CHAPTER XI: GORENSTEIN RINGS, MATLIS DUALITY

§1: GORENSTEIN RINGS

(11.1) Definition: Let (R,m,k) be a local Artinian ring. The socle $\mathcal{S}(R)$ of R is defined by $\mathcal{S}(R) = \mathrm{ann}(m) = \{a \in R \mid ma = 0\}$.

(11.2) Remark: If (R,m,k) is a local Artinian ring, then $\mathcal{S}(R)$ is an ideal of R. Since $\mathrm{Ass}_R(R) = \{m\}$, $\mathcal{S}(R) \neq 0$ if $m \neq 0$. If $I \subseteq R$ is a nonzero ideal of R then $\mathrm{Ass}_R(I) = \{m\}$ and $I \cap \mathcal{S}(R) \neq 0$. Moreover, $\mathcal{S}(R)$ is a finite dimensional k-vector space.

(11.3) Proposition: Let (R,m,k) be a local Noetherian CM-ring of dimension d and let $x_1,\dots,x_d \in m$ and $y_1,\dots,y_d \in m$ be maximal regular sequences of R. Then:
$\dim_k(\mathcal{S}(R/(x_1,\dots,x_d))) = \dim_k(\mathcal{S}(R/(y_1,\dots,y_d)))$.

Proof: By induction on d: If $d=0$, there is nothing to show. If $d=1$, let $x,y \in R$ be regular elements of R. Then xy is regular and it suffices to show that $\dim_k(\mathcal{S}(R/(x))) = \dim_k(\mathcal{S}(R/(xy)))$. Let $a \in R-(x)$ with $ma \subseteq (x)$. Since y is regular, $ay \notin (xy)$ and $may \in (xy)$. Thus multiplication by y defines an injective k-linear map $\sigma: \mathcal{S}(R/(x)) \longrightarrow \mathcal{S}(R/(xy))$ with $\sigma(a+(x)) = ay+(xy)$. We claim that σ is surjective. Let $b \in R-(xy)$ with $mb \in (xy)$. In particular, $xb \in (xy)$ and thus $b \in (y)$ since x is regular. Hence $b = yt$ with a unique $t \in R$. Since y is regular, $mty \subseteq (xy)$ implies $mt \in (x)$ and $t+(x) \in \mathcal{S}(R/(x))$ with $\sigma(t+(x)) = yt+(xy) = b+(xy)$. σ is an isomorphism of k-vector spaces.
For the induction step suppose that the statement has been shown for local CM-rings of dimension $<d$. Let R be a local CM-ring of dimension d and let $x_1,\dots,x_d$ and $y_1,\dots,y_d$ be maximal regular sequences of R. Since x_d is regular on $R/(x_1,\dots,x_{d-1})$ and y_d is regular on $R/(y_1,\dots,y_{d-1})$, $\mathrm{Ass}_R(R/(x_1,\dots,x_{d-1})) \cup \mathrm{Ass}_R(R/(y_1,\dots,y_{d-1})) = \{P_1,\dots,P_s\}$ with $m \neq P_i$ for $1 \leq i \leq s$. Let $c \in m-(P_1 \cup \dots \cup P_s)$. Then c is regular on $R/(x_1,\dots,x_{d-1})$ and on

$R/(y_1,\dots,y_{d-1})$. In particular, $x_1,\dots,x_{d-1},c$ and $y_1,\dots,y_{d-1},c$ are regular sequences and so are $c,x_1,\dots,x_{d-1}$ and $c,y_1,\dots,y_{d-1}$. Set $\overline{R}=R/(c)$. $\overline{R}$ is a CM-ring of dimension $d-1$ with regular sequences $x_1,\dots,x_{d-1}$ and $y_1,\dots,y_{d-1}$. By induction hypothesis: $\dim_k(\mathcal{S}(\overline{R}/(x_1,\dots,x_{d-1})))=\dim_k(\mathcal{S}(\overline{R}/(y_1,\dots,y_{d-1})))$. Let $R'=R/(x_1)$. Again by induction hypothesis: $\dim_k(\mathcal{S}(R'/(x_2,\dots,x_d)))=\dim_k(\mathcal{S}(R'/(x_2,\dots,x_{d-1},c)))$. This implies:

$$\begin{aligned}\dim_k(\mathcal{S}(R/(x_1,\dots,x_d))) &= \dim_k(\mathcal{S}(R/(x_1,\dots,x_{d-1},c)))\\ &= \dim_k(\mathcal{S}(R/(y_1,\dots,y_{d-1},c)))\\ &= \dim_k(\mathcal{S}(R/(y_1,\dots,y_d)))\end{aligned}$$

where the last equality follows by a similar argument as above.

(11.4) Definition: Let (R,m,k) be a local Noetherian CM-ring. The number $r=\dim_k(\mathcal{S}(R/(x_1,\dots,x_d)))$, where $x_1,\dots,x_d$ is a SoP of R, is called the CM-type of R. R is called a Gorenstein ring if $r=1$. Any Noetherian ring R is a Gorenstein ring if R_m is Gorenstein for all maximal ideals $m\subseteq R$.

(11.5) Definition: Let R be a Noetherian ring. An ideal $I\subseteq R$ is called irreducible if for all ideals $K,J\subseteq R$ with $I=K\cap J$ it follows that $I=K$ or $I=J$.

(11.6) Lemma: Let (R,m,k) be a local Noetherian ring and $Q\subseteq R$ an m-primary ideal. Q is irreducible if and only if $\dim_k(\mathcal{S}(R/Q))=1$.

Proof: Obviously, Q is irreducible in R if and only if (0) is irreducible in R/Q. Thus we may assume that R is a local Artinian ring and have to show that $(0)\subseteq R$ is irreducible if and only if $\dim_k(\mathcal{S}(R))=1$. If $(0)=K\cap J$ is reducible with $K\neq(0)$ and $J\neq(0)$, then $K\cap\mathcal{S}(R)\neq 0$ and $J\cap\mathcal{S}(R)\neq 0$ and $\dim_k(\mathcal{S}(R))>1$. Conversely, if $\dim_k(\mathcal{S}(R))\geq 2$, let $K,J\subseteq\mathcal{S}(R)$ be two nonzero subspaces with $K\cap J=(0)$. Hence (0) is reducible.

(11.7) Definition: Let (R,m) be a local Noetherian ring. An ideal $I\subseteq R$ is called a parameter ideal if I is generated by a system of paramelers of R.

(11.8) <u>Corollary</u>: Let (R, m) be a local CM-ring. The following are equivalent:

(a) R is Gorenstein

(b) R contains an irreducible parameter ideal.

(c) Every parameter ideal of R is irreducible.

Recall (7.43): If R is a ring and M an R-module, then

$\operatorname{injdim} M \leq n \iff \operatorname{Ext}^i_R(N, M) = 0$ for all $i > n$ and every R-module N

$\iff \operatorname{Ext}^{n+1}_R(R/I, M) = 0$ for every R-ideal I.

(11.9) <u>Lemma</u>: Let R be a Noetherian ring and M an R-module. Then $\operatorname{injdim} M \leq n$ if and only if $\operatorname{Ext}^{n+1}_R(R/P, M) = 0$ for every prime ideal $P \subseteq R$.

<u>Proof</u>: "$\Leftarrow$": It suffices to show that $\operatorname{Ext}^{n+1}_R(N, M) = 0$ for every finite R-module N. Since R is Noetherian there is a decreasing chain of submodules of N: $N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_{s+1} = (0)$ so that $N_j/N_{j+1} \cong R/P_j$ for $0 \leq j \leq s$ where $P_j \subseteq R$ are prime ideals. We show by decreasing induction that $\operatorname{Ext}^{n+1}_R(N_j, M) = 0$. By assumption $\operatorname{Ext}^{n+1}_R(N_s, M) = 0$. Suppose that for some $0 < j \leq s$ $\operatorname{Ext}^{n+1}_R(N_j, M) = 0$. The exact sequence $0 \to N_j \to N_{j-1} \to N_{j-1}/N_j \to 0$ yields an exact sequence:

$$\underset{\substack{\text{is} \\ \operatorname{Ext}^{n+1}_R(R/P_{j-1}, M) = 0}}{\operatorname{Ext}^{n+1}_R(N_{j-1}/N_j, M)} \longrightarrow \operatorname{Ext}^{n+1}_R(N_{j-1}, M) \longrightarrow \underset{\substack{\| \\ 0 \text{ by ind. hyp.}}}{\operatorname{Ext}^{n+1}_R(N_j, M)}$$

Thus $\operatorname{Ext}^{n+1}_R(N_{j-1}, M) = 0$.

(11.10) <u>Lemma</u>: Let R be a Noetherian ring, M an R-module, and N a finite R-module. If $n \in \mathbb{N}$ with $\operatorname{Ext}^n_R(R/P, M) = 0$ for all $P \in \operatorname{Supp}(N)$, then $\operatorname{Ext}^n_R(N, M) = 0$.

