CHAPTER III: FINITELY GENERATED ALGEBRAS OVER A FIELD

§1: HILBERT'S NULLSTELLENSATZ

(3.1) **Proposition:** Let $K \subseteq L$ be a field extension and $\alpha_1, \ldots, \alpha_n \in L$ algebraic over K. Let $x_1, \ldots, x_n$ be variables over K. For all $1 \leq i \leq n$ there is a polynomial $f_i \in K[x_1, \ldots, x_i]$ which is monic in x_i so that $K(\alpha_1, \ldots, \alpha_n) = K[\alpha_1, \ldots, \alpha_n] \cong K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

Proof: By induction on n: If $n=1$ there is a monic polynomial $f \in K[x]$ with $K(\alpha) = K[\alpha] \cong K[x]/(f)$. f is the minimal polynomial of α over K.

$n-1 \Rightarrow n$: By induction hypothesis $E = K(\alpha_1, \ldots, \alpha_{n-1}) \cong K[x_1, \ldots, x_{n-1}]/(f_1, \ldots, f_{n-1})$ where $f_i \in K[x_1, \ldots, x_i]$ monic in x_i for all $1 \leq i \leq n-1$. Since α_n is algebraic over E, $E(\alpha_n) = E[\alpha_n] \cong E[x_n]/(g)$ where $g \in E[x_n]$ is the minimal polynomial of α_n over E. Consider the surjective homomorphism of rings $\varphi: K[x_1, \ldots, x_n] \longrightarrow K[\alpha_1, \ldots, \alpha_{n-1}][x_n] = E[x_n]$ defined by $\varphi|_K = \mathrm{id}_K$, $\varphi(x_i) = \alpha_i$ for $1 \leq i \leq n-1$, and $\varphi(x_n) = x_n$. There is a polynomial $f_n \in K[x_1, \ldots, x_n]$ which is monic in x_n and with $\varphi(f_n) = g$. Then $K(\alpha_1, \ldots, \alpha_n) = K[\alpha_1, \ldots, \alpha_n] \cong K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

(3.2) **Definition and Remark:** Let R and S be rings. S is called an R-algebra if S is an R-module. If S is an R-algebra, the map $\varphi: R \longrightarrow S$ defined by $\varphi(a) = a1_S$ for all $a \in R$ is a homomorphism of rings. Conversely, every homomorphism of rings $\varphi: R \longrightarrow S$ defines an R-module structure on S and hence makes S into an R-algebra. If $\varphi: R \longrightarrow S$ is a homomorphism of rings, S is called a finitely generated R-algebra if there are elements $b_1, \ldots, b_n \in S$ so that the homomorphism $\psi: R[x_1, \ldots, x_n] \longrightarrow S$ defined by $\psi|_R = \varphi$ and $\psi(x_i) = b_i$ for all $1 \leq i \leq n$ is surjective. Note that a finitely generated R-algebra is, in general, not a finite R-module.

(3.3) **Theorem:** Let $R \subseteq S \subseteq T$ be ring extensions and assume that R is a Noetherian ring and T is a finitely generated R-algebra and a finite S-module. Then S is a finitely generated R-algebra.

Proof: First note that we may replace R by any ring $R \subseteq R_0 \subseteq S$ which is a finitely generated R-algebra. We want to construct a ring R_0 with $R \subseteq R_0 \subseteq S$ such that:

(a) R_0 is a finitely generated R-algebra

(b) T is a finite R_0-module.

Since R_0 is a Noetherian ring, T is a Noetherian R_0-module. S is an R_0-submodule of T. Thus S is a finite R_0-module and a finitely generated R-algebra.

In order to construct R_0 write $T = R[\alpha_1, -, \alpha_n] = S\beta_1 + — + S\beta_m$ for some $\alpha_i, \beta_j \in T$. Then for all $1 \leq i \leq n$ $\alpha_i = \sum_{j=1}^{m} \lambda_{ij}\beta_j$ where $\lambda_{ij} \in S$. Replace R by $R' = R[\lambda_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \subseteq S$. Thus we may assume $\alpha_i \in R[\beta_1, -, \beta_m]$ and therefore $T = R[\beta_1, -, \beta_m] = S\beta_1 + — + S\beta_m$. For all $1 \leq i, j \leq m$ there are elements $\mu_{ijk} \in S$ so that $\beta_i\beta_j = \sum_{k=1}^{m} \mu_{ijk}\beta_k$. Set

$$R_0 = R[\mu_{ijk} \mid 1 \leq i, j, k \leq m] \subseteq S.$$

Then $T = R_0[\beta_1, -, \beta_m] = R_0\beta_1 + — + R_0\beta_m$ is a finite R_0-module.

(3.3) Theorem: Let K be a field and R a finitely generated K-algebra. If R is a field then R is algebraic over K.

Proof: Suppose that R is a field which is not algebraic over K. The field extension $K \subseteq R$ is finitely generated and there is a transcendence basis $\{y_1, -, y_t\} \subseteq R$ of R over K. With $T = K(y_1, -, y_t)$, $K \subseteq T \subseteq R$, T is purely transcendental over K, and R is finite algebraic over T. Thus, K is a Noetherian ring and R a finitely generated K-algebra and a finite T-module. By (3.2) T is a finitely generated K-algebra. Hence there are elements $f_i, g_i \in K[y_1, -, y_t]$ such that

$$T = K(y_1, -, y_t) = K[f_1/g_1, —, f_n/g_n].$$

Since $K[y_1, -, y_t]$ is not a field, there is a g_i with $g_i \notin K$. Set $h = \prod_{i=1}^{n} g_i + 1$.

