

CHAPTER II: PRIMARY DECOMPOSITION

§1: ASS_R(M) AND SUPP_R(M)

Recall from Chapter I: Let M be an R -module. The annihilator of M is the set $\text{ann}_R(M) = \{a \in R \mid am = 0 \ \forall m \in M\}$. For an element $m \in M$ define the annihilator of m by $\text{ann}_R(m) = \{a \in R \mid am = 0\} = 0 :_R Rm$. $\text{ann}_R(M)$ and $\text{ann}_R(m)$ are ideals of R .

(2.1) Remark: Let M be an R -module. Then $\text{NZD}(R) = R - \bigcup_{m \in M - \{0\}} \text{ann}_R(m)$.

(2.2) Proposition: The maximal elements of the set $\Gamma := \{\text{ann}_R(m) \mid m \in M - \{0\}\}$ are prime ideals of R . (Note that Γ may not contain a maximal element.)

Proof: Let $P = \text{ann}_R(m)$ be a maximal element of Γ and $a, b \in R$ with $ab \in P$, $b \notin P$. Then $bm \neq 0$ and $I = \text{ann}_R(bm) \in \Gamma$. Since $P \subseteq I$, by the maximality of P : $P = I$ and $a \in P$.

(2.3) Definition: Let M be an R -module. A prime ideal $P \in R$ for which there is an $m \in M$ with $P = \text{ann}_R(m)$ is called an associated prime ideal of M . The set of the associated prime ideals of M is denoted by $\text{Ass}_R(M) = \{P \in \text{Spec}(R) \mid P \text{ an associated prime ideal of } M\}$.

(2.4) Remark: (a) Let M be a R -module and $m \in M - \{0\}$. Then $R \cdot m \cong R/\text{ann}_R(m)$ as R -modules. Thus $P \in \text{Ass}_R(M)$ if and only if there is an injective R -linear map: $R/P \rightarrow M$.

(b) For every prime ideal $P \in \text{Spec}(R)$: $\text{Ass}_R(R/P) = \{P\}$.

(2.5) Lemma: Let R be a Noetherian ring and M an R -module. Then $M = \{0\} \iff \text{Ass}_R(M) = \emptyset$

Proof: Suppose that $M \neq (0)$. Then $\Gamma = \{\text{ann}_R(m) \mid m \in M - (0)\}$ is a nonempty set of ideals of R . Since R is Noetherian Γ has a maximal element which is prime by (2.2).

(2.6) Lemma: Let R be a Noetherian ring and M an R -module. Then

$$\text{ZD}(M) = \bigcup_{P \in \text{Ass}_R(M)} P.$$

Proof: "z": By definition

" \subseteq ": Let $a \in \text{ZD}(M)$. Then $a \in \text{ann}_R(m)$ for some $m \in M - (0)$. Since R is Noetherian there is a maximal element $\text{ann}_R(x) \in \Gamma$ with $\text{ann}_R(m) \subseteq \text{ann}_R(x)$. Then $\text{ann}_R(x) \in \text{Ass}_R(M)$.

(2.7) Lemma: Let R be a Noetherian ring and M a nonzero finite R -module. There is a normal series of M : $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$ such that for all $0 \leq i \leq n-1$: $M_{i+1}/M_i \cong R/P_i$ for some prime ideal $P_i \subseteq R$.

Proof: By (2.5) there is a prime ideal $P \in \text{Ass}_R(M) \neq \emptyset$. P corresponds to an element $m \in M - (0)$ with $\text{ann}_R(m) = P$. Set $M_1 = Rm \cong R/P$ and $P = P_0$. If $M/M_1 \neq (0)$ replace M by M/M_1 and repeat the argument. This produces an R -submodule \bar{M}_2 of M/M_1 with $\bar{M}_2 \cong R/P_1$ for some prime ideal $P_1 \subseteq R$. Let M_2 be the contraction of \bar{M}_2 to M . Then $M_1 \subsetneq M_2$ and $\bar{M}_2 = M_2/M_1 \cong R/P_1$. This produces an ascending chain of submodules $(0) = M_0 \subsetneq M_1 \subsetneq \dots$ with factor module $M_{i+1}/M_i \cong R/P_i$ for some $P_i \in \text{Spec}(R)$. Since R is Noetherian and M finite, after n steps: $M_n = M$.

(2.8) Remarks: The normal series of (2.7) is not uniquely determined by M . For example, if $M = \mathbb{Z}$ the series: $(0) \subseteq \mathbb{Z}$ and $(0) \subseteq (6) \subseteq (3) \subseteq \mathbb{Z}$ satisfy the conditions of (2.7).

