CHAPTER IV: DIMENSION THEORY

§1: GRADED RINGS AND MODULES

(4.1) Definition: (a) A ring R is called a graded ring if R has a direct sum decomposition as an additive group $R = \bigoplus_{i=0}^{\infty} R_i$ where $R_i \subseteq R$ are subgroups of $(R,+)$ so that for all $i,j \in \mathbb{N}$ $R_i R_j \subseteq R_{i+j}$.

(b) Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring and M an R-module. M is a graded R-module if $(M,+)$ has a direct sum decomposition $M = \bigoplus_{i=0}^{\infty} M_i$ such that $R_i M_j \subseteq M_{i+j}$ for all $i,j \in \mathbb{N}$.

(4.2) Example: Let K be a field. The polynomial ring $R = K[x_1, \ldots, x_n]$ is a graded ring, $R = \bigoplus_{n \geq 0} R_n$ where $R_n = \{ \sum_{|(i)|=n} a_{(i)} x_1^{i_1} \cdots x_n^{i_n} \mid a_{(i)} \in K \}$ is the set of homogeneous polynomials of degree n.

(4.3) Definition: Let R be a graded ring and M a graded R-module. An element $a \in R$ ($m \in M$, resp.) is called homogeneous if $a \in R_i$ ($m \in M_i$) for some $i \in \mathbb{N}$. In this case i is called the degree of a (or m).

(4.4) Remark: Every element $a \in R = \bigoplus_{i \geq 0} R_i$ ($m \in M = \bigoplus_{i \geq 0} M_i$) can be written (uniquely, up to order) as a sum of homogeneous elements $a = \sum a_i$, $a_i \in R_i$ ($m = \sum m_i$, $m_i \in M$). The a_i (or m_i) are called the homogeneous components of a (or m).

(4.5) Definition: Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring and $M = \bigoplus_{i \geq 0} M_i$ a graded R-module. A submodule $N \subseteq M$ is called homogeneous if $N = \bigoplus_{i \geq 0} (N \cap M_i)$. An ideal I of R is homogeneous if I is a homogeneous submodule of the graded R-module R, that is, if $I = \bigoplus_{i \geq 0} (I \cap R_i)$.

(4.6) <u>Proposition</u>: Let $R = \bigoplus_{i\geq 0} R_i$ be a graded ring, $M = \bigoplus_{i\geq 0} M_i$ a graded R-module and $N \subseteq M$ a submodule. The following are equivalent:

(a) N is a homogeneous submodule of M.

(b) N is generated by homogeneous elements.

(c) For all $m = \sum'_{i\geq 0} m_i$ with $m_i \in M$: $m \in N \iff m_i \in N$ for all $i \in \mathbb{N}$.

<u>Proof</u>: (a) $\iff$ (b) and (c) $\Rightarrow$ (a) obvious

(b) $\Rightarrow$ (c): Suppose that N is generated by homogeneous elements $\{n_j\}_{j\in J}$ with $n_j \in M_{d_j}$, i.e. $\deg(n_j) = d_j$. Let $n \in N$, then $n = \sum' a_j n_j$ for some $a_j \in R$. For all j, $a_j = \sum' a_{jk}$ where $a_{jk} \in R_k$. Hence $n = \sum'_{j,k} a_{jk} n_j$ and the elements $a_{jk} n_j$ are homogeneous of degree $d_j + k$. Collect the terms of the same degree $m_i = \sum_{d_j+k=i} a_{jk} n_j \in M_i$. The $m_i \in N$ are the homogeneous components of n.

(4.7) <u>Remark</u>: Let $R = \bigoplus_{i\geq 0} R_i$ be a graded ring, $M = \bigoplus_{i\geq 0} M_i$ a graded R-module, and $N \subseteq M$ a homogeneous submodule. $M/N \cong \bigoplus_{i\geq 0} M_i/(N\cap M_i)$ is a graded R-module.

(4.8) <u>Definition</u>: Let R be a ring. A <u>filtration</u> of R is a chain of ideals:
$\mathcal{F}: R = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$ which satisfies $I_n I_m \subseteq I_{n+m}$ for all $n, m \in \mathbb{N}$.

(4.9) <u>Remark</u> and <u>Definition</u>: Let $\mathcal{F}: R = I_0 \supseteq I_1 \supseteq \ldots$ be a filtration of the ring R. Consider the additive group $gr_{\mathcal{F}}(R) = \bigoplus_{n\geq 0} I_n/I_{n+1}$. Define a multiplication on $gr_{\mathcal{F}}(R)$ as follows: Let $\alpha \in I_n/I_{n+1}$ and $\beta \in I_m/I_{m+1}$ with representatives $a \in I_n$ and $b \in I_m$. Set $\alpha\beta = ab + I_{n+m+1} \in I_{n+m}/I_{n+m+1}$. This multiplication is well defined and extends linearly to a multiplication $gr_{\mathcal{F}}(R)$. $gr_{\mathcal{F}}(R)$ is a commutative ring. It is called <u>the graded ring associated to the filtration</u> $\mathcal{F}$. If $I \subseteq R$ is an ideal, the filtration $\mathcal{F}: R = I^0 \supseteq I^1 \supseteq I^2 \supseteq \ldots\ldots$ is called the I-<u>adic filtration</u> of R. $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is called the <u>associated graded ring</u> (with respect to the I-adic filtration of R).

(4.10) Remark: Let $R = \bigoplus_{n\geq 0} R_n$ be a graded ring. $R_0 \subseteq R$ is a subring of R and $R_+ = \bigoplus_{n\geq 1} R_n$ is an ideal of R with $R/R_+ \cong R_0$.

(4.11) Proposition: Let R be a ring and $I = (a_1, -, a_n) \subseteq R$ a finitely generated ideal. Then $gr_I(R)$ is a homomorphic image of the polynomial ring $(R/I)[x_1, -, x_n]$. In particular, $gr_I(R)$ is a Noetherian ring if R is Noetherian.

Proof: There is a surjective homomorphism of rings $\varphi : (R/I)[x_1, -, x_n] \longrightarrow \bigoplus_{n\geq 0} I^n/I^{n+1}$ defined by $\varphi|_{R/I} = id_{R/I}$ and $\varphi(x_i) = a_i + I^2 \in I/I^2$.

(4.12) Proposition: Let $R = \bigoplus_{n\geq 0} R_n$ be a graded ring. The following are equivalent:

(a) R is Noetherian

(b) R_0 is Noetherian and R is a finitely generated R_0-algebra

Proof: (b) $\Rightarrow$ (a): Hilbert's Basis Theorem

(a) $\Rightarrow$ (b): Since $R_0 \cong R/R_+$, R_0 is Noetherian. Moreover, R_+ is a finitely generated, homogeneous ideal. Assume $R_+ = (a_1, -, a_r)$ where $a_i \in R_{d_i}$ homogeneous. We claim that $R = R_0[a_1, -, a_r]$. We show by induction on n that $R_n \subseteq R_0[a_1, -, a_r]$. Let $b \in R_n$. Then $b = \sum_{i=1}^r f_i a_i$ for some $f_i \in R$. Let g_i be the homogeneous component of degree $n - d_i$ of f_i if $n \geq d_i$ and $g_i = 0$ if $n < d_i$. Then $b = \sum_{i=1}^r g_i a_i$ and by induction hypothesis $g_i \in R_0[a_1, -, a_r]$. Hence $b \in R_0[a_1, -, a_r]$.