Then $h \notin K$ and h is not a unit in $K[y_1, -, y_t]$. Since $1/h \in T$,

$$1/h = \sum_{(i_1,-,i_n)\in\mathbb{N}^n} a_{(i_1,-,i_n)} (f_1/g_1)^{i_1} \cdots (f_n/g_n)^{i_n} \qquad \text{(finite sum)}$$

where $a_{(i_1,-,i_n)} \in K$. Thus there is an $r \in \mathbb{N}$ so that

$$(*) \quad (g_1 \cdots g_n)^r = \Big(\sum_{(i_1,-,i_n)\in\mathbb{N}^n} a_{(i_1,-,i_n)} f_1^{i_1} \cdots f_n^{i_n} g_1^{r-i_1} \cdots g_n^{r-i_n}\Big) h \in K[y_1, -, y_t].$$

$K[y_1, -, y_n]$ is a factorial domain. By $(*)$ h divides $(g_1 \cdots g_n)^r$ which implies that there is an irreducible polynomial $p \in K[y_1, -, y_t]$ which divides h and $(g_1 \cdots g_n)$, a contradiction.

(3.4) Theorem: (Hilbert's Nullstellensatz I) Let K be a field and $m \subseteq K[x_1, -, x_n]$ a maximal ideal in the polynomial ring. Then

(a) $K[x_1, -, x_n]/m$ is an algebraic extension field of K.

(b) m is generated by n elements.

(c) If K is algebraically closed, there are elements $a_i \in K$ with $m = (x_1 - a_1, -, x_n - a_n)$.

Proof: (a) $K[x_1, -, x_n]/m$ is a field and a finitely generated K-algebra. By (3.3) $K[x_1, -, x_n]/m$ is algebraic over K.

(b) By (a) and (3.1).

(c) If K is algebraically closed, the natural map $K \longrightarrow K[x_1, -, x_n]/m$ is surjective. Thus there are $a_i \in K$ with $x_i \equiv a_i \pmod{m}$ implying $m = (x_1 - a_1, -, x_n - a_n)$.

(3.5) Theorem: (Hilbert's Nullstellensatz II) Let K be a field, R a finitely generated K-algebra and $I \subseteq R$ an ideal. Then $\operatorname{rad}(I) = \bigcap_{I \subseteq m;\, m \text{ max.}} m$.

Proof: We may assume $I = (0)$ and have to show $\operatorname{nil}(R) = \operatorname{Jrad}(R)$. The last statement is equivalent to: if $f \notin \operatorname{nil}(R)$ then there is a maximal ideal $m \subseteq R$ with $f \notin m$. If $f \notin \operatorname{nil}(R)$ then $f^n \neq 0$ for all $n \in \mathbb{N}$ and the localization $R_f = S^{-1}R$, where $S = \{1, f, f^2, \ldots\}$, is not the null ring. Let $n \subseteq R_f$ be a maximal ideal and $m = i_{R,f}^{-1}(n)$ its contraction to R. We claim that m is a maximal ideal of R. First note that $R_f = i_{R,f}(R)[1/f]$ is a finitely generated K-algebra. By (3.3) R_f/n is an algebraic extension field of K. Consider the injective homomorphisms: $K \hookrightarrow R/m \hookrightarrow R_f/n = L$. Since $K \subseteq L$ is an algebraic field extension, any intermediate ring $K \subseteq T \subseteq L$ is a field. Hence R/m is a field and m is a maximal ideal of R with $f \notin m$.

(3.5) <u>Definition</u>: Let R be a ring. A finite chain of $n+1$ distinct prime ideals of R: $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ is called a <u>prime</u> <u>ideal</u> <u>chain</u> of <u>length</u> n in R.

(3.6) <u>Definition</u>: Let R be a ring and $M \neq 0$ a finite R-module.

(a) Let $P \subseteq R$ be a prime ideal. The <u>height</u> of P is defined by:
$\operatorname{ht} P = \sup\{n \mid \exists$ a prime ideal chain of length n inside P: $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n \subseteq P\}$.

(b) Let $I \subseteq R$ be an ideal. The <u>height</u> of I is defined by
$\operatorname{ht} I = \inf\{\operatorname{ht} P \mid P \subseteq R$ a prime ideal with $I \subseteq P\}$.

(c) The (Krull) <u>dimension</u> of R is given by: $\dim R = \sup\{\operatorname{ht} P \mid P$ a prime ideal of $R\}$.

(d) The (Krull) <u>dimension</u> of M is given by: $\dim_R M := \dim(R/\operatorname{ann}_R(M))$.

(e) Let $P \subseteq R$ be a prime ideal. $\dim P = \dim R/P$ is called the <u>dimension</u> of P (in R).

(3.7) <u>Remark</u>: Let R be a ring and $P \subseteq R$ a prime ideal. Then $\operatorname{ht} P = \dim R_P = \operatorname{ht} PR_P$.

(3.8) <u>Remark</u>: Let K be a field and R a finitely generated K-algebra and a domain. The quotient field $L = Q(R)$ is a finitely generated field extension of K. We define: $\operatorname{trdeg}_K(R) := \operatorname{trdeg}_K(L)$.

(3.9) <u>Theorem</u>: Let K be a field and R a finitely generated K-algebra and a domain. Then $\dim R = \operatorname{trdeg}_K(R)$.

<u>Proof</u>: Set $d = \dim R$ and $t = \operatorname{trdeg}_K(R)$.