<u>Proof</u>: Homework (use a similar argument as in (11.9))

(11.11) <u>Proposition</u>: Let (R, m) be a local Noetherian ring, M a finite R-module, and $P \in \operatorname{Spec}(R)$ with $P \neq m$. If $\operatorname{Ext}^{n+1}_R(R/Q, M) = 0$ for every prime ideal $Q \supsetneq P$, then $\operatorname{Ext}^n_R(R/P, M) = 0$.

Proof: Let $x \in m - P$. There is an exact sequence $0 \to R/P \xrightarrow{x} R/P \to N = R/(P,x) \to 0$. For every $Q \in \operatorname{Supp}(N)$ we have $P \subsetneq Q$ and thus $\operatorname{Ext}_R^{n+1}(R/Q, M) = 0$. By (11.10) $\operatorname{Ext}_R^{n+1}(N,M)=0$. From the long exact sequence we obtain: $\operatorname{Ext}_R^n(R/P, M) \xrightarrow{x} \operatorname{Ext}_R^n(R/P, M) \to \operatorname{Ext}_R^{n+1}(N,M) = 0$. Thus $\operatorname{Ext}_R^n(R/P, M) = 0$ by Nakayama's Lemma.

(11.12) Proposition: Let (R,m,k) be a local Noetherian ring and $M \neq 0$ a finite R-module. Then $\operatorname{injdim} M = \sup\{n \mid \operatorname{Ext}_R^n(k,M) \neq 0\}$.

Proof: Use (11.9) and (11.11).

(11.13) Corollary: Let (R,m,k) be a local Noetherian ring, $M \neq 0$ a finite R-module with $\operatorname{injdim} M < \infty$, and N a finite R-module with $\operatorname{depth} N = 0$. Then $\operatorname{injdim} M = \sup\{n \mid \operatorname{Ext}_R^n(N,M) \neq 0\}$.

Proof: It suffices to show that if $t = \operatorname{injdim} M < \infty$, then $\operatorname{Ext}_R^t(N,M) \neq 0$. Since $\operatorname{depth} N = 0$, we have $k = R/m \hookrightarrow N$ giving an exact sequence $0 \to k \to N \to U \to 0$. Thus $\operatorname{Ext}_R^t(N,M) \to \operatorname{Ext}_R^t(k,M) \to \operatorname{Ext}_R^{t+1}(U,M)$ is exact. Since $t = \operatorname{injdim} M$, $\operatorname{Ext}_R^{t+1}(U,M) = 0$ and $\operatorname{Ext}_R^t(k,M) \neq 0$ by (11.12). Thus $\operatorname{Ext}_R^t(N,M) \neq 0$.

(11.14) Lemma: Let R be a ring, M and N R-modules, and $x \in \operatorname{ann}(N)$ a NZD on R and M. Then for all $n \geq 0$ $\operatorname{Ext}_R^{n+1}(N,M) \cong \operatorname{Ext}_{R/(x)}^n(N, M/xM)$. The isomorphism is natural in the first variable.

Proof: Let $F = \operatorname{Hom}_R(-, M/xM)$ be the functor from the category of $R/(x)$-modules into itself. We want to show that $R^nF \cong \operatorname{Ext}_R^{n+1}(-,M)$ as contravariant functors on $R/(x)$-mod.

(1) $F \cong \operatorname{Ext}_R^1(-,M)$. In order to prove this, consider the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ and let N be an $R/(x)$-module. Since $\operatorname{Hom}_R(N,M)=0$ we have an exact sequence:

$0 \to \operatorname{Hom}_R(N, M/xM) \to \operatorname{Ext}_R^1(N,M) \xrightarrow{x \equiv 0} \operatorname{Ext}_R^1(N,M)$.

(2) There is a long exact sequence $\operatorname{Ext}_R^\circ(-,M)$ which is natural in the first variable.

(3) Let P be a free $R/(x)$-module. Since x is a NZD on R, $\operatorname{projdim}_R P \leq 1$ and thus $\operatorname{Ext}_R^{n+1}(P, M) = 0$ for all $n+1 \geq 2$.

It follows now by induction on n that $R^n F \cong \operatorname{Ext}_R^{n+1}(-, M)$.

(11.15) <u>Proposition</u>: Let R be a local Noetherian ring, M a finite R-module, and x a regular element on R and M. Then $\operatorname{injdim}_{R/(x)} M/xM = \operatorname{injdim}_R M - 1$.

<u>Proof</u>: Use (11.12) and (11.14)

Recall (9.30): If R is a local Noetherian ring and $M \neq 0$ a finite R-module of finite injective dimension, then $\dim M \leq \operatorname{injdim} M = \operatorname{depth} R$.

(11.16) <u>Lemma</u>: Let (R, m, k) be a local Noetherian ring, M a finite R-module, and $P \subseteq m$ a prime ideal with $ht(m/P) = d$. If $\operatorname{Ext}_R^{i+d}(k, M) = 0$, then $\operatorname{Ext}^i_{R_P}(k(P), M_P) = 0$ where $k(P) = (R/P)_P$.

<u>Proof</u>: By (11.11) $\operatorname{Ext}^i_R(R/P, M) = 0$ and by (7.89)(b) $\operatorname{Ext}^i_{R_P}(k(P), M_P) \cong \operatorname{Ext}^i_R(R/P, M)_P = 0$.

(11.17) <u>Theorem</u>: Let (R, m, k) be a local Noetherian ring of dimension n. The following are equivalent:

(a) $\operatorname{injdim} R < \infty$

(b) $\operatorname{injdim} R = n$

(c) $\operatorname{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\operatorname{Ext}^n_R(k, R) \cong k$

(d) $\operatorname{Ext}^i_R(k, R) = 0$ for some $i > n$.

(e) $\operatorname{Ext}^i_R(k, R) = 0$ for $i < n$ and $\operatorname{Ext}^n_R(k, R) \cong k$

(f) R is a CM-ring and $\operatorname{Ext}^n_R(k, R) \cong k$

(g) R is a CM-ring and every parameter ideal is irreducible

(h) R is a CM-ring and there is an irreducible parameter ideal

(k) R is a Gorenstein ring.

<u>Proof</u>: By (11.8) (g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (k). We will show: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (c) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (k) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b): Suppose that $\operatorname{injdim} R = r < \infty$. Let $P \subseteq R$ be a minimal prime ideal with $\operatorname{ht}(m/P) = n$. Then $PR_P \in \operatorname{Ass}(R_P)$ and $\operatorname{Hom}_{R_P}(k(P), R_P) \neq 0$. By (11.16) $\operatorname{Ext}^n_R(k,R) \neq 0$ and by (11.12) $r \geq n$. In order to show $r \leq n$ we proceed by induction on r. If $r=0$, we are done. Since $\operatorname{injdim} R = r$, the functor $T = \operatorname{Ext}^r_R(-,R)$ is right exact by (7.43). Moreover, by (11.12) $\operatorname{Ext}^r_R(k,R) \neq 0$. If $m \in \operatorname{Ass}(R)$ the exact sequence $0 \to k \to R$ yields an exact sequence $\operatorname{Ext}^r_R(R,R) \to \operatorname{Ext}^r_R(k,R) \to 0$. Since $\operatorname{Ext}^r_R(k,R) \neq 0$, we obtain that $\operatorname{Ext}^r_R(R,R) \neq 0$, a contradiction since $r > 0$. Hence $m \notin \operatorname{Ass}(R)$ and there is a regular element $x \in m$. By (11.15) $S = R/(x)$ is a local Noetherian ring of injective dimension $r-1$. By induction hypothesis $r-1 \leq \dim S \leq n-1$ and $r \leq n$.

(b) $\Rightarrow$ (c): By induction on n. If $n=0$, then $m \in \operatorname{Ass}(R)$ and there is an exact sequence $0 \to k \to R$. Since $\operatorname{injdim} R = 0$, the sequence $R \cong \operatorname{Hom}_R(R,R) \to \operatorname{Hom}_R(k,R) \to 0$ is exact. Thus $\operatorname{Hom}_R(k,R)$ is cyclic and $\operatorname{Hom}_R(k,R) \cong k$. If $n>0$, the same argument as in (a) $\Rightarrow$ (b) yields that R contains a regular element $x \in m$. By (11.15) $S = R/(x)$ is a local Noetherian ring of injective dimension $n-1$. Thus by induction hypothesis and (11.14) $\operatorname{Ext}^{i+1}_R(k,R) \cong \operatorname{Ext}^i_S(k,S) = 0$ for all $i \neq n-1$ and $\operatorname{Ext}^n_R(k,R) \cong \operatorname{Ext}^{n-1}_S(k,S) \cong k$. Since x is a regular element on R, $\operatorname{Hom}_R(k,R) = 0$.

