a \otimes b) - i_\mu(f_{\mu\lambda} \otimes N)(a \otimes b) \mid a \otimes b \in M_\lambda \otimes_R N, \lambda \le \mu\}$, is mapped onto h. This shows the assertion.

§7: INVERSE SYSTEMS

(6.94) Definition: Let R be a ring, Λ a directed set, and $\{M_\lambda, \lambda \in \Lambda\}$ a family of R-modules. Suppose that for each pair $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ there is given an R-linear map $f_{\lambda\mu} : M_\mu \longrightarrow M_\lambda$ so that (i) $f_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ for all $\lambda \in \Lambda$ and (ii) $f_{\lambda\mu} f_{\mu\nu} = f_{\lambda\nu}$ whenever $\lambda \leq \mu \leq \nu$. Then $\{M_\lambda, f_{\lambda\mu}\}$ is called an inverse system of R-modules over Λ. Inverse systems of sets, groups, rings etc. are defined accordingly.

(6.95) Definition: (a) Let $\{M_\lambda, f_{\lambda\mu}\}$ and $\{M'_\lambda, f'_{\lambda\mu}\}$ be inverse systems of R-modules over Λ. A morphism (or an R-linear map) $\varphi : \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M'_\lambda, f'_{\lambda\mu}\}$ is a family of R-linear maps $\varphi = \{\varphi_\lambda : M_\lambda \longrightarrow M'_\lambda \mid \lambda \in \Lambda$ and φ_λ R-linear$\}$ so that for all $\lambda \leq \mu$, $f'_{\lambda\mu} \varphi_\mu = \varphi_\lambda f_{\lambda\mu}$, that is, the diagram:

$$\begin{tikzcd}
M_\mu \arrow[r, "\varphi_\mu"] \arrow[d, "f_{\lambda\mu}"'] & M'_\mu \arrow[d, "f'_{\lambda\mu}"] \\
M_\lambda \arrow[r, "\varphi_\lambda"] & M'_\lambda
\end{tikzcd}$$

commutes.

(b) Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of R-modules over Λ and X an R-module. A morphism (or an R-linear map) $\varphi : X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is a family of R-linear maps $\varphi = \{\varphi_\lambda : X \longrightarrow M_\lambda \mid \lambda \in \Lambda$ and φ_λ R-linear$\}$ so that for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, $\varphi_\lambda = f_{\lambda\mu} \varphi_\mu$, that is, the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi_\lambda"] \arrow[dr, "\varphi_\lambda"'] & M_\mu \arrow[d, "f_{\lambda\mu}"] \\
& M_\lambda
\end{tikzcd}$$

commutes.

(c) Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of R-modules over Λ. An R-module M_∞ together with an R-linear map $\psi : M_\infty \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is called an inverse limit of $\{M_\lambda, f_{\mu\lambda}\}$ if for every R-linear map $\varphi : X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ there is a unique R-linear map $h : X \longrightarrow M_\infty$ so that $\varphi_\lambda = \psi_\lambda h$ for all $\lambda \in \Lambda$, that is, for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ the diagram:

$$\begin{tikzcd}
& & M_\mu \arrow[dd, "f_{\lambda\mu}"] \\
X \arrow[r, "h"] \arrow[urr, bend left, "\varphi_\mu"] \arrow[drr, bend right, "\varphi_\lambda"'] & M_\infty \arrow[ur, "\psi_\mu"] \arrow[dr, "\psi_\lambda"'] & \\
& & M_\lambda
\end{tikzcd}$$

commutes.

Notation: $M_\infty = \varprojlim M_\lambda = \varprojlim_{\Lambda} M_\lambda$.

(6.96) Theorem: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of R-modules over Λ. The inverse limit $\varprojlim M_\lambda$ exists.