(2.9) Theorem: Let M be an R -module and $N \subseteq M$ a submodule. Then

$$\text{Ass}_R(N) \subseteq \text{Ass}_R(M) \subseteq \text{Ass}_R(N) \cup \text{Ass}_R(M/N).$$

Proof: The first inclusion $\text{Ass}_R(N) \subseteq \text{Ass}_R(M)$ is obvious. If $P \in \text{Ass}_R(N)$ then there is an injective R -linear map: $R/P \rightarrow N \subseteq M$ and $P \in \text{Ass}_R(M)$.

For the second inclusion let $P \in \text{Ass}_R(M)$ and $m \in M - (0)$ with $P = \text{ann}_R(m)$. If $m \in N$ then $P \in \text{Ass}_R(N)$. Suppose $m \notin N$ and consider $\bar{m} = m + N \in M/N - (0)$. Then $P \subseteq \text{ann}_R(\bar{m})$. If $P = \text{ann}_R(\bar{m})$ then $P \in \text{Ass}_R(M/N)$ and we are done.

If $P \subsetneq \text{ann}_R(\bar{m})$, there is an $a \in \text{ann}_R(\bar{m}) - P$ and $a\bar{m} = 0$. Hence $am \in N$. We claim that $\text{ann}_R(m) = \text{ann}_R(am)$. Obviously, $\text{ann}_R(m) \subseteq \text{ann}_R(am)$. Let $b \in \text{ann}_R(am)$. Then $bam = 0$ and $ab \in \text{ann}_R(m) = P$. Since P is prime and $a \notin P$: $b \in P$. Thus $P = \text{ann}_R(am)$ with $am \in N$ and therefore $P \in \text{Ass}_R(N)$.

(2.10) Corollary: Let M be an R -module. Suppose that M has a normal series $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$ with factor modules $M_{i+1}/M_i \cong R/P_i$ for $P_i \in \text{Spec}(R)$ for all $0 \leq i \leq n-1$. Then $\text{Ass}_R(M) \subseteq \{P_0, \dots, P_{n-1}\}$.

Proof: By induction on the length n of the normal series. If $n=1$ then $0 = M_0 \subsetneq M_1 = M$ and $M \cong R/P_0$. Thus $\text{Ass}_R(M) = \{P_0\}$. $n-1 \Rightarrow n$: By induction hypothesis:

$\text{Ass}_R(M_{n-1}) \subseteq \{P_0, \dots, P_{n-1}\}$ and by (2.9) $\text{Ass}_R(M) \subseteq \text{Ass}_R(M_{n-1}) \cup \text{Ass}_R(M/M_{n-1}) \subseteq \{P_0, \dots, P_{n-2}\} \cup \{P_{n-1}\} = \{P_0, \dots, P_{n-1}\}$.

(2.11) Corollary: Let R be a Noetherian ring and M a finite R -module. Then $|\text{Ass}_R(M)| < \infty$.

Proof: (2.7) and (2.10).

(2.12) Theorem: Let R be a Noetherian ring, $S \subseteq R$ a multiplicative set, and M an R -module. Then $\text{Ass}_{S^{-1}R}(S^{-1}M) = \{S^{-1}P \mid P \in \text{Ass}_R(M) \text{ and } P \cap S = \emptyset\}$.

Proof: " \supseteq ": Let $P \in \text{Ass}_R(M)$ with $P \cap S = \emptyset$. Then $P = \text{ann}_R(m)$ for some $m \in M$.

Consider $Q = \text{ann}_{S^{-1}R}(m/1) \in S^{-1}R$. Obviously, $S^{-1}P \subseteq Q$. If $(a/s)(m/1) = 0$, then there is a $t \in S$ with $tam = 0$. Thus $ta \in P$. Since P is prime and $t \notin P$, $a \in P$ and $a/s \in S^{-1}P$.

" \subseteq ": Let $Q \in \text{Ass}_{S^{-1}R}(M)$. Then $Q = \text{ann}_{S^{-1}R}(m/s)$ for some $m/s \in S^{-1}M$. Moreover, $Q = S^{-1}P$ for some $P \in \text{Spec}(R)$. Since R is Noetherian, the ideal P is finitely generated. Suppose $P = (p_1, \dots, p_r)$. Then $(p_i/1)(m/s) = 0$ for all $1 \leq i \leq r$ and there is a $t \in S$ with $tp_i m = 0$ for all $1 \leq i \leq r$. Since $m/s \neq 0$, $tm \neq 0$ and $P \subseteq \text{ann}_R(tm) \neq R$.

On the other hand, if $atm = 0$ for some $a \in R$, then $(a/1)(m/s) = 0$ and $a/1 \in Q = \text{ann}_{S^{-1}R}(m/s)$. Hence $a \in P$ and $P = \text{ann}_R(tm) \in \text{Ass}_R(M)$.