(c) $\Rightarrow$ (d): trivial

(d) $\Rightarrow$ (a): By induction on n: If $n=0$, let $\operatorname{Ext}^i_R(k,R) = 0$ for some $i>0$. Since m is the only prime ideal of R, by (11.9) $\operatorname{injdim} R \leq i < \infty$. Let $n>0$ and $i>n$ with $\operatorname{Ext}^i_R(k,R)=0$. We want to show that $\operatorname{Ext}^i_R(R/P,R)=0$ for every prime ideal $P \subseteq R$. Then by (11.9) $\operatorname{injdim} R \leq i$. Assume that there is a prime ideal $P \subseteq R$ with $\operatorname{Ext}^i_R(R/P,R) \neq 0$. Since R is Noetherian, we may assume that $P \in \operatorname{Spec}(R)$ is maximal with $\operatorname{Ext}^i_R(R/P,R) \neq 0$. Then $P \neq m$ and for $x \in m - P$ consider the exact sequence $0 \to R/P \xrightarrow{x} R/P \to R/(P,x) \to 0$. This yields an exact sequence: $\operatorname{Ext}^i_R(R/(P,x),R) \to \operatorname{Ext}^i_R(R/P,R) \xrightarrow{x} \operatorname{Ext}^i_R(R/P,R)$. By assumption $\operatorname{Ext}^i_R(R/Q,R)=0$ for all $Q \in \operatorname{Supp}(R/(P,x))$. Thus by (11.10) $\operatorname{Ext}^i_R(R/(P,x),R)=0$ and x is regular on $\operatorname{Ext}^i_R(R/P,R)$. On the other hand, if $\operatorname{ht}(m/P)=d$, by (11.16) $\operatorname{Ext}^{i-d}_{R_P}(k(P),R_P)=0$. Since $\dim R_P \leq n-d < i-d$, by induction hypothesis $\operatorname{injdim} R_P < \infty$ and by '(a) $\Rightarrow$ (c)': $\operatorname{Ext}^i_{R_P}(k(P),R_P) \cong \operatorname{Ext}^i_R(R/P,R)_P = 0$. Since $\operatorname{Ext}^i_R(R/P,R)$ is a finite R-module, there is an element $x \in m-P$ with $x\operatorname{Ext}^i_R(R/P,R)=0$. Thus $\operatorname{Ext}^i_R(R/P,R)=0$.

(c) $\Rightarrow$ (e): trivial

(e) $\Rightarrow$ (f): Theorem 8.16

(f) $\Rightarrow$ (k): Let $x_1, \ldots, x_n$ be a SoP of R, $I = (x_1, \ldots, x_n)$, and $S = R/I$. Since R is a CM-ring, $x_1, \ldots, x_n$ is a regular sequence of R. Repeated application of (11.14) yields: $\mathrm{Ext}_R^n(k,R) \cong \mathrm{Ext}^{n-1}_{R/(x_1)}(k, R/(x_1)) \cong \ldots \cong \mathrm{Ext}_S^0(k,S) = \mathrm{Hom}_S(k,S) \cong k$. Thus $\dim_k \mathcal{S}(S) = 1$ and R is Gorenstein.

(k) $\Rightarrow$ (c): Since R is a CM-ring, by (8.16) $\mathrm{Ext}_R^i(k,R) = 0$ for $i<n$. If $I = (x_1, \ldots, x_n) \subseteq R$ is a parameter ideal of R and $S = R/I$, then by (11.14) $\mathrm{Ext}_R^n(k,R) \cong \mathrm{Ext}_S^0(k,S) = \mathrm{Hom}_S(k,S) \cong k$, since R is Gorenstein. In order to show that $\mathrm{Ext}_R^i(k,R) = 0$ for all $i>n$, let I and S be as above. By (11.14) it suffices to show that $\mathrm{Ext}_S^i(k,S) = 0$ for $i>0$. Since $\dim S = 0$, by (11.9) $\mathrm{Ext}_S^1(k,S) = 0$ implies that $\operatorname{injdim} S \leq 1$ and it suffices to show that $\mathrm{Ext}_S^1(k,S) = 0$.
Let $S = N_r \supseteq N_{r-1} \supseteq \ldots \supseteq N_1 \supseteq N_0 = 0$ be an ascending chain of ideals with $N_i/N_{i-1} \cong k$ for all $1 \leq i \leq r$. This yields short exact sequences $0 \longrightarrow N_i \longrightarrow N_{i+1} \longrightarrow k \longrightarrow 0$ for all $1 \leq i \leq r-1$ inducing long exact sequences:

$$0 \longrightarrow \mathrm{Hom}_S(k,S) \longrightarrow \mathrm{Hom}_S(N_{i+1},S) \longrightarrow \mathrm{Hom}_S(N_i,S) \xrightarrow{\delta_i} \mathrm{Ext}_S^i(k,S).$$

Since S is Gorenstein, $\mathrm{Hom}_S(k,S) \cong \mathrm{Hom}_S(N_1,S) \cong k$ and by induction on i: $\ell_S(\mathrm{Hom}_S(N_i,S)) \leq i$ for all $1 \leq i \leq r$. Moreover, $\ell_S(\mathrm{Hom}_S(N_i,S)) = i \iff \delta_j = 0$ for all $j \lneq i$. Since $\ell_S(\mathrm{Hom}_S(S,S)) = \ell_S(S) = r$ it follows that $\delta_1 = \delta_2 = \ldots = \delta_{r-1} = 0$. Thus the exact sequence $0 \to N_{r-1} \to N_r = B \to \ldots$ yields a long exact sequence $0 \longrightarrow \mathrm{Ext}_S^1(k,S) \longrightarrow \mathrm{Ext}_S^1(S,S) \longrightarrow \ldots$. Since $\mathrm{Ext}_S^1(S,S) = 0$, $\mathrm{Ext}_S^1(k,S) = 0$ and $\operatorname{injdim} S \leq 1$.

(11.18) <u>Corollary</u>: Let (R, m, k) be a local Noetherian CM-ring of CM-type r and dimension n. Then $r = \dim_k(\mathrm{Ext}_R^n(k,R))$.

<u>Proof</u>: Let $I = (x_1, \ldots, x_n)$ be a parameter ideal of R and $S = R/I$. By (10.14) $\mathrm{Ext}_R^n(k,R) \cong \mathrm{Hom}_S(k,S)$ and $r = \dim_k(\mathcal{S}(S)) = \dim_k(\mathrm{Hom}_S(k,S))$.

(11.19) <u>Theorem</u>: Let (R, m, k) be a local Gorenstein ring and $P \in \mathrm{Spec}(R)$. Then R_P is a local Gorenstein ring.

<u>Proof</u>: Consider a finite injective resolution of R: $0 \to R \to E_0 \to E_1 \to \dots \to E_n \to 0$. By Homework $(E_i)_P$ is an injective R_P-module and $0 \to R_P \to (E_0)_P \to (E_1)_P \to \dots \to (E_n)_P \to 0$ is a finite injective resolution of R_P. Thus $\operatorname{injdim} R_P < \infty$.

(11.20) <u>Theorem</u>: Let (R, m, k) be a local Noetherian ring and $\widehat{R}$ the m-adic completion of R. R is Gorenstein if and only if $\widehat{R}$ is Gorenstein.

<u>Proof</u>: By (7.55) R is CM if and only if $\widehat{R}$ is CM. Let $I = (x_1, \ldots, x_n)$ be a parameter ideal of R. Then $I\widehat{R}$ is a parameter ideal of $\widehat{R}$ and $R/I \cong \widehat{R}/I\widehat{R}$.

Recall from (7.69): Let R be a Noetherian ring, M a finite R-module, and $E^\bullet$ a minimal injective resolution of M. Then $E^i \cong \bigoplus_{P \in \operatorname{Spec}(R)} E(R/P)^{\mu_i(P,M)}$, where $\mu_i(P,M) = \dim_{k(P)} \operatorname{Ext}^i_{R_P}(k(P), M_P)$ is the ith Bass number of M with respect to P.

(11.21) <u>Theorem</u>: Let R be a Noetherian ring and $E^\bullet$ a minimal injective resolution of R. R is Gorenstein if and only if for all $i \geq 0$: $E^i \cong \bigoplus_{\operatorname{ht} P = i} E(R/P)$, or equivalently, for all $P \in \operatorname{Spec}(R)$: $\mu_i(P,R) = \delta_{i, \operatorname{ht} P}$.

<u>Proof</u>: "$\Leftarrow$": Suppose that $\mu_i(P,R) = \delta_{i,\operatorname{ht} P}$ for all $i \geq 0$ and all $P \in \operatorname{Spec}(R)$. By Homework for all $P \in \operatorname{Spec}(R)$, $E^\bullet_P$ is a minimal injective resolution of R_P. Since $E^\bullet_P$ is finite, $\operatorname{injdim} R_P < \infty$ and R_P is Gorenstein.

"$\Rightarrow$": Let $P \in \operatorname{Spec}(R)$. By assumption R_P is Gorenstein, thus by (11.17) $\operatorname{Ext}^i_{R_P}(k(P), R_P) = 0$ for $i \neq \dim R_P = \operatorname{ht} P$ and $\operatorname{Ext}^i_{R_P}(k(P), R_P) \cong k(P)$ if $i = \dim R_P = \operatorname{ht} P$. Thus $\mu_i(P,R) = \delta_{i, \operatorname{ht} P}$.