(a) Let $(0) = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_d \subsetneq R$ be a chain of prime ideals of maximal length d in R. We claim that $\operatorname{trdeg}_K(R) \gneq \operatorname{trdeg}_K(R/P_1) \gneq \dots \gneq \operatorname{trdeg}_K(R/P_d)$. In order to prove this it suffices to show: if $Q \subsetneq P$ are prime ideals of R then $\operatorname{trdeg}_K(R/Q) \gneqq \operatorname{trdeg}_K(R/P)$. We may assume that $Q=(0)$ and have to show for any prime ideal $P \neq (0)$ that $t = \operatorname{trdeg}_K(R) \gneqq \operatorname{trdeg}_K(R/P)$. Let $P \subseteq R$ be a prime ideal, $P \neq (0)$, and $r = \operatorname{trdeg}_K(R/P)$. Choose $a_1, \ldots, a_r \in R$ so that $\{\bar{a}_1, \ldots, \bar{a}_r\} \subseteq R/P$ is a transcendence basis of $Q(R/P)$ over K. This implies that $a_1, \ldots, a_r \in R$ are algebraically independent over K and that $K[a_1, \ldots, a_r] \cap P = (0)$. Let $S = K[a_1, \ldots, a_r] - (0)$ and $L = S^{-1}K[a_1, \ldots, a_r] \subseteq S^{-1}R$. Since R is a finitely generated $K[a_1, \ldots, a_r]$-algebra, $S^{-1}R$ is a finitely generated L-algebra and a domain. Since $S \cap P = \emptyset$, $S^{-1}P \subseteq S^{-1}R$ is a proper, nonzero prime ideal of $S^{-1}R$. If $Q(R)$ is algebraic over L, every intermediate ring $L \subseteq A \subseteq Q(R)$ is a field and $S^{-1}R$ is a field, a contradiction. Thus $\operatorname{trdeg}_L(S^{-1}R) > 0$ and $\operatorname{trdeg}_K(S^{-1}R) = \operatorname{trdeg}_K(L) + \operatorname{trdeg}_L(S^{-1}R) = r + \operatorname{trdeg}_L(S^{-1}R) > r = \operatorname{trdeg}_K(R/P)$.

(b) We show by induction on t that $t \leq d$. If $t=0$, then $R = Q(R)$ is a field. Let $t > 0$ and let $a \in R$ be transcendental over K. Set $L = K(a)$ and $S = K[a] - (0) \subseteq R$. R is a finitely generated $K[a]$-algebra and $S^{-1}R$ is a finitely generated L-algebra. Moreover, $\operatorname{trdeg}_L(S^{-1}R) = t-1$ and by induction hypothesis $\dim S^{-1}R \geq t-1$. Let $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{t-1} \subsetneq S^{-1}R$ be a chain of prime ideals of length $t-1$ in $S^{-1}R$ and let $Q_i = R \cap P_i$ be its contraction to R. Then $(0) = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{t-1}$ is a chain of prime ideals of length $t-1$ in R. Since $Q_{t-1} \cap S = \emptyset$, $Q_{t-1} \cap K[a] = (0)$ and $\bar{a} \in R/Q_{t-1}$ is transcendental over K. By (3.3) R/Q_{t-1} is not a field. Thus there is a prime ideal $P \subseteq R/Q_{t-1}$ with $P \neq (0)$ and $\dim R \geq t$.

(3.10) <u>Corollary</u>: Let K be a field and $K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. Then $\dim K[x_1, \ldots, x_n] = n$.

(3.11) <u>Definition</u>: Let R be a Noetherian ring. R is called <u>catenary</u> if for all prime ideals $P, Q \in \operatorname{Spec}(R)$ with $P \subseteq Q$ all saturated chains of prime ideals $P \subsetneq P_1 \subsetneq \cdots \subsetneq P_s \subsetneq Q$ have the same length. (A chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is <u>saturated</u> if it cannot be extended, that is, there is no prime ideal W with $P_i \subsetneq W \subsetneq P_{i+1}$ for some $0 \leq i \leq n-1$.)

(3.12) Remark: (a) Nagata constructed in 1956 the first non-catenary Noetherian ring.
(b) Ratliff showed in 1972 that a local Noetherian domain R is catenary if and only if $\operatorname{ht} P + \dim R/P = \dim R$ for all $P \in \operatorname{Spec}(R)$.

(3.13) Proposition: Let K be a field, $R = K[x_1, \ldots, x_n]$ the polynomial ring over K, and $I \subseteq R$ an ideal. Then $\operatorname{ht} I + \dim R/I = \dim R$.

Proof: Homework

(3.14) Proposition: Let K be a field. Every finitely generated K-algebra is catenary.

Proof: Homework

§ 2: ALGEBRAIC VARIETIES

(3.15) *Definition:* Let K be a field. The set $\mathbb{A}^n_K = \mathbb{A}^n = K^n = \{(a_1, -, a_n) \mid a_i \in K\}$ is called the *affine n-space*. An element $P = (a_1, -, a_n) \in \mathbb{A}^n_K$ is called a *point* in $\mathbb{A}^n_K$, the a_i are the *coordinates* of P.

(3.16) *Definition:* (a) Let $T \subseteq K[x_1, -, x_n]$ be a subset of the polynomial ring. The set $Z(T) = \{P \in \mathbb{A}^n_K \mid f(P) = 0 \text{ for all } f \in T\} \subseteq \mathbb{A}^n_K$ is the *zero set* of T in $\mathbb{A}^n_K$.
(b) A subset $V \subseteq \mathbb{A}^n_K$ is called an *affine variety* (or *algebraic variety*) if there is a subset $T \subseteq K[x_1, -, x_n]$ with $V = Z(T)$.

(3.17) *Examples:* (a) *Linear varieties:* Let $T = \{f_1, -, f_m\} \subseteq K[x_1, -, x_n]$ where the f_i are linear polynomials.
(b) *Hypersurfaces:* Let $f \in K[x_1, -, x_n] - K$ be a polynomial. The set $Z(f)$ is called a hypersurface in $\mathbb{A}^n_K$. If $n = 3$, $Z(f)$ is called a *surface*. If K is not algebraically closed, (hyper)surfaces may degenerate to a single point, a curve or the empty set. For example, if $K = \mathbb{R}$, then $Z(x^2+y^2+z^2) = \{(0,0,0)\}$ and $Z(x^2+y^2+1) = \emptyset$. Hypersurfaces of order 2 (called quadrics) are described by an equation $f = \sum_{i,k=1}^{n} a_{ik} x_i x_k + \sum_{i=1}^{n} b_i x_i + c = 0$ where $a_{ik}, b_i, c \in K$.
(c) *Plane algebraic curves* are hypersurfaces in $\mathbb{A}^2_K$.
(d) If $T = \{x_1 x_3, x_2 x_3\}$, then $Z(T) \subseteq \mathbb{A}^3_{\mathbb{R}}$ is the (x_1, x_2)-plane together with the x_3-axis.
(e) *Algebraic groups:* For every $A \in GL_n(K)$ consider the point $(A, \det(A^{-1})) \in \mathbb{A}^{n^2+1}_K$. The set of all these points can be identified with the hypersurface:
$H = Z(\det(x_{ik})_{1 \le i,k \le n} Y - 1) \subseteq \mathbb{A}^{n^2+1}_K$. Matrix multiplication provides a group operation on H:

$$\begin{tikzcd}
H \times H \arrow[r] & H \\
{[(A, \det(A^{-1})), (B, \det(B^{-1}))]} \arrow[r] \arrow[u, phantom, sloped, "\in"] & (AB, \det((AB)^{-1})). \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

Varieties which are provided with a group operation are called algebraic groups.