(4.13) Definition: Let R be a ring, $I \subseteq R$ an ideal, M an R-module and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \dots$ a descending chain of submodules of M.

(a) $\mathcal{F}$ is called a filtration of M

(b) $\mathcal{F}$ is called an I-filtration of M if $IM_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$.

(c) $\mathcal{F}$ is called a stable I-filtration of M if $\mathcal{F}$ is an I-filtration and if there is an $n_0 \in \mathbb{N}$ with $IM_n = M_{n+1}$ for all $n \geq n_0$.

(4.14) Remark and Definition: Let M be an R-module and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \dots$ an I-filtration of M. The graded module associated to $\mathcal{F}$: $gr_{\mathcal{F}}(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}$ is naturally a graded $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$-module. The I-filtration: $M \supseteq IM \supseteq I^2M \supseteq \dots$ defined by the powers of I is called the I-adic filtration of M. $gr_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$ is called the associated graded module. $gr_I(M)$ is a graded $gr_I(R)$-module.

(4.15) Definition: Let R be a ring and $I \subseteq R$ an ideal. The graded ring $\mathcal{R} = \bigoplus_{n\geq 0} I^n$ is called the Rees algebra associated to I.

(4.16) Remark: Let R be a ring, $I \subseteq R$ an ideal, M an R-module, and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \dots$ an I-filtration of M. Then $\mathcal{M} = \bigoplus_{n\geq 0} M_n$ is a graded module over $\mathcal{R} = \bigoplus_{n\geq 0} I^n$.

(4.17) Proposition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finite R-module, and $\mathcal{F}: M = M_0 \supseteq M_1 \supseteq \dots$ an I-filtration of M. The following are equivalent:

(a) $\mathcal{F}$ is a stable I-filtration

(b) $\mathcal{M} = \bigoplus_{n\geq 0} M_n$ is a finite $\mathcal{R} = \bigoplus_{n\geq 0} I^n$-module.

Proof: (a) $\Rightarrow$ (b): Let $n_0 \in \mathbb{N}$ with $IM_n = M_{n+1}$ for all $n \geq n_0$. Since every M_i is a finite R-module, $M_0 \oplus M_1 \oplus \dots \oplus M_{n_0}$ is a finite R-module. $\mathcal{M}$ is generated over $\mathcal{R}$ by $M_0 \oplus M_1 \oplus \dots \oplus M_{n_0}$.

(b) $\Rightarrow$ (a): Suppose that $\mathcal{M} = \mathcal{R}m_1 + \dots + \mathcal{R}m_r$ where $m_i \in M_{n_i}$. For $n > \max\{n_1, \dots, n_r\}$ $M_n = (\mathcal{M})_n = \{\sum_{i=1}^r a_i m_i \mid a_i \in I^{n-n_i}\} \subseteq IM_{n-1}$. Thus $M_n \subseteq IM_{n-1} \subseteq M_n$ and $M_n = IM_{n-1}$.

(4.18) Theorem: (Artin-Rees) Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finite R-module, and $N \subseteq M$ a submodule. The I-filtration $\{I^nM \cap N\}_{n\geq 0}$ of N is I-stable, that is, there is an integer $n_0 \in \mathbb{N}$ so that $I^nM \cap N = I^{n-n_0}(I^{n_0}M \cap N)$ for all $n \geq n_0$.

Proof: $\{I^nM \cap N\}_{n\geq 0}$ is an I-filtration and $\mathcal{N} = \bigoplus_{n\geq 0}(I^nM \cap N)$ is a homogeneous submodule of the $\mathcal{R} = \bigoplus_{n\geq 0} I^n$-module $\mathcal{M} = \bigoplus_{n\geq 0} I^n M$. Since $\{I^n M\}_{n\geq 0}$ is a stable I-filtration of M, by (4.17) $\mathcal{M}$ is a finite $\mathcal{R}$-module. Thus $\mathcal{N}$ is a finite $\mathcal{R}$-module and by (4.17) $\{I^n M \cap N\}$ is a stable I-filtration of N provided that $\mathcal{R}$ is Noetherian. This follows from:

(4.19) <u>Lemma</u>: Let R be a Noetherian ring and $I \subseteq R$ an ideal. The Rees algebra $\mathcal{R} = \bigoplus_{n\geq 0} I^n$ is a Noetherian ring.

<u>Proof</u>: Suppose that $I = (a_1, -, a_s)$. $\mathcal{R}$ can be identified with the finitely generated subalgebra $\mathcal{R} \cong R[a_1 x, -, a_s x]$ of the polynomial ring $R[x]$.

(4.20) <u>Corollary</u>: (Krull's intersection theorem) Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R-module. With $N = \bigcap_{n\geq 0} I^n M$, $N = I^n N$ for all $n \in \mathbb{N}$.

<u>Proof</u>: By (4.18) there is an $r \in \mathbb{N}$ so that $I^n(I^r M \cap N) = I^{n+r} M \cap N$ for all $n \in \mathbb{N}$. By definition of N, $I^t M \cap N = N$ for all $t \in \mathbb{N}$ and therefore $I^n N = N$ for all $n \in \mathbb{N}$.

(4.21) <u>Corollary</u>: Let R be a Noetherian ring, $I \subseteq \mathrm{Jac}(R)$ an ideal in the Jacobson radical of R, and M a finite R-module. Then $\bigcap_{n\geq 0} I^n M = 0$.

<u>Proof</u>: Lemma of Nakayama.

(4.22) <u>Corollary</u>: Let R be a Noetherian ring and $I \subseteq \mathrm{Jrad}(R)$ an ideal. Then $\bigcap_{n\geq 0} I^n = (0)$.

(4.23) <u>Corollary</u>: Let R be a Noetherian domain and $I \subsetneq R$ a proper ideal. Then $\bigcap_{n\geq 0} I^n = (0)$.

<u>Proof</u>: Let $m \subseteq R$ be a maximal ideal with $I \subseteq m$. Pass from R to R_m.

§2: HILBERT FUNCTIONS

(4.24) Lemma: Let $R=\bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring and $M=\bigoplus_{n\geq 0} M_n$ a finite graded R-module. For all $n\in\mathbb{N}$ M_n is a finite R_0-module.