§2: MATLIS DUALITY

Let R be a Noetherian ring and E an injective R-module. Then by (7.64) $E \cong \bigoplus_{P \in \operatorname{Spec}(R)} E(R/P)^{\mu_P}$ where $E(R/P)$ is the injective hull of R/P and $\mu_P = \dim_{k(P)} \operatorname{Hom}_{R_P}(k(P), E_P)$. If (R, m, k) is a local Noetherian ring and $E = E(k)$ the injective hull of k, for an R-module M set $M' = \operatorname{Hom}_R(M, E)$. Then $M'' = \operatorname{Hom}_R(M,E)' = \operatorname{Hom}_R(\operatorname{Hom}_R(M,E),E)$ and there is a natural map $\Theta : M \longrightarrow M''$ defined by: for $x \in M$, $\Theta(x) : \operatorname{Hom}_R(M,E) \longrightarrow E$ is given by $\Theta(x)(\varphi) = \varphi(x)$ for all $\varphi \in \operatorname{Hom}_R(M,E)$. Note that Θ is R-linear.

(11.22) Proposition: Assumptions as above and suppose that $M \neq 0$.
(a) For all $x \in M - (0)$ there is a $\varphi \in M'$ with $\varphi(x) \neq 0$. In particular, Θ is injective.
(b) If M is an R-module of finite length, then $\ell_R(M) = \ell_R(M')$ and Θ is an isomorphism.

Proof: (a) The submodule Rx of M is isomorphic to $R/\operatorname{ann}(x)$. Let f be the composition of maps $f : Rx \xrightarrow{\cong} R/\operatorname{ann}(x) \xrightarrow{\text{nat}} k \hookrightarrow E$. Then $f(x) \neq 0$. Since E is injective, f extends to an R-linear map $\varphi : M \longrightarrow E$ with $\varphi(x) \neq 0$.
(b) Let $M_1 \subseteq M$ be a submodule with $\ell_R(M_1) = n-1 = \ell_R(M) - 1$. The exact sequence $0 \to M_1 \to M \to k \to 0$ yields an exact sequence $0 \to k' \to M' \to M_1' \to 0$. Since E is an essential extension of k: $k' = \operatorname{Hom}_R(k,E) \cong \operatorname{Hom}_R(k,k) \cong k$ and thus $\ell_R(M') = \ell_R(M_1) + 1$. The statement follows by induction on $n = \ell_R(M)$.

(11.23) Proposition: Assumptions as above. Let $\widehat{R}$ be the m-adic completion of R.
(a) E is an $\widehat{R}$-module. Moreover, E is the injective hull of the $\widehat{R}$-module k.
(b) $\operatorname{Hom}_R(E,E) = \operatorname{Hom}_{\widehat{R}}(E,E) \cong \widehat{R}$.

Proof: (a) We claim that the natural map $\sigma : E \longrightarrow E \otimes_R \widehat{R}$ defined by $\sigma(x) = x \otimes 1$ is an isomorphism. Let $x \otimes \hat{a} \in E \otimes_R \widehat{R}$. By (7.60) there is an $n \in \mathbb{N}$ with $m^n x = 0$. Since R

is dense in $\hat{R}$, there is an $a_0 \in R$ so that $\hat{a} - a_0 \in m^n\hat{R}$. Thus $\hat{a} = a_0 + \sum_{i=1}^{t} y_i \hat{b}_i$ where $y_i \in m^n \subseteq R$ and $b_i \in \hat{R}$. Then $x \otimes \hat{a} = x \otimes (a_0 + \sum_i y_i \hat{b}_i) = a_0 x \otimes 1 + \sum y_i x \otimes \hat{b}_i = a_0 x \otimes 1$. Hence σ is surjective. In order to show that σ is injective consider the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \otimes_R \hat{R} \\ \varepsilon \uparrow & & \uparrow \ \delta = \varepsilon \otimes \hat{R} \\ k & \xrightarrow{\varrho} & k \otimes_R \hat{R} \end{array}$$

ε and ϱ are injective. Since $\hat{R}$ is flat over R, δ is injective. Thus $\sigma\varepsilon = \delta\varrho$ is injective. Since E is an essential extension of k, σ is injective.

Let F be the injective hull of E as an $\hat{R}$-module. Then F is the injective hull of $\hat{R}$-module k and by (7.60) every element of F is annihilated by some power of $m\hat{R}$. F is an R-module with $E \subseteq F$. Since E is injective over R, there is a submodule C of F with $F = E \oplus C$. Since every element of C is annihilated by some power of $m\hat{R}$, C is an $\hat{R}$-module. But F is indecomposable as $\hat{R}$-module. Thus $C = 0$ and $E = F$.

(b) For $\nu > 0$ set $E_\nu = \{x \in E \mid m^\nu x = 0\}$. E_ν is a module over R and $\hat{R}$, $E_\nu \subseteq E_{\nu+1}$, and $E = \bigcup E_\nu = \varinjlim E_\nu$. By (7.90) $\mathrm{Hom}_R(E,E) = \mathrm{Hom}_R(\varinjlim E_\nu, E) = \varprojlim \mathrm{Hom}_R(E_\nu, E)$. Since $\mathrm{Hom}_R(R/m^\nu, E) \cong E_\nu$, $\mathrm{Hom}_R(E_\nu, E) = E_\nu' = (R/m^\nu)''$. R/m^ν is an R-module of finite length, thus by (11.22): $R/m^\nu \cong (R/m^\nu)''$. Thus $\varprojlim \mathrm{Hom}_R(E_\nu, E) \cong \varprojlim R/m^\nu = \hat{R}$ as R-modules. Since E_ν is also an $\hat{R}$-module, the same argument shows that $\mathrm{Hom}_{\hat{R}}(E,E) \cong \hat{R}$ as $\hat{R}$-modules.

(11.24) <u>Remark</u>: (11.23)(b) shows that every $\hat{R}$-linear map $f: E \longrightarrow E$ is the multiplication by some $a \in \hat{R}$.

(11.25) <u>Theorem:</u> Let (R, m, k) be a local Noetherian ring and $E = E_R(k)$ the injective hull of k. E is an Artinian module over R and $\hat{R}$.

<u>Proof</u>: Note that every R-submodule of E is also an $\hat{R}$-submodule. Thus we may assume that R is complete. If $M \subseteq E$ is a submodule let $M^\perp = \mathrm{ann}(M) = \{a \in R \mid aM = 0\}$

and if $I \subseteq R$ is an ideal set $I^\perp = 0:_E I = \{x \in E \mid Ix = 0\}$. Obviously, $M \subseteq M^{\perp\perp}$. We want to show that $M^{\perp\perp} = M$. Consider the exact sequence $0 \to M \to E \to E/M \to 0$. Since E is injective, the sequence $0 \to (E/M)' \xrightarrow{(*)} E'$ is exact where $(E/M)' = \mathrm{Hom}_R(E/M, E)$ and $E' = \mathrm{Hom}_R(E,E) \cong \widehat{R}$ (II.23). Thus for every $f \in \mathrm{Hom}_R(E,E)$ there is an $\hat{a} \in \widehat{R}$ so that $f(x) = \hat{a}x$ for all $x \in E$. Every $g \in \mathrm{Hom}_R(E/M, E)$ is mapped under $(*)$ into an $f \in \mathrm{Hom}_R(R,R)$ with $f|_M = 0$ and conversely, every $f \in \mathrm{Hom}_R(E,E)$ with $f|_M = 0$ factors through a $g \in \mathrm{Hom}_R(E/M, E)$. Thus $\mathrm{Hom}_R(E/M, E) \cong M^\perp$ and the embedding $(*)$ corresponds to the embedding $0 \to M^\perp \hookrightarrow \widehat{R}$. By (II.22) for all $x \in E - M$ there is an $\varphi \in (E/M)'$ with $\varphi(x+M) \neq 0$. Thus for all $x \in E - M$ there is an $a \in M^\perp$ with $ax \neq 0$. This shows that $M^{\perp\perp} \subseteq M$ and hence $M^{\perp\perp} = M$.

If $I \subseteq R$ is an ideal, the exact sequence $0 \to I \to R \to R/I \to 0$ yields an exact sequence $0 \to (R/I)' \xrightarrow{(\tilde{*})} R' = \mathrm{Hom}_R(R,E) \cong E$. Under $(\tilde{*})$ $(R/I)'$ is mapped onto $I^\perp$. If $a \in R - I$, by (II.22) there is a $\varphi \in (R/I)'$ with $\varphi(a+I) \neq 0$. Let $x = \varphi(1+I) \in I^\perp$. Then $Ix = 0$ and $a \notin \mathrm{ann}(x)$. Since $I^{\perp\perp} = \bigcap_{x \in I^\perp} \mathrm{ann}(x)$ it follows that $I^{\perp\perp} \subseteq I$ and hence $I^{\perp\perp} = I$. This shows that '$\perp$' defines order-reversing bijections between the sets:

$$\{M \mid M \subseteq E \text{ a submodule}\} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \{I \mid I \subseteq R \text{ an ideal}\}.$$

Since R is Noetherian, E is Artinian.