(2.13) Definition: Let M be an R -module. The support of M is the set of prime ideals:

$$\text{Supp}_R(M) = \{P \in R \mid P \text{ a prime ideal and } M_P \neq 0\} \subseteq \text{Spec}(R).$$

(2.14) Definition: Let R be a ring and $I \subseteq R$ an ideal. The set of prime ideals

$$V(I) = \{P \in \text{Spec}(R) \mid I \subseteq P\}$$

is called the variety of I .

(2.15) Proposition: Let M be a finite R -module. Then

$$\text{Supp}_R(M) = \{P \in R \mid P \text{ a prime ideal and } \text{ann}_R(M) \subseteq P\} = V(\text{ann}_R(M)).$$

Proof: " \subseteq ": Let $P \in \text{Spec}(R)$ with $\text{ann}_R(M) \not\subseteq P$. Then there is an element $t \in \text{ann}_R(M) - P$.

Hence $tm = 0$ for all $m \in M$ and $M_P = 0$.

" \supseteq ": Suppose that $M = Rm_1 + \dots + Rm_n$. Then $\text{ann}_R(M) = \bigcap_{i=1}^n \text{ann}_R(m_i) \supseteq$

$\bigcap_{i=1}^n \text{ann}_R(m_i)$. If $P \in \text{Spec}(R)$ with $\text{ann}_R(M) \subseteq P$, then for some $1 \leq i \leq n$:

$\text{ann}_R(m_i) \subseteq P$. This implies that $m_i/1 \neq 0$ in M_P . Therefore $M_P \neq 0$ and $P \in \text{Supp}_R(M)$.

(2.16) Proposition: Let R be a Noetherian ring and M an R -module. Suppose that M has

a normal series: $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ so that for all $0 \leq i \leq n-1$:

$M_{i+1}/M_i \cong R/P_i$ for some $P_i \in \text{Spec}(R)$. Then $\text{Ass}_R(M) \subseteq \{P_0, \dots, P_{n-1}\} \subseteq \text{Supp}_R(M)$.

Moreover, all three sets have the same minimal elements.

Proof: By (2.10) $\text{Ass}_R(M) \subseteq \{P_0, \dots, P_{n-1}\}$. Since $0 \neq (R/P_i)_{P_i} \cong (M_{i+1}/M_i)_{P_i} \cong (M_{i+1})_{P_i}/(M_i)_{P_i}$, $0 \neq (M_{i+1})_{P_i} \subseteq M_{P_i}$ and $P_i \in \text{Supp}_R(M)$.

It remains to show that the three sets have the same minimal elements. It suffices to show that every minimal element of $\text{Supp}_R(M)$ is contained in $\text{Ass}_R(M)$.

Let $P \in \text{Supp}_R(M)$ be minimal. Every prime ideal $Q \in \text{Spec}(R_P)$ is of the form $Q = WR_P$ for some prime ideal $W \subseteq P$ and the localization $(M_P)_Q$ is isomorphic to the localization M_W . By the minimality of P : $(M_P)_Q = 0$ for all $Q \in \text{Spec}(R_P)$ with $Q \neq PR_P$. Hence $\text{Supp}_{R_P}(M_P) = \{PR_P\}$. Since M has a normal series with factors isomorphic to R/P_i for some $P_i \in \text{Spec}(R)$, the R_P -module M_P has a normal series with factors isomorphic to R_P/Q_j for some $Q_j \in \text{Spec}(R_P)$. By the first part of the proposition $\text{Ass}_{R_P}(M_P) \subseteq \text{Supp}_{R_P}(M_P)$. Since $M_P \neq 0$ and R_P Noetherian, by (2.5) $\text{Ass}_{R_P}(M_P) \neq \emptyset$ and hence $\text{Ass}_{R_P}(M_P) = \{PR_P\}$. By (2.12) $P \in \text{Ass}_R(M)$.

(2.17) Corollary: Let R be a Noetherian ring and $I \subseteq R$ an ideal. There are only finitely many minimal prime ideals P_1, \dots, P_r containing I . In particular, R has only finitely many minimal prime ideals.

Proof: Apply (2.16) to the R -module $M = R/I$. Then $\text{Supp}_R(R/I) = V(I)$.

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(2.18) Lemma: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module. The following are equivalent:

- (a) There is an $m \in M - (0)$ with $I m = 0$.
- (b) For all $a \in I$ there is an $m \in M - (0)$ with $a m = 0$.
- (c) There is a prime ideal $P \in \text{Ass}_R(M)$ with $I \subseteq P$.