Proof: Case 1: $M=R$. The proof is by induction on n. By (4.12) R is a finitly generated R_0-algebra. Let $a_1, \text{—}, a_r \in R$ be homogeneous elements with $\deg a_i = d_i > 0$ so that $R=R_0[a_1, \text{—}, a_r]$. For $n>0$: $R_n = a_1 R_{n-d_1} + \dots + a_r R_{n-d_r}$ with $R_i=0$ if $i<0$. By induction hypothesis for $j<n$, R_j is a finite R_0-module. Thus R_n is a finite R_0-module.
Case 2: M any finite graded R-module. M is generated by finitely many homogeneous elements, say $M=Rm_1+\dots+Rm_s$ where $\deg(m_i)=e_i\geq 0$. Then $M_n = R_{n-e_1}m_1+\dots+R_{n-e_s}m_s$ where $R_i=0$ if $i<s$. Since R_j is finitely generated over R_0, M_n is a finite R_0-module.

(4.25) Remark and Definition: Let $R=\bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring with R_0 an Artinian ring and $M=\bigoplus_{n\geq 0} M_n$ a finite graded R-module. By (4.24) every M_n is an R_0-module of finite length, i.e. $\ell_{R_0}(M_n)<\infty$.
(a) The function $H_M:\mathbb{N}\longrightarrow\mathbb{N}$ given by $H_M(n)=\ell_{R_0}(M_n)$ is called the Hilbert function of M.
(b) The power series $h_M(t)=\sum_{i=0}^{\infty} H_M(i)t^i \in \mathbb{Z}[[t]]$ is called the Hilbert series of M.

(4.26) Lemma: Let R be an Artinian ring and

$$(*)\qquad 0\longrightarrow M_1\xrightarrow{\varphi_1} M_2\xrightarrow{\varphi_2}\cdots\cdots\xrightarrow{\varphi_{n-1}} M_n\longrightarrow 0$$

be an exact sequence of finite R-modules. Then $\sum_{i=1}^{n}(-1)^i\ell_R(M_i)=0$.

Proof: By induction on n. (1.96) provides the formula for $n=3$. For $n>3$ split $(*)$ into shorter exact sequences: $0\longrightarrow \ker(\varphi_{n-1})\longrightarrow M_{n-1}\xrightarrow{\varphi_{n-1}} M_n\longrightarrow 0$ and

$$0\longrightarrow M_1\xrightarrow{\varphi_1} M_2\xrightarrow{\varphi_2}\cdots\cdots\xrightarrow{\varphi_{n-3}} M_{n-2}\longrightarrow \operatorname{im}(\varphi_{n-2})\longrightarrow 0.$$

By (1.96) $\ell_R(\ker(\varphi_{n-1}))=\ell_R(M_{n-1})+(-1)\,\ell_R(M_n)$ and by induction hypothesis

$\sum_{i=1}^{n-2} (-1)^i \ell_R(M_i) + (-1)^{n-1} \ell_R(\ker(\varphi_{n-1})) = 0$. The statement follows.

(4,27) Theorem: Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring. Assume that R_0 is Artinian and that $R = R_0[a_1, —, a_r]$ with a_i homogeneous of $\deg a_i = d_i > 0$. Let $M = \bigoplus_{n\geq 0} M_n$ be a finite graded R-module. The Hilbert series $h_M(t)$ is a rational function of the form:

$$h_M(t) = \frac{f(t)}{\prod_{i=1}^{r}(1-t^{d_i})} \in \mathbb{Q}(t)$$

where $f(t) \in \mathbb{Z}[t]$ is a polynomial with integer coefficients.

Proof: By induction on r. If $r = 0$, then $R = R_0$ and M is a finite module over the Artinian ring R_0. Thus M is an Artinian R_0-module and the chain of submodules $M \geq \bigoplus_{n\geq 1} M_n \geq \bigoplus_{n\geq 2} M_n \geq \dots$ is stationary. Hence $M_n = 0$ for all $n \geq n_0$ and $h_M(t) \in \mathbb{Z}[t]$.
$r > 0$: For all $n \in \mathbb{N}$ define $\varphi_{n,r} : M_n \longrightarrow M_{n+d_r}$ by $\varphi_{n,r}(m) = a_r m$. Obviously, $\varphi_{n,r}$ is R_0-linear yielding for all $n \in \mathbb{N}$ an exact sequence of R_0-modules:

$$(*_n) \qquad 0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\varphi_{n,r}} M_{n+d_r} \longrightarrow L_{n+d_r} \longrightarrow 0$$

where $K_n = \ker(\varphi_{n,r})$ and $L_{n+d_r} = \operatorname{coker}(\varphi_{n,r}) = M_{n+d_r}/a_r M_n$. Set $K = \bigoplus_{n\geq 0} K_n$ and $L = \bigoplus_{n\geq 0} L_{n+d_r}$. Then

(a) K is a homogeneous submodule of M

(b) $L = (\bigoplus_{n\geq d_r} M_n)/a_r M$ is a graded R-module

(c) $R/a_r R = R_0[\bar{a}_1, —, \bar{a}_{r-1}] = \bigoplus_{n\geq 1} R'_n$ is a graded ring which is generated over $R_0 = R'_0$ by $r-1$ homogeneous elements.

(d) Since $a_r K = 0$ and $a_r L = 0$, the R-modules K and L are finite $(R/a_r R)$-modules. The induction hypothesis applies to K and L and there are polynomials $g(t), h(t) \in \mathbb{Z}[t]$ so that:

$$h_K(t) = \sum_{n=0}^{\infty} \ell_{R_0}(K_n) t^n = \frac{g(t)}{\prod_{i=1}^{r-1}(1-t^{d_i})} \quad \text{and} \quad h_L(t) = \sum_{n=0}^{\infty} \ell_{R_0}(L_n) t^n = \frac{h(t)}{\prod_{i=1}^{r-1}(1-t^{d_i})}.$$

By (4.26) $\ell(K_n) - \ell(M_n) + \ell(M_{n+d_r}) - \ell(L_{n+d_r}) = 0$ and therefore $t^{d_r}\ell(K_n)t^n - t^{d_r}\ell(M_n)t^n + \ell(M_{n+d_r})t^{n+d_r} - t^{d_r}\ell(L_{n+d_r})t^n = 0$. Taking sums yields $t^{d_r}h_K(t) - t^{d_r}h_M(t) + h_M(t) - t^{d_r}h_L(t) = q(t)$ where $q(t) = \sum_{i=0}^{d_r-1} \ell(M_i)t^i$. This proves the theorem.

(4.28) **Definition:** A graded ring $R = \bigoplus_{n \geq 0} R_n$ is called *homogeneous* if $R = R_0[R_1]$, that is, if R is generated in degree one.

(4.29) **Remark:** The function $(1-t)^{-1}$ can be written as a power series $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n \in \mathbb{Z}[[t]]$. Taking the d^{th} derivative yields $(1-t)^{-d} = \sum_{n=0}^{\infty} \binom{d+n-1}{d-1} t^n \in \mathbb{Z}[[t]]$.

(4.30) **Remark:** The 'binomial' polynomials of $\mathbb{Q}[x]$ are defined as follows: $\binom{x}{0} = 1$ and for $k \in \mathbb{N} - (0)$: $\binom{x}{k} = (1/k!)\, x(x-1) \cdots (x-k+1)$. Obviously, $\binom{x}{k}$ is a polynomial of degree k and the set $\{\binom{x}{k}\}_{k \in \mathbb{N}}$ forms a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}[x]$.