(II.26) <u>Theorem</u>: Let (R, m, k) be a complete local Noetherian ring and $E = E_R(k)$ the injective hull of k.

(a) If M is a Noetherian R-module, then $M' = \mathrm{Hom}_R(M,E)$ is an Artinian R-module and $M'' \cong M$.

(b) If M is an Artinian R-module, then $M' = \mathrm{Hom}_R(M,E)$ is a Noetherian R-module and $M'' \cong M$.

<u>Proof</u>: (a) Let $n \in \mathbb{N}$ with $R^n \to M \to 0$ exact. Since E is injective, the sequence $0 \to M' \to (R^n)' = \mathrm{Hom}_R(R^n, E) \cong \mathrm{Hom}_R(R,E)^n \cong E^n$ is exact. E^n is an Artinian R-module and so is every submodule of E^n. Thus M' is Artinian. Since $(E^n)' = \mathrm{Hom}_R(E^n, E) =$

$\mathrm{Hom}_R(E,E)^n \cong R^n$ there is a commutative diagram with exact rows:

$$\begin{array}{ccccc} R^n & \longrightarrow & M & \longrightarrow & 0 \\ \cong\downarrow\theta' & & \downarrow\theta & & \\ (E^n)'' & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

By (11.22) θ is injective, hence θ is an isomorphism.

(b) We claim that there is an $n\in\mathbb{N}$ so that M can be considered a submodule of E^n. For all $m\in\mathbb{N}$ consider all R-linear maps $\tau: M\longrightarrow E^m$. This yields a set Γ of submodules $\ker(\tau)$ of M. Since M is Artinian, there is an $n\in\mathbb{N}$ and an R-linear map $\varphi: M\longrightarrow E^n$ so that $\ker(\varphi)$ is minimal in Γ. If $\ker(\varphi)\neq 0$, let $x\in\ker(\varphi)$. By (11.22) there is an R-linear map $\sigma: M\longrightarrow E$ with $\sigma(x)\neq 0$. Let $\varrho: M\longrightarrow E^{n+1}$ be defined by $\varrho(y)=(\varphi(y),\sigma(y))$. Then $\ker(\varrho)\subsetneq\ker(\varphi)$, a contradiction. Thus $\ker(\varphi)=0$.

Let $0\longrightarrow M\longrightarrow E^n$ be exact. Then $(E^n)'\longrightarrow M'\longrightarrow 0$ is exact and $(E^n)'\cong R^n$. M' is a Noetherian R-module. In order to show that $M''\cong M$ note that every homomorphic image of an Artinian module is Artinian. The exact sequence $0\longrightarrow M\longrightarrow E^n\longrightarrow E^n/M\longrightarrow 0$ yields a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E^n & \longrightarrow & E^n/M & \longrightarrow & 0 \\ & & \theta\downarrow & & \downarrow\cong & & \downarrow\bar\theta & & \\ 0 & \longrightarrow & M'' & \longrightarrow & (E^n)'' & \longrightarrow & (E^n/M)'' & \longrightarrow & 0 \end{array}$$

By (11.22) θ and $\bar\theta$ are injective. Thus $\bar\theta$ is an isomorphism. Hence θ is an isomorphism by the five-lemma.

§3: THE CANONICAL MODULE

(11.27) Definition: Let (R, m, k) be a local Noetherian ring and M a finite R-module with $\operatorname{depth} M = d$. The number $r(M) = \dim_k \operatorname{Ext}^d_R(k, M)$ is called the type of M.

(11.28) Lemma: Let R be a Noetherian ring, M a finite R-module, and $x_1, \dots, x_n$ an M-sequence. Suppose that N is an R-module with $I = (x_1, \dots, x_n) \subseteq \operatorname{ann}(N)$. Then $\operatorname{Hom}_R(N, M/IM) \cong \operatorname{Ext}^n_R(N, M)$.

Proof: Set $M_0 = M$ and $M_i = M/(x_1, \dots, x_i)M$. We show by induction on i that $\operatorname{Ext}^{n-i}_R(N, M_i) \cong \operatorname{Ext}^{n-i-1}_R(N, M_{i+1})$. If $i = 0$ the exact sequence $0 \to M \xrightarrow{x_1} M \to M_1 \to 0$ yields a long exact sequence $\operatorname{Ext}^{n-1}_R(N, M) \to \operatorname{Ext}^{n-1}_R(N, M_1) \to \operatorname{Ext}^N_R(N, M) \xrightarrow{\varphi} \operatorname{Ext}^n_R(N, M)$ where φ is multiplication by x_1. Since $x_1 \in \operatorname{ann}(N)$, $\varphi = 0$ and $\operatorname{Ext}^{n-1}_R(N, M) = 0$ by (8.14). For the induction step $i \Rightarrow i+1$ consider the exact sequence: $0 \to M_i \xrightarrow{x_{i+1}} M_i \to M_{i+1} \to 0$ and repeat the argument. Thus $\operatorname{Hom}_R(N, M_n) \cong \operatorname{Ext}^n_R(N, M)$.

(11.29) Remark: Let (R, m, k) be a local Noetherian ring, M a finite R-module and $x_1, \dots, x_d$ a maximal M-sequence. Then $r(M) = \dim_k \operatorname{Hom}_R(k, M/IM)$ where $I = (x_1, \dots, x_d)$. $\operatorname{Hom}_R(k, M/IM) \cong 0 :_{M/IM} m$ is called the socle of M/IM.

Recall from Chapter VIII: Let R be a local Noetherian ring and M a finite R-module. M is a maximal CM-module (MCM) if $\operatorname{depth} M = \dim R$.

(11.30) Definition: Let R be a local CM-ring. A finite R-module C is called a canonical module of R if C is a MCM, $r(C) = 1$, and $\operatorname{injdim}_R C < \infty$.

(11.31) Examples: Let (R, m, k) be a local Noetherian ring.

(a) If R is Gorenstein then R is a canonical module of R.

(b) If R is Artinian then $E_R(k)$ is a canonical module of R. Conversely, every canonical module of R is isomorphic to $E_R(k)$.

Proof: (b) The Artinian local ring R is complete. Thus by Matlis duality (11.26) $R' = \mathrm{Hom}_R(R, E_R(k)) \cong E_R(k)$ is a Noetherian R-module. Conversely, if C is a canonical module of R, then by (9.30) $\mathrm{injdim}\, C \leq \mathrm{depth}\, R = 0$ and C is injective. Now use (7.64).

(11.32) Lemma: Let R be a local Noetherian ring, $\varphi: M \longrightarrow N$ an R-linear map of finite R-modules, and $x_1, \ldots, x_n$ an N-regular sequence. Set $I = (x_1, \ldots, x_n)$. If $\varphi \otimes R/I$ is an isomorphism then so is φ.

Proof: By Nakayama's Lemma φ is surjective. Thus there is an exact sequence $C_\bullet: 0 \longrightarrow U \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$. Since $x_1, \ldots, x_n$ is an N-sequence, by Homework $C_\bullet \otimes_R R/I$ is exact. Thus $U \otimes_R R/I \cong \ker(\varphi \otimes R/I) = 0$ and $U = 0$ by Nakayama's Lemma.

(11.33) Lemma: Let R be a local Noetherian ring and M a MCM R-module. Every R-regular sequence is M-regular.

Proof: Obviously, $M \neq 0$. If $P \in \mathrm{Ass}_R(M)$ then by (8.21) $\dim R/P \geq \mathrm{depth}\, M = \dim R$ and P is a minimal prime of R. Hence $P \in \mathrm{Ass}(R)$ and every R-regular element x is M-regular. Since M/xM is a MCM $R/(x)$-module, the assertion follows by induction.

(11.34) Proposition: Let R be a local CM-ring of dimension d, M a finite R-module, and C a MCM R-module.

(a) If M is CM of dimension t and $\mathrm{injdim}_R C < \infty$ then $\mathrm{Ext}^i_R(M, C) = 0$ for $i \neq d-t$ and $\mathrm{Ext}^{d-t}_R(M, C)$ is CM of dimension t.

(b) If $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$ then $\mathrm{depth}\, \mathrm{Hom}_R(M, C) \geq d$.

(c) If $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$, M is MCM, and $x_1, \ldots, x_d$ is an R-regular sequence then $\mathrm{Hom}_R(M,C)\otimes_R R/(\underline{x}) \cong \mathrm{Hom}_{R/(\underline{x})}(M/(\underline{x})M, C/(\underline{x})C)$ via the natural map.