(3.18) <u>Remark</u>: Let $T \subseteq K[x_1,-,x_n]$ be a subset and $I=(T)$ the ideal generated by T. Then $Z(T)=Z(I)$. Every affine variety is the zero set of an ideal in $K[x_1,-,x_n]$. Since $K[x_1,-,x_n]$ is Noetherian, $Z(T)$ is the zero set of finitely many polynomials.

(3.19) <u>Proposition</u>: (a) If $Y_1=Z(T_1), Y_2=Z(T_2) \subseteq \mathbb{A}^n_K$ are affine varieties, so is $Y_1 \cup Y_2$.
(b) If $\{Y_i = Z(T_i)\}_{i\in I}$ is a set of affine varieties, the intersection $\bigcap_{i\in I} Y_i$ is an affine variety.
(c) $\emptyset$ and $\mathbb{A}^n_K$ are affine varieties.

<u>Proof</u>: (a) Set $T=T_1T_2=\{fg \mid f\in T_1 \text{ and } g\in T_2\}$. Obviously, $Y_1\cup Y_2 \subseteq Z(T_1T_2)$. If $P\notin Y_1\cup Y_2$ then there are $f\in T_1$ and $g\in T_2$ with $f(P)\neq 0$ and $g(P)\neq 0$. Thus $(fg)(P)\neq 0$ and $P\notin Z(T_1T_2)$.
(b) $\bigcap_{i\in I} Y_i = Z\left(\bigcup_{i\in I} T_i\right)$
(c) $Z(1)=\emptyset$ and $Z(0)=\mathbb{A}^n_K$.

(3.20) <u>Definition</u> and <u>Remark</u>: Define the <u>Zariski-topology</u> on $\mathbb{A}^n_K$ as follows: A subset $Y\subseteq \mathbb{A}^n_K$ is closed if and only if $Y=Z(T)$ is an affine variety. By (3.19) this defines a topology on $\mathbb{A}^n_K$ with open sets $\mathcal{U}=\mathbb{A}^n_K - Y$ where Y is an affine variety.

(3.21) <u>Example</u>: Let $K=\mathbb{C}$. The closed sets of $\mathbb{C}=\mathbb{A}^1_{\mathbb{C}}$ are the varieties $Z(I)$ where $I\subseteq \mathbb{C}[x]$ is an ideal. Thus $I=(f)$ for some $f=(x-a_1) - (x-a_n) \in \mathbb{C}[x]$ or $f=0,1$. In the Zariski-topology the closed sets of $\mathbb{A}^1_{\mathbb{C}}$ are $\emptyset, \mathbb{C}$ and the finite subsets of $\mathbb{C}$.

(3.22) <u>Definition</u>: Let $Y\subseteq \mathbb{A}^n_K$ be a subset. The set

$$\mathcal{J}(Y)=\{f\in K[x_1,-,x_n] \mid f(P)=0 \text{ for all } P\in Y\} \subseteq K[x_1,-,x_n]$$

is called the <u>ideal</u> of Y. (Note that $\mathcal{J}(Y)$ is an ideal.)

(3.23) <u>Proposition</u>: Let K be a field.
(a) $T_1 \subseteq T_2 \subseteq K[x_1,-,x_n] \implies Z(T_1)\supseteq Z(T_2)$

(b) If $I \subseteq K[x_1, -, x_n]$ is an ideal then $Z(I) = Z(\mathrm{rad}(I))$.

(c) $Y_1, Y_2 \subseteq \mathbb{A}^n_K \Rightarrow \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$

(d) $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n_K \Rightarrow \mathcal{J}(Y_1) \supseteq \mathcal{J}(Y_2)$

(e) If $Y \subseteq \mathbb{A}^n_K$, then $\mathcal{J}(Y) = \mathrm{rad}(\mathcal{J}(Y))$, that is, $\mathcal{J}(Y)$ is a reduced ideal.

(f) If $Y \subseteq \mathbb{A}^n_K$, then $Z(\mathcal{J}(Y)) = \overline{Y}$ where $\overline{Y}$ denotes the closure of Y in the Zariski topology. In particular, if $Y = Z(T)$ is an affine variety, then $Z(\mathcal{J}(Y)) = Y$.

<u>Proof</u>: (a), (c), (d), and (e) are obvious.

(b) By (a) $Z(\mathrm{rad}(I)) \subseteq Z(I)$. Let $P \in Z(I)$ and $f \in \mathrm{rad}(I)$. Then $f^r \in I$ for some $r \in \mathbb{N}$ and $f^r(P) = f(P)^r = 0$. Hence $f(P) = 0$ and $P \in Z(\mathrm{rad}(I))$.

(f) Obviously, $Y \subseteq Z(\mathcal{J}(Y))$ and hence $\overline{Y} \subseteq Z(\mathcal{J}(Y))$, since $Z(\mathcal{J}(Y))$ is closed in $\mathbb{A}^n_K$. Let $W \subseteq \mathbb{A}^n_K$ be a closed subset with $Y \subseteq W$. Then $W = Z(I) \supseteq Y$. By (c) $\mathcal{J}(W) = \mathcal{J}(Z(I)) \subseteq \mathcal{J}(Y)$ and thus $I \subseteq \mathcal{J}(Y)$. By (a) $W = Z(I) \supseteq Z(\mathcal{J}(Y))$. This shows that $\overline{Y} = Z(\mathcal{J}(Y))$.