(4.31) **Remark:** Let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian graded ring and $M = \bigoplus_{n \geq 0} M_n$ a finite graded R-module. Then $h_M(t)$ can be uniquely written as

$$h_M(t) = \frac{g(t)}{(1-t)^d} \in \mathbb{Q}(t)$$

where $g(t) \in \mathbb{Z}[t]$ with $g(1) \neq 0$ and $d = d(M)$. Write $g(t) = \sum_{j=0}^{s} e_j (t-1)^j$ with $e_j \in \mathbb{Z}$ and $e_s \neq 0$. Set $P_M(x) = \sum_{j=0}^{d-1} e_j (-1)^j \binom{x+d-1-j}{d-1-j} \in \mathbb{Q}[x]$.

(4.32) **Theorem:** Assumptions as in (4.31). Then for all $n \geq s-d+1$

$$H_M(n) = \ell_{R_0}(M_n) = P_M(n).$$

Proof: If $d = 0$, then $P_M(t) = 0$ and $h_M(t) = g(t)$. Suppose $d \geq 1$, then

$$h_M(t) = \sum_{j=0}^{d-1} (-1)^j e_j \cdot \frac{1}{(1-t)^{d-j}} + \sum_{j=d}^{s} (-1)^j e_j (1-t)^{j-d}.$$

Since $\frac{1}{(1-t)^{d-j}} = \sum_{i=0}^{\infty} \binom{i+d-1-j}{d-1-j} t^i$, the statement follows.

(4.33) **Definition:** The polynomial $P_M(x)$ of (4.31) is called the *Hilbert polynomial* of M. The *multiplicity* of M is defined to be:

$$e(M) = \begin{cases} e_0 & \text{if } d \geq 1 \\ \ell_{R_0}(M) & \text{if } d = 0 \text{ or } M = 0. \end{cases}$$

Note that $P_M(x) = e(M)/(d-1)!\, x^{d-1}$ + terms of lower degree and that $e(M) > 0$ if $M \neq 0$.

Also note that the degree $d-1$ of $P_M(x)$ is strictly less than r, the number of generators of R over R_0.

(4.34) Examples: (a) Let R_0 be an Artinian ring and $R = R_0[x_0, -, x_r]$ the polynomial ring in $r+1$ variables over R_0. Consider the natural grading $R = \bigoplus_{n \geq 0} R_n$ where $R_n = \{ \sum_i a_{(i)} x_0^{i_0} \ldots x_r^{i_r} \mid |i| = n \text{ and } a_{(i)} \in R_0 \}$. The number of monomials of degree n in $r+1$ variables is $\binom{n+r}{r}$. Therefore $\ell_{R_0}(R_n) = \ell_{R_0}(R_0) \binom{n+r}{r}$ and $P_R(x) = \ell_{R_0}(R_0) \binom{x+r}{r} = (\ell_{R_0}(R_0)/r!)(x+r)(x+r-1) \ldots (x+1)$ with $e(R) = \ell_{R_0}(R_0)$.

(b) Let K be a field and $f = \sum_{|i|=s} a_{(i)} x_0^{i_0} \ldots x_s^{i_s} \in K[x_0, \ldots, x_s]$ a homogeneous polynomial of degree $s \geq 1$. Set $R = K[x_0, -, x_r] = \bigoplus_{n \geq 0} R_n$ where R_n is the set of homogeneous polynomials of degree n. Let $S = R/(f) = \bigoplus_{n \geq 0} S_n$ where $S_n = R_n$ if $n < s$ and $S_n = R_n / R_{n-s} f$ if $n \geq s$. Hence for $n \geq s$: $\ell_K(S_n) = \ell_K(R_n) - \ell_K(R_{n-s}) = \binom{n+r}{r} - \binom{n-s+r}{r}$. Suppose that $\binom{x+r}{r} = \frac{1}{r!} x^r + c_{r-1} x^{r-1} + \ldots + c_0$ with $c_i \in \mathbb{Q}$, then

$$P_S(x) = \frac{1}{r!}\left(x^r - (x-s)^r\right) + c_{r-1}\left(x^{r-1} - (x-s)^{r-1}\right) + \ldots + c_1(x - (x-s)) =$$

$$= s/(r-1)!\, x^{r-1} + \text{terms of lower degree.}$$

In particular, $\deg(P_S(x)) = r-1 = \dim S - 1$ and $e(S) = s$.

(4.35) Remark: If $f(x) \in \mathbb{Q}[x]$ is a polynomial with $f(n) > 0$ for all but finitely many $n \in \mathbb{N}$, the leading coefficient of f is greater than 0. In particular, the Hilbert polynomial $P_M(x)$ has a positive leading coefficient.

(4.36) Definition: Let R be a Noetherian semilocal ring. An ideal $I \subseteq R$ is called an ideal of definition of R if there is an $m \in \mathbb{N}$ such that $\operatorname{Jrad}(R)^m \subseteq I \subseteq \operatorname{Jrad}(R)$ (or equivalently, if $\operatorname{rad}(I) = \operatorname{Jrad}(R)$.)

(4.37) Lemma: Let R be a Noetherian semilocal ring and $I \subseteq R$ an ideal. The following conditions are equivalent:

(a) I is an ideal of definition of R.

(b) $I \subseteq \operatorname{Jrad}(R)$ and R/I is Artinian.

(c) $I \subseteq \operatorname{Jrad}(R)$ and R/I has finite length

(d) $\operatorname{Supp}_R(R/I) = \operatorname{mSpec}(R)$.

Proof: Homework

(4.38) Remark: If R is a local Noetherian ring with maximal ideal m, conditions (a)-(d) of (4.37) are equivalent to I is m-primary.

(4.39) Lemma: Let R be a semilocal Noetherian ring, $I \subseteq R$ an ideal of definition, and M a finite R-module. Suppose that $M = M_0 \supseteq M_1 \supseteq \dots$ is an I-filtration of M. Then $\ell_R(M/M_n) < \infty$ for all $n \in \mathbb{N}$.

Proof: For all $n \in \mathbb{N}$, $I^n M \subseteq M_n$, and M/M_n is a finite R/I^n-module. Since I is an ideal of definition, so is I^n, and R/I^n is Artinian. Thus $\ell_R(M/M_n) = \ell_{(R/I^n)}(M/M_n) < \infty$.