Proof: For all $\lambda \in \Lambda$ let $p_\lambda : \prod_{\lambda\in\Lambda} M_\lambda \longrightarrow M_\lambda$ be the λth projection. Set

$$M_\infty = \{x \in \prod M_\lambda \mid p_\lambda(x) = f_{\lambda\mu}(p_\mu(x)) \text{ whenever } \lambda \leq \mu\} \subseteq \prod M_\lambda .$$

M_∞ is a submodule of $\prod M_\lambda$. For all $\lambda \in \Lambda$ let $\psi_\lambda : M_\infty \longrightarrow M_\lambda$ be the restriction of p_λ to M_∞. For all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ and all $x \in M_\infty$: $\psi_\lambda(x) = p_\lambda(x) = f_{\lambda\mu}(p_\mu(x)) = f_{\lambda\mu}(\psi_\mu(x))$. Thus $\psi_\lambda = f_{\lambda\mu}\psi_\mu$ and $\{\psi_\lambda\} : M_\infty \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ is an R-linear map.
In order to show that M_∞ together with $\{\psi_\lambda\}$ is an inverse limit of $\{M_\lambda, f_{\lambda\mu}\}$ let $\varphi = \{\varphi_\lambda\} : X \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ be an R-linear map, that is, $f_{\lambda\mu}\varphi_\mu = \varphi_\lambda$ whenever $\lambda \leq \mu$.
By the universal property of the direct product there is an R-linear map $g : X \longrightarrow \prod M_\lambda$ with $p_\lambda g = \varphi_\lambda$ for all $\lambda \in \Lambda$. If $\lambda \leq \mu$, then $p_\lambda(g(x)) = \varphi_\lambda(x) = f_{\lambda\mu}\varphi_\mu(x) = f_{\lambda\mu}(p_\mu g(x))$ and $g(x) \in M_\infty$. Thus g induces an R-linear map $h : X \longrightarrow M_\infty$ with $\psi_\lambda h = \varphi_\lambda$ for all $\lambda \in \Lambda$. The uniqueness of g yields the uniqueness of h.

(6.97) Remark: The inverse limit $\varprojlim M_\lambda$ is uniquely determined up to isomorphism.

(6.98) Example: Let $p \in \mathbb{Z}$ be a prime number and let for all $n, k \in \mathbb{N}$, $n \geq 1$, $f_{n,n+k} : \mathbb{Z}/(p^{n+k}) \longrightarrow \mathbb{Z}/(p^n)$ be the natural map. The set $\{\mathbb{Z}/(p^n), f_{n,n+k}\}$ is an inverse system. Its inverse limit $\varprojlim \mathbb{Z}/(p^n) = \mathbb{Z}_p$ is the ring of p-adic numbers (integers).

(6.99) Remark: Let $\varphi = \{\varphi_\lambda\} : \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M'_\lambda, f'_{\lambda\mu}\}$ be an R-linear map of inverse systems and let $\psi_\lambda : \varprojlim M_\lambda \longrightarrow M_\lambda$ and $\psi'_\lambda : \varprojlim M'_\lambda \longrightarrow M'_\lambda$ be the structure maps of the inverse limits. For all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ there is a commutative diagram:

$$\begin{tikzcd}
 & M_\mu \arrow[r, "\varphi_\mu"] \arrow[dd, "f_{\lambda\mu}"] & M'_\mu \arrow[dd, "f'_{\lambda\mu}"] \\
\varprojlim M_\lambda \arrow[ur, "\psi_\mu"] \arrow[dr, "\psi_\lambda"'] & & \\
 & M_\lambda \arrow[r, "\varphi_\lambda"] & M'_\lambda
\end{tikzcd}$$

This induces an R-linear map $\varprojlim \varphi_\lambda : \varprojlim M_\lambda \longrightarrow \varprojlim M'_\lambda$.

There is a different way to think of $\varprojlim \varphi_\lambda$. $\varprojlim M_\lambda \subseteq \prod M_\lambda$ and $\varprojlim M'_\lambda \subseteq \prod M'_\lambda$ and $\varphi = \{\varphi_\lambda\}$ induces an R-linear map $\prod \varphi_\lambda$ on the direct products:

$$\begin{tikzcd}
\prod M_\lambda \arrow[r, "\prod\varphi_\lambda"] & \prod M'_\lambda \\
\varprojlim M_\lambda \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\varprojlim \varphi_\lambda"] & \varprojlim M'_\lambda \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Then $\varprojlim \varphi_\lambda = \prod \varphi_\lambda |_{\varprojlim M_\lambda}$.