Proof: (b) \Rightarrow (c): Since $I \subseteq \text{ZD}(M)$, by (2.6) $I \subseteq \bigcup_{P \in \text{Ass}_R(M)} P$. By (2.11) the set $\text{Ass}_R(M)$ is finite. Thus $I \subseteq P$ for some $P \in \text{Ass}_R(M)$.

(c) \Rightarrow (a): Let $P \in \text{Ass}_R(M)$ with $I \subseteq P$. Then $P = \text{ann}_R(m)$ for some $m \in M - (0)$.

(2.19) Lemma: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finite R -module. The following are equivalent:

- (a) For some integer $n > 0$: $I^n M = 0$ and $M \neq 0$.
- (b) $I \subseteq \bigcap_{P \in \text{Ass}_R(M)} P$
- (c) $\text{Supp}_R(M) \subseteq V(I)$.

Proof: (a) \Leftrightarrow (b): By (2.11) $\text{Ass}_R(M) \neq \emptyset$ is a finite set. Moreover, by (2.16) the minimal prime ideals of $\text{Ass}_R(M)$ and $\text{Supp}_R(M)$ are identical. Therefore:

$$\bigcap_{P \in \text{Ass}_R(M)} P = \bigcap_{P \in \text{Ass}_R(M)} \text{min } P = \bigcap_{P \in \text{Supp}_R(M)} \text{min } P = \bigcap_{P \in \text{Supp}_R(M)} P = K.$$

By (2.15) $\text{Supp}_R(M) = V(\text{ann}_R(M))$ and hence $\text{rad}(\text{ann}_R(M)) = K$. Since I is finitely generated: (b) $\Leftrightarrow I \subseteq K = \text{rad}(\text{ann}_R(M)) \Leftrightarrow I^n \subseteq \text{ann}_R(M)$ for some $n \in \mathbb{N} \Leftrightarrow I^n M = 0 \Leftrightarrow$ (a).

(c) \Leftrightarrow (a): Since $\text{Supp}_R(M) = V(\text{ann}_R(M))$: (c) $\Leftrightarrow V(\text{ann}_R(M)) \subseteq V(I) \Leftrightarrow \text{rad}(I) \subseteq \text{rad}(\text{ann}_R(M)) \Leftrightarrow I^n \subseteq \text{ann}_R(M)$ for some $n \in \mathbb{N}$, since I is finitely generated.

(2.20) Lemma: Let R be a Noetherian ring, $a \in R$, and M a finite R -module. The following are equivalent:

- (a) The R -linear map $t_a: M \rightarrow M$ with $t_a(m) = am$ for all $m \in M$ is nilpotent.
 (b) $a \in \bigcap_{P \in \text{Ass}_R(M)} P$.

Proof: (a) \Rightarrow (b): Since M is finite and t_a nilpotent, there is an $n \in \mathbb{N}$ with $a^n m = 0$ for all $m \in M$. Thus $a^n \in \text{ann}_R(M) \subseteq \text{rad}(\text{ann}_R(M)) = \bigcap_{P \in \text{Ass}_R(M)} P$.

(b) \Rightarrow (a): If $a \in \bigcap_{P \in \text{Ass}_R(M)} P = \text{rad}(\text{ann}_R(M))$, then $a^n \in \text{ann}_R(M)$ for some $n \in \mathbb{N}$.

(2.21) Proposition: Let R be a Noetherian ring and M a finite R -module. The following are equivalent:

- (a) For all $a \in R$ the R -linear map $t_a: M \rightarrow M$ is either injective or nilpotent.
 (b) $|\text{Ass}_R(M)| = 1$.

Proof: (a) $\Leftrightarrow \forall a \in R: a \in \bigcap_{P \in \text{Ass}_R(M)} P$ or $a \in \text{NZD}(M) \stackrel{(2.6)}{=} R - \bigcup_{P \in \text{Ass}_R(M)} P \Leftrightarrow$
 $R = (\bigcap_{P \in \text{Ass}_R(M)} P) \cup (R - \bigcup_{P \in \text{Ass}_R(M)} P) \Leftrightarrow \bigcup_{P \in \text{Ass}_R(M)} P \subseteq \bigcap_{P \in \text{Ass}_R(M)} P \Leftrightarrow$
 $|\text{Ass}_R(M)| = 1$.

(2.22) Remark: If (a) or (b) of (2.21) holds then $\text{Ass}_R(M) = \{P\}$ with $P = \{a \in R \mid t_a \text{ nilpotent}\}$.

(By (2.6): $\text{ZD}(M) = \bigcup_{P \in \text{Ass}_R(M)} P$.)

(2.23) Definition: Let R be a Noetherian ring, M a finite R -module, and $N \subseteq M$ a submodule.