(3.24) <u>Remark</u>: By (3.23) there are order-reversing maps:

$$\{\text{affine varieties in } \mathbb{A}^n_K\} \underset{Z}{\overset{\mathcal{J}}{\rightleftarrows}} \{\text{reduced ideals in } K[x_1, -, x_n]\}$$

with $Z \circ \mathcal{J} = \mathrm{id}_{\{\text{aff. var}\}}$. The question is if there is a 1-1 correspondence between the set of reduced ideals of $K[x_1, -, x_n]$ and the set of affine varieties of $\mathbb{A}^n_K$. In general the answer is no, but if K is algebraically closed then also $\mathcal{J} \circ Z = \mathrm{id}_{\{\text{red. id.}\}}$.

(3.25) <u>Theorem</u>: (Hilbert's Nullstellensatz) Let K be an algebraically closed field and $I \subseteq K[x_1, -, x_n]$ an ideal. Then $\mathcal{J}(Z(I)) = \mathrm{rad}(I)$.

<u>Proof</u>: Obviously, $\mathrm{rad}(I) \subseteq \mathcal{J}(Z(I))$. In order to show the other inclusion set $J = \mathcal{J}(Z(I))$ and $R = K[x_1, -, x_n]$. By (3.23)(f) $Z(J) = Z(I)$ and therefore

$$(*) \quad (a_1, -, a_n) \in Z(J) \iff (a_1, -, a_n) \in Z(I).$$

By Taylor's formula for any $f \in R$: $f(a_1, -, a_n) = 0 \iff f \in (x_1 - a_1, -, x_n - a_n)$.

Thus $(*)$ is equivalent to: $\mathcal{J} \subseteq (x_1-a_1, \ldots, x_n-a_n) \Longleftrightarrow I \subseteq (x_1-a_1, \ldots, x_n-a_n)$. This implies:
$\{(x_1-a_1, \ldots, x_n-a_n) \mid \mathcal{J} \subseteq (x_1-a_1, \ldots, x_n-a_n)\} = \{(x_1-a_1, \ldots, x_n-a_n) \mid I \subseteq (x_1-a_1, \ldots, x_n-a_n)\}$.
Since K is algebraically closed, by (3.4) every maximal ideal of R is of the form $(x_1-a_1, \ldots, x_n-a_n)$ for some $a_i \in K$. Hence
$\{m \subseteq R \mid m$ a maximal ideal and $\mathcal{J} \subseteq m\} = \{m \subseteq R \mid m$ a maximal ideal and $I \subseteq m\}$.
By (3.5) $\bigcap_{m \text{ max.}; \mathcal{J} \subseteq m} m = \operatorname{rad}(\mathcal{J}) = \mathcal{J} = \bigcap_{m \text{ max.}; I \subseteq m} m = \operatorname{rad}(I)$.

(3.26) <u>Proposition</u>: Every algebraic variety is an intersection of finitely many hypersurfaces.

<u>Proof</u>: If $Y \subseteq \mathbb{A}^n_K$ is an algebraic variety then $Y = Z(I)$ for some ideal $I \subseteq K[x_1, \ldots, x_n]$. Since $I = (f_1, \ldots, f_m)$ is finitely generated, $Y = Z(f_1, \ldots, f_m) = Z(f_1) \cap \ldots \cap Z(f_m)$.

(3.27) <u>Proposition</u>: Let K be a field. Every decreasing chain of varieties $Y_1 \supseteq Y_2 \supseteq \ldots$ in $\mathbb{A}^n_K$ is stationary.

<u>Proof</u>: Since $K[x_1, \ldots, x_n]$ is Noetherian, the increasing chain of ideals $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y_2) \subseteq \ldots$ is stationary, i.e., there is an $N \in \mathbb{N}$ such that $\mathcal{J}(Y_N) = \mathcal{J}(Y_{N+k})$ for all $k \in \mathbb{N}$. Then by (3.23)(f): $Z(\mathcal{J}(Y_N)) = Y_N = Z(\mathcal{J}(Y_{N+k})) = Y_{N+k}$.

(3.28) <u>Definition</u>: A topological space X is called <u>irreducible</u> if whenever $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$, then $X = A_1$ or $X = A_2$. A subset $X' \subseteq X$ is irreducible if X' is irreducible in the induced topology.

(3.29) <u>Lemma</u>: Let X be a topological space. The following are equivalent:
(a) X is irreducible
(b) If $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ are nonempty open subsets, then $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$.
(c) Any nonempty open subset of X is dense in X.

<u>Proof</u>: Home work

(3.30) <u>Lemma</u>: Let X be a topological space and $X' \subseteq X$ a subset. The following are equivalent:

(a) X' is irreducible.

(b) If $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ are open subsets with $X' \cap \mathcal{U}_i \neq \emptyset$, then $X' \cap \mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$.

(c) The closure $\overline{X'}$ of X' is irreducible.

<u>Proof</u>: (b) $\Longleftrightarrow$ (c): Immediately with: if $\mathcal{U} \subseteq X$ is an open subset, then $\mathcal{U} \cap \overline{X'} \neq \emptyset \Longleftrightarrow \mathcal{U} \cap X' \neq \emptyset$.

(3.31) <u>Definition</u>: An <u>irreducible</u> <u>component</u> of a topological space is a maximal irreducible subset.

(3.32) <u>Proposition</u>: (a) Irreducible components are closed.

(b) Every irreducible subset is contained in an irreducible component.

(c) A topological space is the union of its irreducible components.