Let R be a Noetherian semilocal ring, $I \subseteq R$ an ideal of definition, and M a finite R-module. Suppose that I is r generated, $I = (a_1, \_, a_r)$, and consider the associated graded ring and module: $gr_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and $gr_I(M) = \bigoplus_{n \geq 0} I^nM/I^{n+1}M$. $gr_I(R)$ is a Noetherian graded ring which is as (R/I)-algebra generated in degree one by $a_1 + I^2, \_, a_r + I^2 \in I/I^2$. Furthermore, $gr_I(M)$ is a finite graded $gr_I(R)$-module. By (4.31) and (4.32) there is a polynomial $P_M^I \in \mathbb{Q}[x]$ of degree $\leq r-1$ so that for all $n \geq n_0$: $P_M^I(n) = \ell_R(I^nM/I^{n+1}M) = \ell_{(R/I)}(I^nM/I^{n+1}M)$. Using the exact sequence $0 \longrightarrow I^nM/I^{n+1}M \longrightarrow M/I^{n+1}M \longrightarrow M/I^nM \longrightarrow 0$ induction on n yields that

$$(*) \qquad \ell_R(M/I^{n+1}M) = \sum_{i=0}^{n} \ell_R(I^iM/I^{i+1}M).$$

$(*)$ implies that there is a polynomial $Q_M^I \in \mathbb{Q}[x]$ so that for large n

$$Q_M^I(n) = \ell_R(M/I^{n+1}M).$$

In the remainder of this section we show that the degree of P_M^I and Q_M^I does not depend on the choice of the ideal of definition I. The next section relates the degree of P_M^I (or Q_M^I) to the dimension of M ($\dim M = \dim R/\mathrm{ann}(M)$).

(4.40) *Lemma*: Let $f: \mathbb{N} \longrightarrow \mathbb{N}$ be a function. The following are equivalent:

(a) There is a polynomial $Q(x) \in \mathbb{Q}[x]$ of degree $\leq r+1$ with $f(n) = Q(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0 > r$.

(b) There is a polynomial $P(x) \in \mathbb{Q}[x]$ of degree $\leq r$ with $f(n+1) - f(n) = P(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0 > r$.

Proof: (a) $\Rightarrow$ (b): Set $P(x) = Q(x+1) - Q(x)$.

(b) $\Rightarrow$ (a): First note that by induction on n: $\sum_{\nu=0}^{n} \binom{d+\nu-1}{d-1} = \binom{d+n}{d}$. Suppose that $P(x) = \sum_{k=0}^{r} a_k \binom{x}{k}$ where $a_k \in \mathbb{Q}$. By assumption for all $n \geq n_0 > r$: $f(n+1) - f(n) = P(n)$ and therefore:

$$\begin{aligned}
f(n+1) &= P(n) + f(n) = P(n) + P(n-1) + f(n-1) = \ldots = \\
&= \sum_{\rho=0}^{n-n_0} P(n-\rho) + f(n_0) \\
&= \sum_{k=0}^{r} a_k \Big(\sum_{\rho=0}^{n-n_0} \binom{n-\rho}{k} \Big) + f(n_0) \\
&= \sum_{k=0}^{r} a_k \Big(\sum_{\lambda=n_0}^{n} \binom{\lambda}{k} \Big) + f(n_0) \\
&= \sum_{k=0}^{r} a_k \Big(\sum_{\sigma=n_0-k}^{n-k} \binom{k+\sigma}{k} \Big) + f(n_0) \\
&= \sum_{k=0}^{r} a_k \Big(\sum_{\sigma=0}^{n-k} \binom{k+\sigma}{k} - \sum_{\sigma=0}^{n_0-k-1} \binom{k+\sigma}{k} \Big) + f(n_0) \\
&= \sum_{k=0}^{r} a_k \binom{n+1}{k+1} + c
\end{aligned}$$

where $c \in \mathbb{Q}$ is a constant. The polynomial $Q(x) = \sum_{k=0}^{r} a_k \binom{x}{k+1} + c \in \mathbb{Q}[x]$ has degree $\leq r+1$ and satisfies $Q(n) = f(n)$ for all $n \geq n_0 > r$. Note that $\deg Q(x) = \deg P(x) + 1$.

(4.41) *Proposition* and *Definition*: Let R be a Noetherian semilocal ring, $I \subseteq R$ an ideal of definition, and M a finite R-module. There is a polynomial $Q_M^I \in \mathbb{Q}[x]$ such that for all but finitely many $n \in \mathbb{N}$: $Q_M^I = \ell_R(M/I^{n+1}M)$. Q_M^I is called the *Hilbert-Samuel* polynomial of M with respect to I. The degree of Q_M^I is denoted by $d = d(M)$.

Proof: The exact sequence $0 \longrightarrow I^nM/I^{n+1}M \longrightarrow M/I^{n+1}M \longrightarrow M/I^nM \longrightarrow 0$ yields that $P_M^I(n) = \ell_R(I^nM/I^{n+1}M) = \ell_R(M/I^{n+1}M) - \ell_R(M/I^nM) = Q_M^I(n) - Q_M^I(n-1)$. The statement follows by (4.40).

(4.42) Proposition: Let R be a Noetherian semilocal ring, $I, J \subseteq R$ ideals of definition, and M a finite R-module. Then $\deg Q_M^I = d(M) = \deg Q_M^J$. The degree of the Hilbert-Samuel polynomial is independent of the choice of the ideal of definition.

Proof: Let $\mathfrak{m} = \operatorname{Jrad}(R)$ and $a, b \in \mathbb{N}$ such that $\mathfrak{m}^a \subseteq I \subseteq \mathfrak{m}$ and $\mathfrak{m}^b \subseteq J \subseteq \mathfrak{m}$. Then $I^b \subseteq J$ and $J^a \subseteq I$ and for all $n \in \mathbb{N}$: $I^{b(n+1)}M \subseteq J^{n+1}M$ and $J^{a(n+1)}M \subseteq I^{n+1}M$. This implies that $\ell_R(M/I^{b(n+1)}M) \geq \ell_R(M/J^{n+1}M)$ and $\ell_R(M/J^{a(n+1)}M) \geq \ell_R(M/I^{n+1}M)$. Therefore for sufficiently large n: $Q_M^I(bn+b-1) \geq Q_M^J(n)$ and $Q_M^J(an+a-1) \geq Q_M^I(n)$. Thus

$$\deg Q_M^I(bx+b-1) = \deg Q_M^I(x) \geq \deg Q_M^J(x) = \deg Q_M^J(ax+a-1) \geq \deg Q_M^I(x).$$

(4.43) Theorem: Let R be a Noetherian semilocal ring, and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ an exact sequence of finite R-modules. Then $d(M) = \max(d(M'), d(M''))$. Moreover, if $I \subseteq R$ is an ideal of definition, then $Q_{M'}^I$ and $Q_M^I - Q_{M''}^I$ have the same degree and the same leading coefficient.