(a) N is called primary in M if $|\text{Ass}_R(M/N)| = 1$. More precisely, if $\text{Ass}_R(M/N) = \{P\}$ then N is called P -primary in M .

(b) N is called an irreducible submodule of M if $N \neq M$ and whenever $N = N_1 \cap N_2$ with $N_1, N_2 \subseteq M$ submodules, then $N = N_1$ or $N = N_2$. N is called reducible if N is not irreducible.

(2.24) Proposition: Let R be a Noetherian ring.

- (a) A prime ideal $P \subseteq R$ is P -primary.

(b) Let $P \subseteq R$ be a prime ideal and $Q \subseteq R$ a P -primary ideal. Then there is an $n \in \mathbb{N}$ with $P^n \subseteq Q$ and $Q \subseteq P$.

(c) An ideal $Q \subseteq R$ is primary if and only if for all $a, b \in R$ with $ab \in Q$ and $a \notin Q$ there is an $n \in \mathbb{N}$ with $b^n \in Q$.

(d) If $Q \subseteq R$ is primary, then Q is P -primary where $P = \text{rad}(Q)$.

Proof: (a) By (2.4) $\text{Ass}_R(R/P) = \{P\}$.

(b) Let $Q \subseteq R$ be a P -primary ideal. Then $\text{Ass}_R(R/Q) = \{P\}$ and by (2.16) P is the only minimal prime ideal containing Q . Thus $\text{rad}(Q) = P$. Since P is finitely generated there is an $n \in \mathbb{N}$ with $P^n \subseteq Q$.

(c) " \rightarrow ": Suppose that $Q \subseteq R$ is primary and let $a, b \in R$ with $ab \in Q$ and $a \notin Q$. The map $t_b: R/Q \rightarrow R/Q$ is either nilpotent or injective. Since $t_b(\bar{a}) = \overline{ab} = 0$ and $\bar{a} \neq 0$ in R/Q , t_b is nilpotent and $b^n \in Q$ for some $n \in \mathbb{N}$.

" \leftarrow ": Let $b \in R$ so that $t_b: R/Q \rightarrow R/Q$ is not injective. Then there is an $a \in R - Q$ with $t_b(\bar{a}) = 0$, or equivalently, $ab \in Q$. By assumption $b^n \in Q$ for some $n \in \mathbb{N}$ and t_b is nilpotent.

(d) follows from (b).

(2.25) Remark: If $P \subseteq R$ is a prime ideal and $Q \subseteq P$ an ideal with $\text{rad}(Q) = P$, then in general Q is not P -primary. In particular, the powers of P are not P -primary (in general). An exception is the case where $P = \mathfrak{m} \subseteq R$ is maximal.

(2.26) Lemma: Let R be a Noetherian ring, $\mathfrak{m} \subseteq R$ a maximal ideal, and $Q \subseteq R$ an ideal with $\text{rad}(Q) = \mathfrak{m}$. Then Q is \mathfrak{m} -primary.

Proof: $\text{Ass}_R(R/Q) = \{\mathfrak{m}\}$.

(2.27) Theorem: Let R be a Noetherian ring, M an R -module and $\Phi \in \text{Ass}_R(M)$ a

subset. Then there is a submodule $N \subseteq M$ with $\text{Ass}_R(M/N) = \bar{\Phi}$ and $\text{Ass}_R(N) = \text{Ass}_R(M) - \bar{\Phi}$.

Proof: Consider the set $\mathcal{M} = \{L \subseteq M \mid L \text{ a submodule and } \text{Ass}_R(L) \subseteq \text{Ass}_R(M) - \bar{\Phi}\}$. \mathcal{M} is partially ordered by inclusion and since $\text{Ass}_R(0) = \emptyset$, $(0) \in \mathcal{M}$ and $\mathcal{M} \neq \emptyset$. Let $\mathcal{K} \subseteq \mathcal{M}$ be a chain. Then $L_0 = \bigcup_{L \in \mathcal{K}} L$ is a submodule of M . If $P \in \text{Ass}_R(L_0)$ then $P = \text{ann}_R(m)$ for some $m \in L_0$. Since $m \in L$ for some $L \in \mathcal{K}$, $P \in \text{Ass}_R(L)$ and therefore $P \in \text{Ass}_R(M) - \bar{\Phi}$. Hence $L_0 \in \mathcal{M}$ and \mathcal{M} is inductively ordered. By Zorn's Lemma there is a maximal element $N \in \mathcal{M}$.