<u>Proof</u>: (a) follows from (3.30). Since every point is irreducible, it suffices to show (b). Let $X' \subseteq X$ be an irreducible subset. Consider $\mathcal{M} = \{Y \subseteq X \mid X' \subseteq Y \text{ and } Y \text{ irreducible}\}$. $\mathcal{M}$ is partially ordered by inclusion with $\mathcal{M} \neq \emptyset$, since $X' \in \mathcal{M}$. Let $\mathcal{K} = \{Y_\lambda\}_{\lambda \in \Lambda}$ be a chain in $\mathcal{M}$. Set $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$. We claim that Y is irreducible. Let $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ be open subsets with $\mathcal{U}_i \cap Y \neq \emptyset$. Then there are $\lambda, \mu \in \Lambda$ with $\mathcal{U}_1 \cap Y_\lambda \neq \emptyset$ and $\mathcal{U}_2 \cap Y_\mu \neq \emptyset$. Suppose $Y_\lambda \subseteq Y_\mu$, then $\mathcal{U}_1 \cap Y_\mu \neq \emptyset$ and $\mathcal{U}_2 \cap Y_\mu \neq \emptyset$. Since Y_μ is irreducible, $\mathcal{U}_1 \cap \mathcal{U}_2 \cap Y_\mu \neq \emptyset$ and thus $\mathcal{U}_1 \cap \mathcal{U}_2 \cap Y \neq \emptyset$. Therefore $Y \in \mathcal{M}$. By Zorn's Lemma $\mathcal{M}$ has a maximal element and X' is contained in an irreducible component.

(3.33) <u>Example</u>: $\mathbb{R}^n$ and $\mathbb{C}^n$ with the ordinary topology are Hausdorff spaces. Their irreducible components are points.

(3.34) Definition: A topological space X is called Noetherian if every descending chain $Y_1 \supseteq Y_2 \supseteq \dots$ of closed subsets $Y_i \subseteq X$ is stationary, or equivalently, the set of closed subsets of X satisfies the d.c.c.

(3.35) Remark: (a) For every field K the affine space $\mathbb{A}^n_K$ is Noetherian in the Zariski topology. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are Noetherian in that topology. Note that $\mathbb{R}^n$ and $\mathbb{C}^n$ are not Noetherian in the ordinary topology.

(b) Let X be a Noetherian space. Then the a.c.c. holds for the set of open subsets of X, or equivalently, every chain of open subsets $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots$ is stationary.

(3.36) Proposition: A Noetherian topological space has only finitely many irreducible components. No irreducible component is contained in the union of the others.

Proof: Let X be a Noetherian topological space. Consider the set:
$\mathcal{M} = \{V \subseteq X \mid V$ is closed and V is not a union of finitely many irreducible subsets of $X\}$.
Suppose that $\mathcal{M} \neq \emptyset$. Since X is Noetherian, every nonempty set of closed subsets of X has a minimal element. Let $Y \in \mathcal{M}$ be a minimal element. Then Y is not irreducible and there are closed subsets $Y_i \subseteq Y$ with $Y = Y_1 \cup Y_2$ and $Y_i \neq Y$. By the minimality of Y every Y_i is the union of finitely many irreducible subsets. Thus Y is the union of finitely many irreducible subsets, a contradiction. Hence $\mathcal{M} = \emptyset$ and $X = X_1 \cup \dots \cup X_n$ with $X_i \subseteq X$ irreducible and closed. We may assume that the X_i are irreducible components of X. Let Y be an irreducible component of X. Then $Y = \bigcup_{i=1}^{n} (X_i \cap Y)$ and hence $Y = X_i \cap Y$ for some i. Thus $Y = X_i$ and $X_1, —, X_n$ are the only irreducible components of X. A similar argument shows that no component is contained in the union of the other components.

(3.37) Corollary: An affine variety $Y \subseteq \mathbb{A}^n_K$ has only finitely many irreducible components $Y_1, —, Y_s$ and $Y = Y_1 \cup \dots \cup Y_s$.

(3.38) Proposition: An affine variety $Y \subseteq \mathbb{A}^n_K$ is irreducible if and only if $\mathcal{J}(Y) \subseteq K[x_1, -, x_n]$ is a prime ideal.

Proof: "$\Longrightarrow$": Suppose that Y is irreducible and let $f, g \in K[x_1, -, x_n]$ with $fg \in \mathcal{J}(Y)$. With $H = Z(f)$ and $L = Z(g)$ $H \cup L = Z(fg) \supseteq Y = Z(\mathcal{J}(Y))$. Thus $Y = Y \cap (H \cup L) = (Y \cap H) \cup (Y \cap L)$. Since Y is irreducible, $Y \subseteq H$ or $Y \subseteq L$ and therefore $\mathcal{J}(H) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(L) \subseteq \mathcal{J}(Y)$. Thus $f \in \mathcal{J}(Y)$ or $g \in \mathcal{J}(Y)$.

"$\Longleftarrow$": Suppose that $\mathcal{J}(Y)$ is a prime ideal. Let $Y_1, Y_2 \subseteq Y$ be affine varieties with $Y = Y_1 \cup Y_2$. Then $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$ and $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) \subseteq \mathcal{J}(Y)$, since $\mathcal{J}(Y)$ is prime. Thus $\mathcal{J}(Y_1) = \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) = \mathcal{J}(Y)$ and $Y_1 = Z(\mathcal{J}(Y_1)) = Z(\mathcal{J}(Y)) = Y$ or $Y_2 = Z(\mathcal{J}(Y_2)) = Y$.

(3.39) Definition: Let $Y \subseteq \mathbb{A}^n_K$ be an affine variety. The ring $A(Y) = K[x_1, -, x_n]/\mathcal{J}(Y)$ is called the coordinate ring of Y or the affine K-algebra of Y.

(3.40) Remark: Let $Y \subseteq \mathbb{A}^n_K$ be an affine variety with coordinate ring $A(Y) = K[x_1, -, x_n]/\mathcal{J}(Y)$. For all $F \in K[x_1, -, x_n]$ define $\varphi_F : Y \longrightarrow K$ by $\varphi_F(P) = F(P)$. Consider the set $D(Y) = \{ \varphi : Y \longrightarrow K \mid \exists\ F \in K[x_1, -, x_n] \text{ with } \varphi = \varphi_F \}$. $D(Y)$ is a ring under the obvious operations and the map $\Phi : K[x_1, -, x_n] \longrightarrow D(Y)$ with $\Phi(F) = \varphi_F$ is a surjective homomorphism of rings with kernel $\mathcal{J}(Y)$. Then $D(Y) \cong A(Y)$.