Proof: We may assume that M' is a submodule of M and that $M'' = M/M'$. Let $I \subseteq R$ be an ideal of definition. Since $M''/I^nM'' \cong M/M'+I^nM$ the sequence

$$0 \longrightarrow (M'+I^nM)/I^nM \longrightarrow M/I^nM \longrightarrow M/M'+I^nM \longrightarrow 0$$

is exact and $\ell_R(M/I^nM) = \ell_R((M'+I^nM)/I^nM) + \ell_R(M/M'+I^nM) = \ell_R(M'/M' \cap I^nM) + \ell_R(M''/I^nM'')$. Let $\varphi: \mathbb{N} \longrightarrow \mathbb{N}$ be defined by $\varphi(n) = \ell_R(M'/M' \cap I^nM)$ for all $n \in \mathbb{N}$. Then for all $n \geq n_0$: $Q_M^I(n-1) = Q_{M''}^I(n-1) + \varphi(n)$ and there is a polynomial $F(x) \in \mathbb{Q}[x]$ with $F(n) = \varphi(n)$ for all $n \geq n_0$. Since $Q_{M''}^I$ and F have positive leading coefficients, $d(M) = \deg Q_M^I = \max(\deg Q_{M''}^I, \deg F)$. By Artin-Rees (4.18) there is an integer $c \in \mathbb{N}$ so that for all $n \geq c$:

$$I^{n+1}M' \subseteq M' \cap I^{n+1}M \subseteq I^{n+1-c}M'.$$

Hence for all $n \geq \max(n_0, c)$ $Q_{M'}^I(n) \geq F(n+1) \geq Q_{M'}^I(n-c)$. Thus $\deg Q_{M'}^I = \deg F$ and F and $Q_{M'}^I$ have the same leading coefficient.

(4.44) Corollary: Let R be a Noetherian semilocal ring, M a finite R-module, and $M' \subseteq M$ a submodule. Then $d(M') \leq d(M)$ and $d(M/M') \leq d(M)$.

(4.45) Corollary: Let R be a Noetherian semilocal ring, M a finite R-module, and $a \in R$ a NZD on M. Then $d(M/aM) < d(M)$.

Proof: Consider the exact sequence $0 \longrightarrow M \xrightarrow{\tau} M \longrightarrow M/aM \longrightarrow 0$ where $\tau(m) = am$ for all $m \in M$. Then $M' = \operatorname{im}(\tau) = aM \cong M$ and $Q^I_{M'} = Q^I_M$. By (4.43) Q^I_M and $Q^I_M - Q^I_{M''}$ (where $M'' = M/aM$) have the same degree and the same leading coefficient. Hence $\deg Q^I_{M''} < \deg Q^I_M$.

§3: DIMENSION

(4.46) Definition: Let R be a Noetherian semilocal ring and M a nonzero finite R-module. Define:
$s_R(M) := \inf\{n \in \mathbb{N} \mid \exists\, a_1, \dots, a_n \in \operatorname{Jrad}(R) \text{ so that } \ell_R(M/(a_1M + \dots + a_nM)) < \infty\}$.

(4.47) Remark: Assumptions as in (4.46). If $\operatorname{Jrad}(R) = (a_1, \dots, a_r)$ then $\ell_R(M/\operatorname{Jrad}(R)M) < \infty$ and $s_R(M) \leq r$. In particular, $s_R(M) < \infty$.

Let R be a Noetherian semilocal ring, $I \subseteq R$ an ideal of definition, and M a finite R-module. The Hilbert-Samuel polynomial $Q^I_M(x) \in \mathbb{Q}[x]$ of M with respect to I satisfies for large n: $Q^I_M(n) = \ell_R(M/I^{n+1}M)$. By (4.42) $d_R(M) = \deg Q^I_M$ does not depend on the choice of the ideal of definition I. By (4.43) for an exact sequence of finite R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$: $d_R(M) = \max\{d_R(M'), d_R(M'')\}$ and $\deg(Q^I_M - Q^I_{M'} - Q^I_{M''}) < d_R(M)$. In the following we want to compare $\dim(M)$, $d_R(M)$, and $s_R(M)$.

(4.48) Lemma: Let R be a Noetherian semilocal ring and M a nonzero finite R-module. Then:
(a) For all $a \in \operatorname{Jrad}(R)$, $s_R(M/aM) \geq s_R(M) - 1$.
(b) If $a \in \operatorname{Jrad}(R)$ with $a \notin P$ for all minimal primes $P \in \operatorname{Supp}_R(M)$ then $d_R(M/aM) \leq d_R(M) - 1$.

Proof: Set $\overline{M} = M/aM$ and note that by Nakayama $\overline{M} \neq 0$.
(a) Suppose that $s = s_R(\overline{M})$ and that $a_1, \dots, a_s \in \operatorname{Jrad}(R)$ with $\ell_R(\overline{M}/(a_1, \dots, a_s)\overline{M}) < \infty$. Since $\overline{M}/(a_1, \dots, a_s)\overline{M} \cong M/(a, a_1, \dots, a_s)M$, it follows that $s_R(\overline{M}) + 1 \geq s_R(M)$.
(b) Obviously, $\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(\overline{M})$. Let $\{P_1, \dots, P_r\}$ be the set of minimal prime ideals of $\operatorname{Supp}_R(M) = V(\operatorname{ann}_R(M))$. By assumption $a \notin P_i$ for all $1 \leq i \leq r$. If $P \supseteq \operatorname{ann}_R(\overline{M})$ then $a \in P$ and $P_i \subsetneq P$ for some $1 \leq i \leq r$. Hence $\dim(R/P) < \dim(R/P_i) \leq \dim(R/\operatorname{ann}(M)) = \dim M$. This shows that $\dim_R(M/aM) \leq \dim_R M - 1$.

(4.49) <u>Theorem</u>: Let R be a Noetherian semilocal ring and M a nonzero finite R-module. Then
$$\dim_R M = d_R(M) = s_R(M).$$