Proof: (a) By (8.20) $\mathrm{Ext}^i_R(M,C)=0$ for all $i<\operatorname{depth} C - \dim M = d-t$. Since $\operatorname{ann}(M)\subseteq \operatorname{ann}(\mathrm{Ext}^i_R(M,C))$, $\dim \mathrm{Ext}^{d-t}_R(M,C)\le \dim M$. We want to show by induction on t that $\mathrm{Ext}^i_R(M,C)=0$ for $i>d-t$ and that $\operatorname{depth}\mathrm{Ext}^{d-t}_R(M,C)=t$. If $t=\operatorname{depth} M=0$ then by (11.13) $\operatorname{injdim} C = d = \max\{i \mid \mathrm{Ext}^i_R(M,C)\neq 0\}$. Hence $\mathrm{Ext}^i_R(M,C)=0$ for $i>d$ and $\mathrm{Ext}^d_R(M,C)\neq 0$. Moreover, $\operatorname{depth}\mathrm{Ext}^d_R(M,C)=0$ since $\dim \mathrm{Ext}^d_R(M,C)=0$. If $t\ge 1$ let $x\in m_R$ be an M-regular element. The exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ yields a long exact sequence:

$$\dots\to \mathrm{Ext}^i_R(M/xM,C)\to \mathrm{Ext}^i_R(M,C)\xrightarrow{x}\mathrm{Ext}^i_R(M,C)\to \mathrm{Ext}^{i+1}_R(M/xM,C)\to\dots$$

M/xM is a CM-module of dimension $t-1$. For $i\neq d-t$, by induction hypothesis $\mathrm{Ext}^{i+1}_R(M/xM,C)=0$, hence by Nakayama's Lemma $\mathrm{Ext}^i_R(M,C)=0$. For $i=d-t$ we have an exact sequence $0\to \mathrm{Ext}^{d-t}_R(M,C)\xrightarrow{x}\mathrm{Ext}^{d-t}_R(M,C)\to \mathrm{Ext}^{d-t+1}_R(M/xM,C)\to 0$. Thus $\mathrm{Ext}^{d-t+1}_R(M/xM,C)\cong \mathrm{Ext}^{d-t}_R(M,C)/x\,\mathrm{Ext}^{d-t}_R(M,C)$. By induction hypothesis $\operatorname{depth}\mathrm{Ext}^{d-t+1}_R(M/xM,C)=t-1$ and hence $\operatorname{depth}\mathrm{Ext}^{d-t}_R(M,C)=t$.

(b) Let $F_\bullet$ be a finite free R-resolution of M. Since $\mathrm{Ext}^i_R(M,C)=0$ for all $i>0$, the sequence $0\to \mathrm{Hom}_R(M,C)\to \mathrm{Hom}_R(F_\bullet,C)$ is exact. This yields an exact sequence:

$$(*)\quad 0\to \mathrm{Hom}_R(M,C)\to \mathrm{Hom}_R(F_0,C)\to \dots\to \mathrm{Hom}_R(F_{d-1},C)\to B_d\to 0$$

with $\mathrm{Hom}_R(F_i,C)\cong C^{b_i}$, $\operatorname{depth}\mathrm{Hom}_R(F_i,C)=d$, and $\operatorname{depth} B_d\ge 1$. Splitting $(*)$ into short exact sequences $0\to B_i\to \mathrm{Hom}_R(F_i,C)\to B_{i+1}\to 0$ and applying (8.22) yields that $\operatorname{depth} B_i\ge \min\{d, \operatorname{depth} B_{i+1}+1\}$. Thus $\operatorname{depth}\mathrm{Hom}_R(M,C)=d$.

(c) Let $x\in m_R$ be R-regular. By (11.33) x is M- and C-regular and M/xM, C/xC are MCM over $R/(x)$. From the exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ we obtain:

$0=\mathrm{Ext}^i_R(M,C)\to \mathrm{Ext}^{i+1}_R(M/xM,C)\to \mathrm{Ext}^{i+1}_R(M,C)=0$ and thus $\mathrm{Ext}^{i+1}_R(M/xM,C)=0$ for all $i\ge 1$. By (11.14) $\mathrm{Ext}^i_{R/(x)}(M/xM,C/xC)\cong \mathrm{Ext}^{i+1}_R(M/xM,C)=0$ for all $i\ge 1$ and the $R/(x)$-modules M/xM and C/xC satisfy the assumptions. We proceed by induction on d. If $x_1=x\in m_R$ is R-regular, the exact sequence $0\to C\xrightarrow{x} C\to C/xC\to 0$ induces the long exact sequence:

$0\to \mathrm{Hom}_R(M,C)\xrightarrow{x}\mathrm{Hom}_R(M,C)\to \mathrm{Hom}_R(M,C/xC)=\mathrm{Hom}_{R/(x)}(M/xM,C/xC)\to \mathrm{Ext}^1_R(M,C)=0$.

Thus $\mathrm{Hom}_{R/(x)}(M/xM,C/xC)\cong \mathrm{Hom}_R(M,C)\otimes_R R/(x)$. This shows the case $d=1$. If $d>1$, Let $x_1, \dots, x_d$ be an R-regular sequence. Then by induction hypothesis:

$$\mathrm{Hom}_{R/(\underline{x})}(M/(\underline{x})M, C/(\underline{x})C) \cong \mathrm{Hom}_{R/(x_1)}(M/x_1M, C/x_1C) \otimes_{R/(x_1)} R/(\underline{x}) \cong \mathrm{Hom}_R(M,C) \otimes_R R/(\underline{x}).$$

(11.35) <u>Theorem</u>: Let (R,m) be a local CM-ring of dimension d, $\underline{x} = x_1, \dots, x_d$ an R-regular sequence, and C, C' canonical modules of R.

(a) $\underline{x}$ is regular on C and $C/(\underline{x})C \cong E_{R/(\underline{x})}(k)$

(b) $C \cong C'$, in particular, the canonical module, denoted by ω_R, is unique up to isomorphism.

<u>Proof</u>: (a) By (11.33) $\underline{x}$ is regular on C. Thus $r(C/(\underline{x})C) = r(C) = 1$ by (11.29) and $\mathrm{injdim}_{R/(\underline{x})} C/(\underline{x})C < \infty$ by (11.15). Hence $C/(\underline{x})C$ is a canonical module of the Artinian local ring $R/(\underline{x})$ and by (11.31) $C/(\underline{x})C \cong E_{R/(\underline{x})}(k)$.

(b) Set $\overline{R} = R/(\underline{x})$. C and C' are MCM modules and $\mathrm{injdim}\, C' < \infty$. By (11.34)(a) $\mathrm{Ext}^i_R(C,C') = 0$ for all $i > 0$ and by (11.34)(c) $\mathrm{Hom}_R(C,C') \otimes_R \overline{R} \cong \mathrm{Hom}_{\overline{R}}(C/(\underline{x})C, C'/(\underline{x})C')$ via the natural map. By (a) there is an isomorphism $\varphi \in \mathrm{Hom}_{\overline{R}}(C/(\underline{x})C, C'/(\underline{x})C')$. Thus there is a $\psi \in \mathrm{Hom}_R(C,C')$ with $\psi \otimes_R \overline{R} = \varphi$. By (11.32) ψ is an isomorphism since $\underline{x}$ is regular on C'.

(11.36) <u>Theorem</u>: Let R be a local Noetherian ring. The following are equivalent:

(a) R is Gorenstein

(b) R is CM, ω_R exists, and $\omega_R \cong R$.

<u>Proof</u>: By (11.18) R is Gorenstein if and only if R is CM, $r(R) = 1$, and $\mathrm{injdim}_R R < \infty$.

(11.37) <u>Theorem</u>: Let R be a local CM-ring of dimension d and C a finite R-module. The following are equivalent:

(a) $C \cong \omega_R$

(b) For every t, $0 \leq t \leq d$, and every CM R-module M of dimension t:

(i) $\mathrm{Ext}^{d-t}_R(M,C)$ is a CM-module of dimension t

(ii) $\mathrm{Ext}^i_R(M,C)=0$ for all $i\neq d-t$

(iii) there is a natural isomorphism $M \xrightarrow{\cong} \mathrm{Ext}^{d-t}_R(\mathrm{Ext}^{d-t}_R(M,C),C)$

(c) For every MCM R-module M:

(i) $\mathrm{Hom}_R(M,C)$ is a MCM R-module

(ii) $\mathrm{Ext}^i_R(M,C)=0$ for all $i>0$

(iii) the natural map $M \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$ is an isomorphism.