We claim that $\text{Ass}_R(M/N) = \bar{\Phi}$. Let $P \in \text{Ass}_R(M/N)$. Then there is an injective R -linear map $\varphi: R/P \rightarrow M/N$. Its image corresponds to a submodule $N' \subseteq M$ with $N \subseteq N'$ and $N'/N \cong R/P$. Thus $\text{Ass}_R(N'/N) = \{P\}$ and by (2.9) $\text{Ass}_R(N') \subseteq \text{Ass}_R(N) \cup \{P\}$. Since N is maximal in \mathcal{M} , $N' \notin \mathcal{M}$, and $P \in \bar{\Phi}$. This shows that $\text{Ass}_R(M/N) \subseteq \bar{\Phi}$ and by construction $\text{Ass}_R(N) \subseteq \text{Ass}_R(M) - \bar{\Phi}$. By (2.9) $\text{Ass}_R(M) \subseteq \text{Ass}_R(N) \cup \text{Ass}_R(M/N)$. Hence $\text{Ass}_R(N) = \text{Ass}_R(M) - \bar{\Phi}$ and $\text{Ass}_R(M/N) = \bar{\Phi}$.

(2.28) Definition: Let R be a Noetherian ring, M an R -module, and $N \subseteq M$ a submodule. A finite family of primary submodules $\{Q_1, \dots, Q_n\}$ with $N = \bigcap_{i=1}^n Q_i$ is called a primary decomposition of N in M .

(2.29) Example: Let $R = M = \mathbb{Z}$ and $N = (n)$ for some $n \in \mathbb{Z} - \{0\}$. The integer $n \neq \pm 1$ has a prime decomposition $n = (\pm 1) p_1^{\alpha_1} \dots p_r^{\alpha_r}$ where p_i are distinct positive prime numbers. Then $(n) = (p_1)^{\alpha_1} \dots (p_r)^{\alpha_r}$ is a primary decomposition of the ideal (n) in \mathbb{Z} .

(2.30) Theorem: (Existence of primary decompositions) Let R be a Noetherian ring, M a finite R -module, and $N \subseteq M$ a submodule of M . There are finitely many primary submodules Q_1, \dots, Q_r of M such that:

(a) $N = Q_1 \cap \dots \cap Q_r$

(b) For all $1 \leq i \leq r$ Q_i is P_i -primary.

$$(c) \text{Ass}_R(M/N) = \{P_1, \dots, P_r\}.$$

Proof: $N = \bigcap_{i=1}^r Q_i$ is a primary decomposition of N in M if and only if $(0) = \bigcap_{i=1}^r Q_i/N$ is a primary decomposition of (0) in M/N . Thus we may assume that $N = (0)$ and $\text{Ass}_R(M) = \{P_1, \dots, P_r\}$. By (2.27) there are submodules Q_i of M so that $\text{Ass}_R(M/Q_i) = \{P_i\}$ and $\text{Ass}_R(Q_i) = \{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_r\}$. With $L = Q_1 \cap \dots \cap Q_r$ for all $1 \leq i \leq r$, $\text{Ass}_R(L) \subseteq \text{Ass}_R(Q_i)$. Since $\bigcap_{i=1}^r \text{Ass}_R(Q_i) = \emptyset$, $L = (0)$ and $(0) = Q_1 \cap \dots \cap Q_r$.

(2.31) Lemma: Let R be a ring and M_1, \dots, M_r R -modules. Then

$$\text{Ass}_R\left(\bigoplus_{i=1}^r M_i\right) = \bigcup_{i=1}^r \text{Ass}_R(M_i).$$

Proof: By induction on r . For the induction step consider the exact sequences:

$$0 \rightarrow M_r \rightarrow \bigoplus_{i=1}^r M_i \rightarrow \bigoplus_{i=1}^{r-1} M_i \rightarrow 0$$

$$\text{and } 0 \rightarrow \bigoplus_{i=1}^{r-1} M_i \rightarrow \bigoplus_{i=1}^r M_i \rightarrow M_r \rightarrow 0$$

Then $\text{Ass}_R\left(\bigoplus_{i=1}^r M_i\right) \subseteq \text{Ass}_R\left(\bigoplus_{i=1}^{r-1} M_i\right) \cup \text{Ass}_R(M_r)$ and $\text{Ass}_R\left(\bigoplus_{i=1}^{r-1} M_i\right) \subseteq \text{Ass}_R\left(\bigoplus_{i=1}^r M_i\right)$.

Moreover, $\text{Ass}_R(M_r) \subseteq \text{Ass}_R\left(\bigoplus_{i=1}^r M_i\right)$ and by induction hypothesis:

$$\text{Ass}_R\left(\bigoplus_{i=1}^r M_i\right) = \bigcup_{i=1}^r \text{Ass}_R(M_i).$$

(2.32) Proposition: Let R, M , and N be as in (2.30). Suppose that $N = Q_1 \cap \dots \cap Q_r$ is a primary decomposition of N in M with Q_i P_i -primary. Then $\text{Ass}_R(M/N) \subseteq \{P_1, \dots, P_r\}$.