(3.41) Definition: Let X be a topological space and $Y \subseteq X$ a closed irreducible subset.

(a) The dimension of X is defined by: $\dim X = \sup\{ n \in \mathbb{N} \mid \exists$ a chain of nonempty closed irreducible subsets in X : $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \}$.

(b) If $Y \neq \emptyset$, the codimension of Y in X is defined by: $\operatorname{codim}_X(Y) = \sup\{ n \in \mathbb{N} \mid \exists$ a chain of closed irreducible subsets $X_i \subseteq X$: $Y = Y_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \}$

(c) If $A \subseteq X$ is any nonempty closed subset, the codimension of A in X is defined by: $\operatorname{codim}_X(A) = \sup\{ \operatorname{codim}_X Y \mid Y$ an irreducible component of $A \}$.

(d) $\dim(\emptyset) = -1$ and $\operatorname{codim}_X(\emptyset) = \infty$.

(3.42) Proposition: Let X be a topological space and $Y \subseteq X$ a closed irreducible subset.

(a) If $\{X_\lambda\}_{\lambda \in \Lambda}$ is the set of irreducible components of X, then $\dim X = \sup_{\lambda \in \Lambda} \{\dim X_\lambda\}$.

(b) If $A_i \subseteq X$ are closed subsets with $X = A_1 \cup \ldots \cup A_n$, then $\dim X = \sup \{\dim A_i \mid 1 \leq i \leq n\}$.

(c) $\dim Y + \operatorname{codim}_X(Y) \leq \dim X$ if $Y \neq \emptyset$.

(d) If X is irreducible with $\dim X < \infty$, then $\dim Y \neq \dim X \iff Y \neq X$.

Proof: (b) If $Y \subseteq X$ is irreducible, then $Y = (A_1 \cap Y) \cup \ldots \cup (A_n \cap Y)$ and $Y = A_j \cap Y$ for some $1 \leq j \leq n$. Thus $Y \subseteq A_j$. Apply (a).

(c) Start with a chain of closed irreducible subsets ending in Y and continue with a chain starting in Y.

(3.43) Proposition: Let K be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an affine variety. Then $\dim Y = \dim A(Y)$.

Proof: Case 1: Y is irreducible. Then $Y = Z(P)$ for a prime ideal $P \subseteq K[x_1, -, x_n]$. $\dim Y$ is the supremum of the length n of chains $(*)$ $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n = Y$ where the Y_i are closed irreducible subsets of $\mathbb{A}^n_K$ (resp. Y). Then $Y_i = Z(P_i)$ for prime ideals $P_i \subseteq K[x_1, -, x_n]$. By Hilbert's Nullstellensatz $\mathcal{J}(Y_i) = \mathcal{J}(Z(P_i)) = P_i$ and $(*)$ corresponds to a chain of prime ideals of length n $P = P_n \subsetneq P_{n-1} \subsetneq \cdots \subsetneq P_0$. Thus $\dim Y \leq \dim A(Y)$. For the other inclusion let $P = Q_m \subsetneq Q_{m-1} \subsetneq \cdots \subsetneq Q_0$ be a chain of prime ideals in $K[x_1, -, x_n]$. By Hilbert's Nullstellensatz $Z(Q_i) \neq Z(Q_j)$ for $i \neq j$. This yields a chain $Z(Q_0) \subsetneq Z(Q_1) \subsetneq \cdots \subsetneq Z(Q_m) = Y$ of irreducible closed subsets of length m. Thus $\dim A(Y) \leq \dim Y$.

Case 2: $Y \subseteq \mathbb{A}^n_K$ an arbitrary affine variety. Then $Y = Y_1 \cup \ldots \cup Y_r$ where $Y_i \subseteq \mathbb{A}^n_K$ are irreducible affine varieties. Then $Y_i = Z(P_i)$ for some prime ideals $P_i \subseteq K[x_1, -, x_n]$ and $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup \ldots \cup Y_r) = \bigcap_{i=1}^r \mathcal{J}(Y_i) = \bigcap_{i=1}^r P_i$. In particular, the set $\{P_1, -, P_r\}$ contains all prime ideals which are minimal over $\mathcal{J}(Y)$. Thus $\dim A(Y) = \sup\{\dim P_i \mid 1 \leq i \leq r\} = \sup\{\dim A(Y_i) \mid 1 \leq i \leq r\} = \sup\{\dim Y_i \mid 1 \leq i \leq r\} = \dim Y$.

(3.44) Corollary: Let K be an algebraically closed field and $Y \subseteq \mathbb{A}^n_K$ an irreducible affine variety. Then

(a) $\dim Y = \dim A(Y) = \operatorname{trdeg}_K(A(Y))$

(b) $\dim Y + \operatorname{codim}_{\mathbb{A}^n}(Y) = n.$

Proof: Homework

§3: THE SPECTRUM OF A RING

(3.45) Definition: Let R be a ring. Define:

(a) $\mathrm{Spec}(R) := \{P \subseteq R \mid P \text{ a prime ideal}\}$, the spectrum of R.

(b) $\mathrm{mSpec}(R) := \{m \subseteq R \mid m \text{ a maximal ideal}\}$, the maximal spectrum of R

(c) Let $I \subseteq R$ be an ideal and $X = \mathrm{Spec}(R)$ or $X = \mathrm{mSpec}(R)$. The set $V(I) = \{P \in X \mid I \subseteq P\}$ is called the variety of I.

In the following R is a ring and $X = \mathrm{Spec}(R)$ or $X = \mathrm{mSpec}(R)$.

(3.46) Remark: (a) If $I_1, I_2 \subseteq R$ are ideals, then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.

(b) If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of R, then $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$.