(d) (i) C is a MCM module

(ii) $\mathrm{injdim}_R\, C<\infty$

(iii) the natural map $R \xrightarrow{\cong} \mathrm{End}_R(C)=\mathrm{Hom}_R(C,C)$ is an isomorphism

Proof: (a) $\Rightarrow$ (b): (i) and (ii) follow from (11.34). In order to prove (iii) notice that let $\mathrm{ann}(M)=d-t$. Since R is CM, there is a regular sequence $\underline{x}=x_1,\dots,x_{d-t}\in \mathrm{ann}(M)$. By (11.33) $\underline{x}$ is a regular sequence on C. We claim that $C/(\underline{x})C$ is the canonical module of $R/(\underline{x})$. $C/(\underline{x})C$ is a MCM $R/(\underline{x})$-module of dimension t and by (11.14) $\mathrm{Ext}^d_R(k,C)\cong \mathrm{Ext}^t_{R/(\underline{x})}(k,C/(\underline{x})C)\cong k$ and by (11.15) $\mathrm{injdim}_{R/(\underline{x})}(C/(\underline{x})C)<\infty$. Thus $C/(\underline{x})C$ is the canonical module of $R/(\underline{x})$. By (11.28) $\mathrm{Ext}^{d-t}_R(\mathrm{Ext}^{d-t}_R(M,C),C)\cong \mathrm{Hom}_R(\mathrm{Ext}^{d-t}_R(M,C),C/(\underline{x})C)$ and $\mathrm{Ext}^{d-t}_R(M,C)\cong \mathrm{Hom}_R(M,C/(\underline{x})C)$. Thus there are natural isomorphisms: $\mathrm{Ext}^{d-t}_R(\mathrm{Ext}^{d-t}_R(M,C),C)\cong \mathrm{Hom}_R(\mathrm{Hom}_R(M,C/(\underline{x})C),C/(\underline{x})C)$ $\cong \mathrm{Hom}_{R/(\underline{x})}(\mathrm{Hom}_{R/(\underline{x})}(M/(\underline{x})M,C/(\underline{x})C),C/(\underline{x})C)$ and we may replace R by $R/(\underline{x})$ and assume M is a MCM R-module, i.e. $t=d$. Let $\varphi_M: M\longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$ be the natural map defined by $\varphi_M(m):\mathrm{Hom}_R(M,C)\longrightarrow C$ with $\varphi_M(m)(f)=f(m)$. Let $\underline{y}=y_1,\dots,y_d$ be a regular R-sequence and set $\overline{R}=R/(\underline{y})$. By (i) $\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$ is MCM and by (11.33) $\underline{y}$ is a regular sequence on this module. By (11.32) it suffices to show that $\varphi_M\otimes\overline{R}$ is an isomorphism. By (ii) $\mathrm{Ext}^i_R(M,C)=0$ for all $i>0$ and thus by (11.34)(c) $\mathrm{Hom}_R(M,C)\otimes\overline{R}\cong \mathrm{Hom}_{\overline{R}}(M/(\underline{y})M,C/(\underline{y})C)$. Similarly, $\mathrm{Hom}_R(M,C)$ is MCM and $\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)\otimes_R\overline{R}\cong \mathrm{Hom}_{\overline{R}}(\mathrm{Hom}_R(M,C)\otimes\overline{R},C/(\underline{y})C)$. Thus $\mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)\otimes\overline{R}$ $\cong \mathrm{Hom}_{\overline{R}}(\mathrm{Hom}_{\overline{R}}(M/(\underline{y})M,C/(\underline{y})C),C/(\underline{y})C)$ and $\varphi_M\otimes\overline{R}=\varphi_{M\otimes\overline{R}}$. By (11.35) $C\otimes_R\overline{R}\cong E_{\overline{R}}(k)$. The natural map $\varphi_{M\otimes\overline{R}}: M\otimes\overline{R}\longrightarrow (M\otimes\overline{R})''$ is an isomorphism by (11.22).

(b) $\Rightarrow$ (c): trivial

(c) $\Rightarrow$ (d): (i) and (iii) follow from (c) applied to $M=R$. In order to prove (ii) let N be a finite R-module and let M be a (finite) dth syzygy of N. By (8.22) M is a MCM R-module since R is CM of dimension d. By assumption (c) $\operatorname{Ext}^1_R(M,C)=0$ and by (7.37) $\operatorname{Ext}^1_R(M,C)\cong\operatorname{Ext}^{d+1}_R(N,C)=0$ since M is a dth syzygy of N. By (7.43) $\operatorname{injdim}_R C\le d<\infty$.
(d) $\Rightarrow$ (a): It remains to show that $r(C)=1$. Let $\underline{x}=x_1,\dots,x_d$ be an R-sequence, $\overline{R}=R/(\underline{x})$, and $E=E_{\overline{R}}(k)$. Since C is MCM, $\underline{x}$ is C-regular and $r(C)=r(C/(\underline{x})C)$ by (11.29). By (11.15) $\operatorname{injdim}_{\overline{R}} C/(\underline{x})C<\infty$ and $C/(\underline{x})C$ is an injective $\overline{R}$-module, hence $C/(\underline{x})C\cong E^r$ with $r=r(C/(\underline{x})C)$. Since C is MCM, by (11.34)(a),(c) $R\xrightarrow{\cong}\operatorname{End}_R(C)$ specializes to an isomorphism $\overline{R}\xrightarrow{\cong}\operatorname{End}_{\overline{R}}(C/(\underline{x})C)$. But $\operatorname{End}_{\overline{R}}(C/(\underline{x})C)\cong\operatorname{Hom}_{\overline{R}}(E^r,E^r)\overset{(1)}{\cong}\operatorname{Hom}_{\overline{R}}(E,E)^{r^2}\overset{(2)}{\cong}\overline{R}^{r^2}$ where (1) follows by (7.87) and (2) by (11.23). Thus $\overline{R}\cong\overline{R}^{r^2}$ and $r=1$.

(11.37) shows that $\operatorname{Ext}^{d-t}_R(-,\omega_R)$ is a contravariant functor on the category of finite CM-modules of dimension t and defines a duality on this category, in particular, $\operatorname{Hom}_R(-,\omega_R)$ is a contravariant exact functor on the category of MCM R-modules and defines a duality on this category. Also, (11.36) and (11.37) show that among CM-rings Gorenstein rings are exactly those rings for which $\operatorname{Hom}_R(-,R)$ is a contravariant exact functor and a duality on the category of MCM R-modules; in particular, over a local Gorenstein ring every MCM R-module is reflexive (i.e. the natural map $M\longrightarrow\operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$ is an isomorphism).

Recall: If (R,m,k) is a local Noetherian ring and M a finite R-module, then $\mu(M)=\dim_k(k\otimes_R M)$ is the minimal number of generators of M.

(11.38) <u>Proposition</u>: Let (R,m,k) be a local CM-ring of dimension d with a canonical module ω_R.
(a) Let M be a CM R-module of dimension t, then $\mu(\operatorname{Ext}^{d-t}_R(M,\omega_R))=r(M)$ and $r(\operatorname{Ext}^{d-t}_R(M,\omega_R))=\mu(M)$.
(b) ω_R is a faithful R-module with $\mu(\omega_R)=r(R)$ and $r(\omega_R)=1$.

<u>Proof</u>: (a) As in the proof of (11.37) we can reduce to the case where $\dim R = 0$. Then $\omega_R \cong E_R(k) = E$ and by (11.22) $\mathrm{Hom}_R(M,E) = M'$ and $M'' = \mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E) \cong M$. By (6.24) $\mathcal{S}(M') \cong \mathrm{Hom}_R(k, \mathrm{Hom}_R(M,E)) \cong \mathrm{Hom}_R(k \otimes_R M, E)$ and $r(M') = \ell_R(k \otimes_R M) = \mu(M)$. By (11.22): $r(M) = r((M')') = \mu(M')$.

(b) By (11.37) $\mathrm{End}_R(\omega_R) \cong R$ and ω_R is faithful. The rest follows from (a) with $M = R$.

(11.39) <u>Theorem</u>: Let R be a local CM-ring with a canonical module ω_R and $P \in \mathrm{Spec}(R)$. Then R_P has a canonical module and $\omega_{R_P} \cong (\omega_R)_P$.

<u>Proof</u>: The conditions of (11.37)(d) are preserved under localization.

(11.40) <u>Lemma</u>: Let (R, m, k) be a local CM-ring of dimension d and C a finite R-module. The following are equivalent:

(a) C is a canonical module of R.

(b) $\mu_i(m, C) = \delta_{id}$ for all i.

<u>Proof</u>: By (7.67) $\mu_i(m,C) = \dim_k \mathrm{Ext}^i_R(k,C)$ and by (8.19) C is MCM if and only if $\mu_i(m,C) = 0$ for $i < d$ and $\mu_d(m,C) \neq 0$. In this case $r(C) = \mu_d(m,C)$. Moreover, $\mathrm{injdim}_R C < \infty$ if and only if $\mathrm{injdim}_R C \leq d$ which is equivalent to $\mu_i(m,C) = 0$ for $i > d$ by (11.12).

(11.41) <u>Theorem</u>: Let R be a local CM-ring and C a finite R-module. The following are equivalent:

(a) $C \cong \omega_R$

(b) $\mu_i(P,C) = \delta_{i\,\mathrm{ht} P}$ for all $i \geq 0$ and all $P \in \mathrm{Spec}(R)$

(c) Let $I^\bullet$ be a minimal injective R-resolution of C. Then $I^i = \bigoplus E_R(R/P)$, where P runs over all prime ideals of height i for all $i \geq 0$.

<u>Proof</u>: (a) $\Leftrightarrow$ (b): By (11.39) $C \cong \omega_R$ if and only if $C_P \cong \omega_{R_P}$ for all $P \in \mathrm{Spec}(R)$. By (11.40)

this is equivalent to $\mu_i(PR_p, C_p) = \delta_{i\,\dim R_p} = \delta_{i\,\mathrm{ht}\,p} = \mu_i(P,C)$.
(b)$\Leftrightarrow$(c): Use (7.69).