<u>Proof</u>: (1) We may replace R by $\overline{R} = R/\mathrm{ann}(M)$ and assume that $\mathrm{ann}_R(M) = 0$.
<u>Pf</u> of (1): (a) Obviously, $\dim_R M = \dim_{\overline{R}} M$.
(b) $d_R(M) = d_{\overline{R}}(M)$: If $I \subseteq \mathrm{Jrad}(R)$ is an ideal of definition of R, then $\overline{I} = I + \mathrm{ann}_R(M) \subseteq \overline{R}$ is an ideal of definition of $\overline{R}$ and $\ell_R(M/I^nM) = \ell_{\overline{R}}(M/\overline{I}M)$. Thus $Q_M^I = Q_M^{\overline{I}}$ and $d_R(M) = d_{\overline{R}}(M)$.
(c) $s_R(M) = s_{\overline{R}}(M)$: Let $\mathrm{mSpec}(R) = \{m_1, \ldots, m_n\}$. If $\mathrm{ann}_R(M) \subseteq m_i$ for all $1 \le i \le n$ then $s_R(M) = s_{\overline{R}}(M)$, since every $\overline{a} \in \mathrm{Jrad}(\overline{R})$ is image of an element $a \in \mathrm{Jrad}(R)$. Suppose $\mathrm{ann}_R(M) \subseteq m_1 \cap \ldots \cap m_r$ and $\mathrm{ann}_R(M) \nsubseteq m_{r+1} \cup \ldots \cup m_n$. If $a_1, \ldots, a_s \in \mathrm{Jrad}(R)$ with $\ell_R(M/(a_1, \ldots, a_s)M) < \infty$, then $\overline{a}_1, \ldots, \overline{a}_s \in \mathrm{Jrad}(\overline{R})$ with $\ell_{\overline{R}}(M/(\overline{a}_1, \ldots, \overline{a}_s)M) < \infty$. Thus $s_R(M) \ge s_{\overline{R}}(M)$. In order to show equality note that $\mathrm{ann}_R(M) + m_{r+1} \cap \ldots \cap m_n = R$ and let $q \in \mathrm{ann}_R(M)$ and $t \in m_{r+1} \cap \ldots \cap m_n$ with $q + t = 1$. Suppose that $\overline{a}_1, \ldots, \overline{a}_s \in \mathrm{Jrad}(\overline{R})$ with $\ell_{\overline{R}}(M/(\overline{a}_1, \ldots, \overline{a}_s)M) < \infty$ and let $a_1, \ldots, a_s \in m_1 \cap \ldots \cap m_r$ be preimages of the $\overline{a}_i$. Then $a_i = a_i q + a_i t$ and $a_i M = a_i t M$ where $a_i t \in \mathrm{Jrad}(R)$ for all $1 \le i \le s$. Thus $\ell_{\overline{R}}(M/(\overline{a}_1, \ldots, \overline{a}_s)M) = \ell_R(M/(a_1, \ldots, a_s)M) = \ell_R(M/(a_1 t, \ldots, a_s t)M) < \infty$. Thus $s_{\overline{R}}(M) \ge s_R(M)$.
(2) $\dim_R M \le d_R(M)$
<u>Pf</u> of (2): By induction on $d = d_R(M)$. Let $m = \mathrm{Jrad}(R)$. If $d_R(M) = 0$, then $\ell_R(M/m^{n+1}M)$ is a constant for all $n \ge n_0$. Thus $m^{n+1}M = m^nM$ for $n \ge n_0$ and by Nakayama $m^nM = 0$. Hence $m^n \subseteq \mathrm{ann}_R(M)$ and by (1.101) $R/\mathrm{ann}(M)$ is an Artinian ring and $\dim_R M = 0$.
For the induction step suppose that $d_R(M) = d+1$ and let $P_0 \subseteq R$ be a minimal prime ideal with $\dim R/P_0 = \dim R = \dim_R M$. (Note that possibly $\dim R = \infty$.) Since $\mathrm{ann}_R(M) = 0$, $\mathrm{Supp}_R(M) = \mathrm{Spec}(R)$, and P_0 is a minimal prime ideal of $\mathrm{Supp}_R(M)$. Hence $P_0 \in \mathrm{Ass}_R(M)$ and there is a submodule $N \subseteq M$ with $N \cong R/P_0$. By (4.43) $d_R(N) \le d_R(M)$ and it suffices to show that $d_R(N) \ge \dim_R N = \dim_R M$. Let $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ be a chain of prime ideals of R. We need to show that $n \le d_R(N)$. If $n = 0$, we are done. If $n \ge 1$, pick $a \in P_1 - P_0$. Since $N \cong R/P_0$, the element a is a NZD on N and $N/aN \cong R/P_0 + (a)$. Moreover,

$P_1/P_0+(a) \subsetneq P_2/P_0+(a) \subsetneq \cdots \subsetneq P_n/P_0+(a)$ is a chain of prime ideals of length $n-1$ in $R/P_0+(a)$. Hence $\dim_R N/aN = \dim R/P_0+(a) \geq n-1$. By (4.48) $d_R(N/aN) \leq d_R(N)-1 < d_R(M)$ and the induction hypothesis applies to N/aN. Therefore $d_R(N)-1 \geq d_R(N/aN) \geq \dim_R N/aN \geq n-1$ and $d_R(N) \geq n$. This shows that $d_R(M) \geq \dim_R M$ and, in particular, that $\dim_R M < \infty$.

(3) $d_R(M) \leq s_R(M)$

Pf of (3): Suppose $s = s_R(M)$ and let $a_1, \ldots, a_s \in \operatorname{Jrad}(R)$ with $\ell_R(M/(a_1, -, a_s)M) < \infty$. Set $I = (a_1, \ldots, a_s)$. We want to show that I is an ideal of definition or equivalently, that R/I is an Artinian ring. Let $P \in \operatorname{Spec}(R)$ with $I \subseteq P$. Since $\operatorname{ann}_R(M) = 0$, the localization $M_P \neq 0$ and by Nakayama $(M/IM)_P \cong M_P/IM_P \neq 0$. Thus $P \in \operatorname{Supp}_R(M/IM)$. Let $Q \subseteq P$ be a minimal prime ideal in $\operatorname{Supp}_R(M/IM)$. Then $Q \in \operatorname{Ass}_R(M/IM)$ and there is a submodule $N \subseteq M/IM$ with $N \cong R/Q$. Since $\ell_R(M/IM) < \infty$, $\ell_R(N) = \ell_R(R/Q) = \ell_{R/Q}(R/Q) < \infty$. By (1.97) R/Q satisfies the d.c.c. and R/Q is an Artinian ring. By (1.98) every prime ideal of R/Q is maximal. Thus $P = Q$, P is a maximal ideal of R, $\operatorname{rad}(I) = \operatorname{Jrad}(R)$, and R/I is an Artinian ring. By (4.33) $\deg P_M^I \leq s-1$ and therefore $d_R(M) = \deg Q_M^I \leq s$.

(4) $s_R(M) \leq \dim_R M$

Pf of (4): By (2) $\dim_R M < \infty$. The proof is by induction on $n = \dim_R M = \dim R$. If $n = 0$, then R is Artinian and $\ell_R(M) = 0$. Thus $s_R(M) = 0$. For the induction step suppose that $\dim_R M = n+1$. Let $P_1, -, P_r$ be the minimal prime ideals of R with $\dim R/P_i = \dim R = \dim_R M$ for all $1 \leq i \leq r$. Since $\dim_R M \geq 1$, none of the P_i is maximal in R and $m = \operatorname{Jrad}(R) \not\subseteq \bigcup_{i=1}^r P_i$. Let $a \in m - \bigcup_{i=1}^r P_i$. By (4.48)(a) $s_R(M/aM) \geq s_R(M)-1$ and by the choice of a, $\dim R/aR \leq \dim R - 1 = \dim_R M - 1$. Since $\dim_R M/aM \leq \dim R/aR \leq \dim R - 1 = n$, the induction hypothesis applies to M/aM and $s_R(M)-1 \leq s_R(M/aM) \leq \dim_R M/aM \leq \dim R/aR \leq \dim R - 1 = n$. This shows $s_R(M) \leq n+1 = \dim_R M$.

(4.50) Corollary: Let R be a Noetherian semilocal ring. The dimension, $\dim R$, of R is the least number of generators of an ideal of definition of R.

(4.51) Definition: Let R be a local Noetherian ring with maximal ideal m and M a finite

R-module with $\dim_R M = n$. A set of elements $\{a_1, -, a_n\} \subseteq m$ is called a system of parameters of M if $\ell_R(M/(a_1, -, a_n)M) < \infty$.

(4.52) Remark: Systems of parameters always exist.

In the following (R, m, k) denotes a local ring R with maximal ideal m and with residue class field $R/m = k$.