Proof: The R -linear map $\varphi: M/N \rightarrow \bigoplus_{i=1}^r M/Q_i$ defined by $\varphi(m+N) = (m+Q_1, \dots, m+Q_r)$ is injective. Thus $\text{Ass}_R(M/N) \subseteq \text{Ass}_R\left(\bigoplus_{i=1}^r M/Q_i\right) = \bigcup_{i=1}^r \text{Ass}_R(M/Q_i) = \{P_1, \dots, P_r\}$.

(2.33) Remark: Primary decompositions are not unique. For example, let R be a Noetherian ring and $P \subsetneq Q$ prime ideals of R . P and $P \cap Q$ are primary decompositions of P .

(2.34) Lemma: Let R be a Noetherian ring, M a finite R -module and $Q_1, \dots, Q_n \subseteq M$

P -primary submodules. Then $Q = Q_1 \cap \dots \cap Q_n$ is a P -primary submodule of M .

Proof: Consider the injective R -linear map $M/Q \rightarrow \bigoplus_{i=1}^n M/Q_i$ from (2.32). Then $\emptyset \neq \text{Ass}_R(M/Q) \subseteq \bigcup_{i=1}^n \text{Ass}_R(M/Q_i) = \{P\}$.

(2.35) Definition: Let R be a Noetherian ring, M a finite R -module and $N \subseteq M$ a submodule.

A primary decomposition $N = \bigcap_{i=1}^n Q_i$ of N in M is called a shortest primary decomposition if:

(a) Q_i is P_i -primary and $P_i \neq P_j$ whenever $i \neq j$.

(b) For all $1 \leq i \leq n$: $\bigcap_{j=1, j \neq i}^n Q_j \not\subseteq Q_i$.

(2.36) Remark: Under the assumptions of (2.35) every submodule N has a shortest primary decomposition.

(2.37) Proposition: Assumptions as in (2.35). Let $N = \bigcap_{i=1}^n Q_i$ be a shortest primary decomposition of N in M . Then:

(a) $\text{Ass}_R(Q_i/N) = \bigcup_{j=1, j \neq i}^n \text{Ass}_R(M/Q_j)$ for all $1 \leq i \leq n$.

(b) $\text{Ass}_R(M/N) = \bigcup_{i=1}^n \text{Ass}_R(M/Q_i)$.

Proof: (b) By (2.32) $\text{Ass}_R(M/N) = \{P_1, \dots, P_n\}$ where Q_i is P_i -primary. For $1 \leq i \leq n$ set $L_i = \bigcap_{j=1, j \neq i}^n Q_j$. Then $L_i \cap Q_i = N$ and $L_i \neq N$ since $N = \bigcap Q_i$ is a shortest primary decomposition. Thus $(0) \neq L_i/N = L_i/L_i \cap Q_i \cong L_i + Q_i/Q_i \subseteq M/Q_i$ and hence $\text{Ass}_R(L_i/N) = \{P_i\}$. Since $L_i/N \subseteq M/N$, $P_i \in \text{Ass}_R(M/N)$.

(a) Obviously, for all $1 \leq i \leq n$: $N = \bigcap_{j=1, j \neq i}^n (Q_j \cap Q_i)$. Since there is an injective R -linear map $\varphi: Q_i/N \rightarrow \bigoplus_{j=1, j \neq i}^n (Q_i/Q_i \cap Q_j)$, $\text{Ass}_R(Q_i/N) \subseteq \bigcup_{j=1, j \neq i}^n \text{Ass}_R(Q_i/Q_i \cap Q_j)$. Since $Q_i \not\subseteq Q_j$, $(0) \neq Q_i/Q_i \cap Q_j \cong (Q_i + Q_j)/Q_j \subseteq M/Q_j$ and $\text{Ass}_R(Q_i/Q_i \cap Q_j) = \{P_j\}$.

Therefore $\text{Ass}_R(Q_i/N) \subseteq \bigcup_{j=1, j \neq i}^n \text{Ass}_R(M/Q_j)$.

For the other inclusion consider the exact sequence: $0 \rightarrow Q_i/N \rightarrow M/N \rightarrow M/Q_i \rightarrow 0$.

By (b) $\text{Ass}_R(M/N) = \{P_1, \dots, P_n\} \subseteq \text{Ass}_R(Q_i/N) \cup \text{Ass}_R(M/Q_i) = \text{Ass}_R(Q_i/N) \cup \{P_i\}$. Hence $\{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n\} \subseteq \text{Ass}_R(Q_i/N)$.