(11.42) <u>Proposition</u>: Let R be a local CM-ring and C a finite R-module.
(a) Let $\underline{x}$ be a regular sequence on R and C. Then $C \cong \omega_R$ if and only if $C/(\underline{x})C \cong \omega_{R/(\underline{x})}$.
(b) $C \cong \omega_R$ if and only if $\widehat{C} \cong \omega_{\widehat{R}}$.

<u>Proof</u>: (a) Homework
(b) By (7.89) $\mathrm{Ext}^i_{\widehat{R}}(\widehat{R}/\widehat{m}, \widehat{C}) \cong \mathrm{Ext}^i_R(R/m, C) \otimes_R \widehat{R}$. Thus $\mu_i(\widehat{m}, \widehat{C}) = \mu_i(m, C)$ and the assertion follows by (11.40).

(11.43) <u>Theorem</u>: Let $\varphi: R \longrightarrow S$ be a local homomorphism of local CM-rings so that S is a finite R-module and $g = \dim R - \dim S$. If R has a canonical module, then so does S an $\omega_S \cong \mathrm{Ext}^g_R(S, \omega_R)$.

<u>Proof</u>: Set $I = \ker \varphi$. Then $R/I \hookrightarrow S$ is an integral extension and $\dim R/I = \dim S$. Since R is CM, by (8.31) $\dim R = \mathrm{ht}\, I + \dim R/I$ and $g = \dim R - \dim R/I = \mathrm{ht}\, I$. Let $\underline{x} = x_1, \dots, x_g \in I$ be an R-regular sequence. By (11.33) $\underline{x}$ is a regular sequence on ω_R and by (11.14) $\mathrm{Ext}^g_R(S, \omega_R) \cong \mathrm{Hom}_{R/(\underline{x})}(S, \omega_R/(\underline{x})\omega_R)$. Moreover, by (11.42) $\omega_R/(\underline{x})\omega_R \cong \omega_{R/(\underline{x})}$. $R/(\underline{x})$ is a CM-ring of dimension $\dim R - g = \dim S$. Thus we may replace R by $R/(\underline{x})$ and assume that $\dim R = \dim S$. We have to show that $\mathrm{Hom}_R(S, \omega_R)$ is a canonical module of S.

Let $\underline{y} = y_1, \dots, y_d$ be a SoP of R. Then $\underline{y}$ is a SoP of S, since φ is local, finite, and $\dim S = d$. Since S is CM, $\underline{y}$ is a regular sequence on S. Thus S is a MCM R-module. By (11.34) $\mathrm{Hom}_R(S, \omega_R)$ is a MCM R-module and by (11.33) $\underline{y}$ is a regular sequence on $\mathrm{Hom}_R(S, \omega_R)$. By (11.42)(a) it suffices to show that $\omega_{\overline{S}} \cong \mathrm{Hom}_R(S, \omega_R) \otimes_S \overline{S}$, where $\overline{S} = S/(\underline{y})$. Set $\overline{R} = R/(\underline{y})$. By (11.34) $\mathrm{Hom}_R(S, \omega_R) \otimes_S \overline{S} \cong \mathrm{Hom}_R(S, \omega_R) \otimes_R \overline{R} \cong \mathrm{Hom}_{\overline{R}}(S \otimes_R \overline{R}, \omega_R \otimes_R \overline{R}) \cong \mathrm{Hom}_{\overline{R}}(\overline{S}, \omega_{\overline{R}})$. Thus we may replace R, S by $\overline{R}, \overline{S}$ and may assume that $\dim R = \dim S = 0$.

Let k, ℓ be the residue fields of R and S. Then $\omega_R = E_R(k)$ and we have to show that $E_S(\ell) \cong \operatorname{Hom}_R(S, \omega_R)$ as S-modules. There is an adjoint isomorphism $\operatorname{Hom}_S(M, \operatorname{Hom}_R(S,\omega_R)) \cong \operatorname{Hom}_R(M \otimes_S S, \omega_R)$ for all S-modules M by (6.24). Since ω_R is an injective R-module, $\operatorname{Hom}_R(-,\omega_R)$ is exact. Thus $\operatorname{Hom}_S(-, \operatorname{Hom}_R(S,\omega_R))$ is exact and $\operatorname{Hom}_R(S,\omega_R)$ is an injective S-module. Thus $\operatorname{Hom}_R(S,\omega_R) \cong E_S(\ell)^r$. By (11.22) $\ell_R(\operatorname{Hom}_R(S,\omega_R)) = \ell_R(S)$ and $\ell_R(E_S(\ell)^r) = r\,\ell_S(E_S(\ell)) \cdot \dim_k \ell = r\,\ell_S(S) \dim_k \ell = r\,\ell_R(S)$. Thus $r=1$.

(11.44) <u>Corollary</u>: Every complete local CM-ring has a canonical module.

<u>Proof</u>: By (10.40) every complete local ring is factor ring of a RLR. Use (11.43).

(11.45) <u>Examples</u>: Let S be a local CM-ring.

(a) Assume $S \cong R/I$ with R a local Gorenstein ring and I an R-ideal of grade g. Let $\underline{x} = x_1, \dots, x_g \in I$ be an R-regular sequence. Then $\operatorname{Ext}^g_R(S,R) \cong \operatorname{Hom}_{R/(\underline{x})}(R/I, R/(\underline{x})) \cong \omega_S$.

(b) Assume that S is complete and contains a field. Let k be a coefficient field of S, $x_1, \dots, x_d$ a SoP of S, and set $R = k[\![x_1, \dots, x_d]\!] \subseteq S$. Then R is a power series ring and S is finite over R. By (11.43) $\omega_S \cong \operatorname{Hom}_R(S,R)$.

Let R be a ring and M an R-module. We construct a ring extension $R \subseteq R * M$ of R, called the <u>trivial extension</u> of R by M as follows: As an R-module $R * M = R \oplus M$ and multiplication is defined by $(a,x)(b,y) = (ab, ay + bx)$ for all $a,b \in R$ and $x,y \in M$. Note that $M \subseteq R * M$ is an ideal with $M^2 = 0$ and $R*M/M \cong R$.

(11.46) <u>Theorem</u>: Let R be a local CM-ring. Then R has a canonical module if and only if R is a factor ring of a local Gorenstein ring.

<u>Proof</u>: "$\Longleftarrow$": Use (11.43).

"$\Longrightarrow$": It is enough to show that $S = R * \omega_R$ is a local Gorenstein ring. Let $d = \dim R$.

Since $R \subseteq S$ is a finite ring extension, S is a Noetherian ring with $\dim S = d$. Since $\omega_R^2 = 0$ in S and $S/\omega_R \cong R$ local, the ring S is local. Let $\underline{x} = x_1, \ldots, x_d$ be an R-sequence. Then $\underline{x}$ is regular on ω_R and hence on $S = R * \omega_R$. Thus S is CM. It remains to show that $r(S) = 1$, or equivalently, $r(S/(\underline{x})S) = 1$. Since $S/(\underline{x})S \cong R/(\underline{x}) * \omega_{R/(\underline{x})}$ we may replace R by $R/(\underline{x})$ and assume that $\dim R = 0$. In this case $\omega_R = E_R(k)$. It remains to show that $r(S) = r(R * E_R(k)) = 1$.

Let $(a,x) \in \mathcal{S}(S)$. Then for all $b \in m$: $(b,0)(a,x) = (ab, bx) = (0,0)$ and $a \in \mathcal{S}(R)$ and $x \in \mathcal{S}(E_R(k))$. If $a \neq 0$ the exact sequence $R \xrightarrow{a} R \to R/(a) \to 0$ induces an exact sequence: $0 \to \mathrm{Hom}_{R/(a)}(R/(a), E_R(k)) \to \mathrm{Hom}_R(R, E_R(k)) \xrightarrow{a} \mathrm{Hom}_R(R, E_R(k))$. By a similar argument as in the proof of (11.43) (via the adjoint isomorphism) $\mathrm{Hom}_{R/(a)}(R/(a), E_R(k))$ is an injective $R/(a)$-module. Since $\mathcal{S}(\mathrm{Hom}_{R/(a)}(R/(a), E_R(k))) \cong k$, we have that $\mathrm{Hom}_{R/(a)}(R/(a), E_R(k)) \cong E_{R/(a)}(k)$ and the sequence $0 \to E_{R/(a)}(k) \to E_R(k) \xrightarrow{a} E_R(k)$ is exact. Moreover, $\ell(E_{R/(a)}(k)) = \ell(R/(a)) < \ell(R) = \ell(E_R(k))$ and multiplication by a on $E_R(k)$ cannot be the zero map. Thus there is a $y \in E_R(k)$ with $ay \neq 0$ and $(0,y)(a,x) = (0, ay) \neq (0,0)$. Thus $\mathcal{S}(R * E_R(k)) \cong \mathcal{S}(E_R(k))$ and $r(R * E_R(k)) = 1$.