(6.100) <u>Theorem</u>: Let $\{M_\lambda, f_{\lambda\mu}\}$, $\{M'_\lambda, f'_{\lambda\mu}\}$, $\{M''_\lambda, f''_{\lambda\mu}\}$ be inverse systems of R-modules over $\Lambda = \mathbb{N}$. Suppose that $\alpha : \{M'_\lambda, f'_{\lambda\mu}\} \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ and $\beta : \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M''_\lambda, f''_{\lambda\mu}\}$ are R-linear maps of inverse systems with $0 \longrightarrow M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda$ exact for all $\lambda \in \mathbb{N}$. Then

$$0 \longrightarrow \varprojlim M'_\lambda \xrightarrow{\varprojlim \alpha_\lambda} \varprojlim M_\lambda \xrightarrow{\varprojlim \beta_\lambda} \varprojlim M''_\lambda \qquad \text{is exact.}$$

<u>Proof</u>: Remark (6.99) implies that $\varprojlim \alpha_\lambda$ is injective and that $\varprojlim \beta_\lambda \varprojlim \alpha_\lambda = 0$. Let $x \in \varprojlim M_\lambda$ with $(\varprojlim \beta_\lambda)(x) = 0$. Then for all $\lambda \in \mathbb{N}$ $p_\lambda((\varprojlim \beta_\lambda)(x)) = \beta_\lambda(p_\lambda(x)) = 0$ and there is a $y_\lambda \in M'_\lambda$ with $\alpha_\lambda(y_\lambda) = p_\lambda(x)$. Let $y \in \prod M'_\lambda$ with $p_\lambda(y) = y_\lambda$. We claim that $y \in \varprojlim M'_\lambda$. For all $\lambda, \mu \in \mathbb{N}$ there is a commutative diagram:

$$\begin{tikzcd}
M'_{\lambda+\mu} \arrow[r, "\alpha_{\lambda+\mu}"] \arrow[d, "f'_{\lambda,\lambda+\mu}"'] & M_{\lambda+\mu} \arrow[d, "f_{\lambda,\lambda+\mu}"] \\
M'_\lambda \arrow[r, "\alpha_\lambda"] & M_\lambda
\end{tikzcd}$$

Thus $\alpha_\lambda f'_{\lambda,\lambda+\mu}(y_{\lambda+\mu}) = f_{\lambda,\lambda+\mu}\alpha_{\lambda+\mu}(y_{\lambda+\mu}) = f_{\lambda,\lambda+\mu}(p_{\lambda+\mu}(x)) = p_\lambda(x) = \alpha_\lambda(y_\lambda)$. Since α_λ is injective, $y_\lambda = f_{\lambda,\lambda+\mu}(y_{\lambda+\mu})$. Thus $y \in \varprojlim M'_\lambda$ with $(\varprojlim \alpha_\lambda)(y) = x$.

(6.101) <u>Definition</u>: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of R-modules over $\Lambda = \mathbb{N}$. $\{M_\lambda, f_{\lambda\mu}\}$ satisfies the <u>Mittag-Leffler condition</u> ML if for every λ the decreasing chain of submodules $\{f_{\lambda\mu}(M_\mu) \mid \lambda \leq \mu\}$ stabilizes, i.e. there is an $n_\lambda \in \mathbb{N}$ with $f_{\lambda n_\lambda}(M_{n_\lambda}) = f_{\lambda\mu}(M_\mu)$ for all $\mu \geq n_\lambda$.

(6.102) <u>Remark</u>: Let $\{M_\lambda, f_{\lambda\mu}\}$ be an inverse system of R-modules over $\Lambda = \mathbb{N}$ with

$f_{\lambda\mu}: M_\mu \longrightarrow M_\lambda$ surjective for all $\lambda, \mu \in \mathbb{N}$. Then $\{M_\lambda, f_{\lambda\mu}\}$ satisfies ML.

(6.103) Proposition: Let $\{M_\lambda\}$ be a direct (an inverse) system of R-modules (rings) over $\mathbb{N}$ and let (n_i) be a strictly increasing sequence of integers. Then

$$\varinjlim M_\lambda \cong \varinjlim M_{n_i} \quad \text{and} \quad \varprojlim M_\lambda \cong \varprojlim M_{n_i}.$$