(4.53) Corollary: Let (R, m, k) be a local Noetherian ring. Then $\dim R \leq \dim_k m/m^2 = \mu(m)$.

Proof: By (1.35) and (4.49).

(4.54) Definition: Let (R, m, k) be a local Noetherian ring.

(a) $\dim_k m/m^2 = \operatorname{edim} R$ is called the embedding dimension of R.

(b) R is called a regular local ring (RLR) if $\dim R = \operatorname{edim} R = \dim_k m/m^2$.

(4.55) Theorem: Let R be a Noetherian ring and $P \subseteq R$ a prime ideal. The following are equivalent:

(a) $\operatorname{ht} P \leq n$

(b) There are elements $a_1, -, a_n \in P$ so that P is a minimal prime ideal over $(a_1, -, a_n)$.

Proof: Note that R_p is a local Noetherian ring with maximal ideal PR_p and $\dim R_p = \operatorname{ht} P$.

(a) $\Rightarrow$ (b): Since $\operatorname{ht} P = \dim R_p \leq n$, by (4.49) there are $s \in R - P$ and $a_1, -, a_n \in P$ such that $(a_1/s, -, a_n/s)R_p$ is PR_p-primary. Then P is a minimal prime ideal over $(a_1, -, a_n)$.

(b) $\Rightarrow$ (a): Let $I = (a_1, -, a_n)$ and P minimal over I. Then IR_p is PR_p-primary. Hence IR_p is an ideal of definition of R_p. By (4.49) $\dim R_p = \operatorname{ht} P \leq n$.

(4.56) Corollary: (Krull's principal ideal theorem) Let R be a Noetherian ring and $a \in R - R^*$ a nonunit of R. Every minimal prime ideal over aR has height ≤ 1.

(4.57) <u>Corollary</u>: (Generalized principal ideal theorem) Let R be a Noetherian ring and $I=(a_1,\ldots,a_n)\subsetneq R$ an ideal. Every minimal prime ideal over I has height $\leq n$, in particular, $\operatorname{ht} I \leq n$.

(4.58) <u>Corollary</u>: Let R be a Noetherian ring and $P_0 \subsetneq P_1 \subsetneq P_2$ a chain of prime ideals of R. There are infinitely many prime ideals $Q \in \operatorname{Spec}(R)$ with $P_0 \subsetneq Q \subsetneq P_2$.

<u>Proof</u>: We may assume that $P_0 = 0$ and that R is a local Noetherian domain with maximal ideal $m = P_2 R_{P_2}$. By assumption $\operatorname{ht} m = \dim R \geq 2$. By (4.57) every element $a \in m-(0)$ is contained in a prime ideal $Q \subseteq R$ with $\operatorname{ht} Q = 1$. Thus $m = \bigcup_{Q \text{ prime}, \operatorname{ht} Q = 1} Q$. If there are only finitely many prime ideals Q of height one, then $m = Q$ for some height one prime ideal Q, a contradiction.

(4.59) <u>Theorem</u>: Let R be a Noetherian ring, $R[x_1,\ldots,x_n]$ the polynomial ring in n variables over R. Then $\dim R[x_1,\ldots,x_n] = \dim R + n$.

Obviously, it suffices to show that $\dim R[x] = \dim R + 1$. The proof uses several Lemmas.

(4.60) <u>Lemma</u>: Let R be a Noetherian ring, x a variable over R, and $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_r$ a chain of prime ideals of R. Then $P_0R[x] \subsetneq P_1R[x] \subsetneq \ldots \subsetneq P_rR[x] \subsetneq P_rR[x]+(x)$ is a chain of prime ideals of $R[x]$.

<u>Proof</u>: $R[x]/P_iR[x] \cong (R/P_i)[x]$ and $R[x]/(P_r+(x))R[x] \cong \left(R[x]/P_rR[x]\right)/\left((P_r+(x))R[x]/P_rR[x]\right) \cong (R/P_r)[x]/x(R/P_r)[x] \cong R/P_r$.

(4.61) <u>Corollary</u>: $\dim R[x] \geq \dim R + 1$

(4.62) <u>Lemma</u>: Let $P \subsetneq Q \subseteq R[x]$ be prime ideals with $P \cap R = Q \cap R$. Then $P = (P \cap R)R[x]$.

Proof: We may assume that $P \cap R = Q \cap R = 0 \in \operatorname{Spec}(R)$, that is, $P \cap R = 0$ and R is a domain. If $P \neq 0$, there is a chain of prime ideals $0 \subsetneq P \subsetneq Q$ in $R[x]$ with $Q \cap R = 0$. With $S = R - (0)$, $S^{-1}(R[x]) = Q(R)[x] = K[x]$, where $K = Q(R)$ is the quotient field of R. $0 \subsetneq PK[x] \subsetneq QK[x]$ is a chain of prime ideals in $K[x]$ and $\dim K[x] \geq 2$, a contradiction.

(4.63) **Lemma**: Let $I \subseteq R$ be an ideal and $P \in \operatorname{Spec}(R)$ a prime ideal which is minimal over I. Then $PR[x]$ is a minimal prime ideal over $IR[x]$.

Proof: By (4.62)

(4.64) **Lemma**: Let R be a Noetherian ring and $P \in \operatorname{Spec}(R)$. Then $\operatorname{ht} P = \operatorname{ht} PR[x]$.

Proof: Let $\operatorname{ht} P = n$. By (4.55) there are elements $a_1, \dots, a_r \in P$ so that P is a minimal prime ideal over $I = (a_1, \dots, a_n)$. By (4.63) $PR[x]$ is a minimal prime over $IR[x]$. Thus $\operatorname{ht} PR[x] \leq n$. Let $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P$ be a chain of prime ideals of R. Then $P_0 R[x] \subsetneq \dots \subsetneq P_n R[x] = PR[x]$ is a chain of prime ideals in $R[x]$ and $\operatorname{ht} PR[x] \geq n$.

Proof of (4.59): We have to show that $\dim R[x] \leq \dim R + 1$. Let $Q_0 \subsetneq Q_1 \subsetneq \dots \subsetneq Q_r$ be a chain of prime ideals in $R[x]$ and set $P_i = Q_i \cap R$. Suppose that $P_i = P_{i+1}$ for some i and let j be maximal with $P_j = P_{j+1}$. By (4.62) $Q_j = P_j R[x]$ and by (4.64) $\operatorname{ht} P_j = \operatorname{ht} Q_j$. This shows that $\operatorname{ht} P_j \geq j$. Moreover, $P_j = P_{j+1} \subsetneq P_{j+2} \subsetneq \dots \subsetneq P_r$ is a chain of prime ideals of R of length $r - j - 1$. Thus $\operatorname{ht} P_r \geq r - 1$ and $\dim R \geq r - 1$.

(4.65) **Corollary**: Let K be a field. Then

(a) $\dim K[x_1, \dots, x_n] = n$

(b) For all $1 \leq r \leq n$: $\operatorname{ht}(x_1, \dots, x_r) = r$.

Proof: Homework