(c) $V(R) = \emptyset$ and $V(0) = X$.

The topology on X defined by: $Y \subseteq X$ closed if and only if $Y = V(I)$ for some ideal $I \subseteq R$ is called the Zariski topology on X. Note that the Zariski topology on $\mathrm{mSpec}(R)$ is induced by the Zariski topology on $\mathrm{Spec}(R)$.

(3.47) Definition: Let $T \subseteq X$ be a subset. The ideal $\mathcal{J}(T) = \bigcap_{P \in T} P$ if $T \neq \emptyset$ and $\mathcal{J}(T) = R$ if $T = \emptyset$ is called the (vanishing) ideal of T.

(3.48) Proposition: (a) For any subset $T \subseteq X$: $V(\mathcal{J}(T)) = \overline{T}$ where $\overline{T}$ is the closure of T in X.

(b) Let $X = \mathrm{Spec}(R)$ and $I \subseteq R$ an ideal. Then $\mathcal{J}(V(I)) = \mathrm{rad}(I)$. There is a one-to-one correspondence between the closed subsets of $\mathrm{Spec}(R)$ and the reduced ideals of R.

Proof: (a) Since $T \subseteq V(\mathcal{J}(T))$, $\overline{T} \subseteq V(\mathcal{J}(T))$. Let $V(I)$ be a closed subset of X with $T \subseteq V(I)$. Then $I \subseteq \bigcap_{P \in T} P = \mathcal{J}(T)$ and $V(\mathcal{J}(T)) \subseteq V(I)$. Hence $\overline{T} \supseteq V(\mathcal{J}(T))$.

(b) $\mathcal{J}(V(I)) = \bigcap_{P \in V(I)} P = \bigcap_{P \text{ prime};\, I \subseteq P} P = \mathrm{rad}(I)$

(3.49) **Proposition:** If R is a Noetherian ring, $\mathrm{Spec}(R)$ and $\mathrm{mSpec}(R)$ are Noetherian topological spaces.

Proof: The topology on $\mathrm{mSpec}(R)$ is induced by the topology of $\mathrm{Spec}(R)$. Thus it suffices to show that $\mathrm{Spec}(R)$ is a Noetherian space. Let $V(I_1) \supseteq V(I_2) \supseteq \dots$ be a descending chain of closed subsets of X. Then $\mathcal{J}(V(I_1)) \subseteq \mathcal{J}(V(I_2)) \subseteq \dots$ is an ascending chain of ideals in R. There is an $n \in \mathbb{N}$ with $\mathcal{J}(V(I_n)) = \mathcal{J}(V(I_{n+k}))$ for all $k \in \mathbb{N}$. Therefore $V(\mathcal{J}(V(I_n))) = V(I_n) = V(I_{n+k}) = V(\mathcal{J}(V(I_{n+k})))$ for all $k \in \mathbb{N}$.

(3.50) **Proposition:** Let $Y \subseteq X$ be a nonempty closed subset. Y is irreducible if and only if $\mathcal{J}(Y)$ is a prime ideal of R.

Proof: "$\Longrightarrow$": Suppose that Y is irreducible and let $fg \in \mathcal{J}(Y)$. Then $Y = V(\mathcal{J}(Y)) \subseteq V(fg)$. Since $V(fg) = V(f) \cup V(g)$, $Y = (Y \cap V(f)) \cup (Y \cap V(g))$. Thus $Y \subseteq V(f)$ or $Y \subseteq V(g)$ and $f \in \mathcal{J}(Y)$ or $g \in \mathcal{J}(Y)$.

"$\Longleftarrow$": Suppose that $\mathcal{J}(Y)$ is a prime ideal. Let $Y = Y_1 \cup Y_2$ with Y_1 and Y_2 closed subsets of X. Then $\mathcal{J}(Y) \subseteq \mathcal{J}(Y_1)$, $\mathcal{J}(Y) \subseteq \mathcal{J}(Y_2)$, and $\mathcal{J}(Y) = \mathcal{J}(Y_1 \cup Y_2) = \mathcal{J}(Y_1) \cap \mathcal{J}(Y_2)$. Since $\mathcal{J}(Y)$ is prime, $\mathcal{J}(Y_1) \subseteq \mathcal{J}(Y)$ or $\mathcal{J}(Y_2) \subseteq \mathcal{J}(Y)$. Hence $Y = Y_1$ or $Y = Y_2$.

(3.51) **Corollary:** Let R be a ring. Then $\dim R = \dim \mathrm{Spec}(R)$.

(3.52) **Corollary:** Let R be a ring with the property that every prime ideal of R is the intersection of maximal ideals. Then $\dim R = \dim \mathrm{mSpec}(R)$.

Proof: In this case for every ideal $I \subseteq R$: $\mathcal{J}(V(I)) = \bigcap_{P \text{ max};\, I \subseteq P} P = \mathrm{rad}(I)$.

(3.53) **Remark:** Let K be a field and $R = K[x_1, \dots, x_n]$ the polynomial ring over K. By (3.5) every prime ideal of R is the intersection of maximal ideals and by (3.52)

$\dim R = n = \dim \mathrm{mSpec}(R) = \dim \mathrm{Spec}(R)$. If K is algebraically closed there is a homeomorphism of topological spaces:

$$\begin{array}{ccc} \mathbb{A}^n_K & \xrightarrow{\sim} & \mathrm{mSpec}(R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (a_1, -, a_n) & \longmapsto & (x_1 - a_1, -, x_n - a_n). \end{array}$$

Similarly, for an affine variety $Y = Z(I) \subseteq \mathbb{A}^n_K$ the topological spaces Y and $\mathrm{mSpec}(R/\mathrm{rad}\, I) = \mathrm{mSpec}(A(Y))$ are homeomorphic via $(a_1, -, a_n) \mapsto (x_1 - a_1, -, x_n - a_n)$ and $\dim Y = \dim \mathrm{mSpec}(A(Y)) = \dim \mathrm{Spec}(A(Y)) = \dim A(Y)$.