Proof: Homework

(6.104) Theorem: Let $\{M_\lambda, f_{\lambda\mu}\}$, $\{M'_\lambda, f'_{\lambda\mu}\}$, $\{M''_\lambda, f''_{\lambda\mu}\}$ be inverse systems of R-modules over $\mathbb{N}$ with $f'_{\lambda\mu}: M'_\mu \longrightarrow M'_\lambda$ surjective for all $\lambda \leq \mu$. Suppose that $\alpha: \{M'_\lambda, f'_{\lambda\mu}\} \longrightarrow \{M_\lambda, f_{\lambda\mu}\}$ and $\beta: \{M_\lambda, f_{\lambda\mu}\} \longrightarrow \{M''_\lambda, f''_{\lambda\mu}\}$ are R-linear maps of inverse systems with $0 \longrightarrow M'_\lambda \xrightarrow{\alpha_\lambda} M_\lambda \xrightarrow{\beta_\lambda} M''_\lambda \longrightarrow 0$ exact for all $\lambda \in \mathbb{N}$. Then

$$0 \longrightarrow \varprojlim M'_\lambda \xrightarrow{\varprojlim \alpha_\lambda} \varprojlim M_\lambda \xrightarrow{\varprojlim \beta_\lambda} \varprojlim M''_\lambda \longrightarrow 0$$

is exact.

Proof: We only have to show that $\varprojlim \beta_\lambda$ is surjective. Let $z = (z_\lambda) \in \varprojlim M''_\lambda$. We construct an $x = (x_\lambda) \in \prod M_\lambda$ inductively so that $x \in \varprojlim M_\lambda$ and $(\varprojlim \beta_\lambda)(x) = z$. Let $x_0 \in M_0$ with $\beta_0(x_0) = z_0$ and suppose that $x_i \in M_i$ for $0 \leq i \leq n-1$ have been constructed so that

(a) $\beta_i(x_i) = z_i$ for $0 \leq i \leq n-1$

(b) $f_{i,i-1}(x_i) = x_{i-1}$ for $1 \leq i \leq n-1$.

Consider the commutative diagram:

$$\begin{tikzcd}
M'_n \arrow[r, "\alpha_n"] \arrow[d, "f'_{n-1,n}"] & M_n \arrow[r, "\beta_n"] \arrow[d, "f_{n-1,n}"] & M''_n \arrow[d, "f''_{n-1,n}"] \\
M'_{n-1} \arrow[r, "\alpha_{n-1}"] & M_{n-1} \arrow[r, "\beta_{n-1}"] & M''_{n-1}
\end{tikzcd}$$

Let $\tilde{x}_n \in M_n$ with $\beta_n(\tilde{x}_n) = z_n$. Then

$\beta_{n-1}(f_{n-1,n}(\tilde{x}_n) - x_{n-1}) = \beta_{n-1} f_{n-1,n}(\tilde{x}_n) - \beta_{n-1}(x_{n-1}) = f''_{n-1,n}(z_n) - \beta_{n-1}(x_{n-1}) = z_{n-1} - z_{n-1} = 0.$

Hence there is a $y_{n-1} \in M'_{n-1}$ with $\alpha_{n-1}(y_{n-1}) = f_{n-1,n}(\tilde{x}_n) - x_{n-1}$. Since $f'_{n-1,n}$ is surjective there is a $y_n \in M'_n$ so that $f'_{n-1,n}(y_n) = y_{n-1}$. Set $x_n = \tilde{x}_n - \alpha_n(y_n)$. Then

(a) $\beta_n(x_n) = \beta_n(\tilde{x}_n) - \beta_n \alpha_n(y_n) = \beta_n(\tilde{x}_n) = z_n$.

(b) $$\begin{aligned} f_{n-1,n}(x_n) = f_{n-1,n}(\tilde{x}_n - \alpha_n(y_n)) &= f_{n-1,n}(\tilde{x}_n) - f_{n-1,n}\,\alpha_n(y_n) \\ &= f_{n-1,n}(\tilde{x}_n) - \alpha_{n-1}\, f'_{n-1,n}(y_n) \\ &= f_{n-1,n}(\tilde{x}_n) - \alpha_{n-1}(y_{n-1}) \\ &= x_{n-1}\,. \end{aligned}$$

Thus with $x = (x_n) \in \prod M_n$, $x \in \varprojlim M_n$ and $(\varprojlim \beta_n)(x) = z$.

(6.105) <u>Remark</u>: Theorem (6.104) also holds true under the weaker assumption that $\{M'_\lambda, f'_{\lambda\mu}\}$ satisfies ML.