(2.38) Theorem: Let R be a Noetherian ring, M a finite R -module and $N \subseteq M$ a submodule. A primary decomposition $N = \bigcap_{i=1}^n Q_i$ is a shortest primary decomposition of N in M if and only if for all $1 \leq i \leq n$:

(a) $\text{Ass}_R(M/Q_i) \subseteq \text{Ass}_R(M/N)$

(b) $\text{Ass}_R(M/Q_i) \neq \text{Ass}_R(M/Q_j)$ if $i \neq j$.

Proof: " \Rightarrow ": By (2.37) and the definition of a shortest primary decomposition.

" \Leftarrow ": Set $\text{Ass}_R(M/Q_i) = \{P_i\}$. It remains to show that for all $1 \leq i \leq n$: $\bigcap_{j=1, j \neq i}^n Q_j \not\subseteq Q_i$.

Suppose that $\bigcap_{j=1, j \neq i}^n Q_j \subseteq Q_i$ for some $1 \leq i \leq n$ and $N = \bigcap_{j=1, j \neq i}^n Q_j$. Consider

the injective R -linear map $M/N \longrightarrow \bigoplus_{j=1, j \neq i}^n M/Q_j$. Then $\text{Ass}_R(M/N) \subseteq$

$\{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n\}$ and $\text{Ass}_R(M/Q_i) = \{P_i\} \not\subseteq \text{Ass}_R(M/N)$, a contradiction.

(2.39) Summary: Let R be a Noetherian ring, M a finite R -module, and $N \subseteq M$ a submodule. N has a shortest primary decomposition in M : $N = \bigcap_{i=1}^n Q_i$. By (2.37)

$\text{Ass}_R(M/N) = \bigcup_{i=1}^n \text{Ass}_R(M/Q_i) = \{P_1, \dots, P_n\}$ where $\text{Ass}_R(M/Q_i) = \{P_i\}$. Thus the prime

ideals P_i are uniquely determined by M and N . In general, the primary

components Q_i are not unique — at least not for all $1 \leq i \leq n$.

(2.40) Theorem: Let R be a Noetherian ring, M a finite R -module, and $N \subseteq M$ a submodule

with shortest primary decomposition $N = \bigcap_{i=1}^n Q_i$. Suppose that Q_i is P_i -primary and

let P be a minimal prime ideal in $\text{Ass}_R(M/N)$ with $P = P_j$ for some $1 \leq j \leq n$. Then

$Q_j = i_{M, P_j}^{-1}(N_{P_j})$ where $i_{M, P_j}: M \longrightarrow M_{P_j}$ is the natural map. The primary components

of N belonging to minimal prime ideals of $\text{Ass}_R(M/N)$ are unique.

Proof: Let $P \in \text{Ass}_R(M/N)$ be minimal. We may assume $P = P_i$ and set $Q = Q_i$. The shortest

primary decomposition of $N = \prod_{i=1}^n Q_i$ in M corresponds to the shortest primary decomposition of $(0) = \prod_{i=1}^n Q_i/N$ in M/N . Using the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{i_{M,P}} & M_P \\ \text{nat} \downarrow & & \downarrow \text{nat} \\ M/N & \xrightarrow{i_{M/N,P}} & (M/N)_P \cong M_P/N_P \end{array}$$

We may assume that $N=0$. By (2.37) $\text{Ass}_R(Q) = \{P_{e_1}, \dots, P_{e_n}\}$ and by (2.12) and the minimality of P : $\text{Ass}_{R_P}(M_P) = \{PR_P\}$ and $\text{Ass}_{R_P}(Q_P) = \emptyset$, since $P_i \cap (R-P) \neq \emptyset$ for all $2 \leq i \leq n$. Thus $Q_P = 0$ in M_P and $Q \subseteq i_{M,P}^{-1}(Q_P) = i_{M,P}^{-1}(0)$. Set

$U = i_{M,P}^{-1}(Q_P) = i_{M,P}^{-1}(0)$. Since $U/Q \subseteq M/Q$, $\text{Ass}_R(U/Q) \subseteq \text{Ass}_R(M/Q) = \{P\}$.

If $U/Q \neq 0$, then $\text{Ass}_R(U/Q) = \{P\}$ and by (2.12) $\text{Ass}_{R_P}(U_P/Q_P) = \{PR_P\}$,

a contradiction since $U_P = Q_P = 0$ and $\text{Ass}_{R_P}(U_P/Q_P) = \emptyset$. Thus $U = Q = i_{M,P}^{-1}(0)$.

(2.41) Definition: Assumptions as in (2.40). A prime ideal $P \in \text{Ass}_R(M/N)$ is called embedded if P is not minimal in $\text{Ass}_R(M/N)$. A primary component belonging to P is called an embedded primary component of N .