

CHAPTER X: COMPLETION

§1: COMPLETION

Let R be a ring, M an R -module, I a directed index set, and $\mathcal{F} = \{M_i \mid i \in I\}$ a family of submodules of M with $M_i \supseteq M_j$ for $i \leq j$. For every $x \in M$ let $\{x + M_i \mid i \in I\}$ be a system of neighborhoods of x and define a topology on M by: $U \subseteq M$ open $\iff \forall x \in U \exists M_i \in \mathcal{F}$ with $x + M_i \subseteq U$. This topology is called a linear topology on M .

(10.1) Remark: (a) $M \times M \xrightarrow{+} M$ is continuous.

(b) For all $a \in R$ multiplication by a : $M \xrightarrow{a} M$ is continuous.

(c) For $M = R$: $M \times M \xrightarrow{\cdot} M$ is continuous.

(10.2) Remark: (a) Every M_i is open and closed.

(b) $\bar{0} = \bigcap_{i \in I} M_i$

(c) M is separated (Hausdorff) if and only if $\bigcap_{i \in I} M_i = 0$.

Proof: (a) For all $x \in M_i$, $x + M_i \subseteq M_i$ and M_i is open. Moreover, for all $y \in M$ the set $y + M_i$ is open. Then $M - M_i = \bigcup_{y \notin M_i} (y + M_i)$ is open as a union of open sets. Thus M_i is closed.

(b) If $x \in \bar{0}$ then $(x + M_i) \cap \{0\} \neq \emptyset$ for all $i \in I$. Thus $x \in M_i$ for all i and $x \in \bigcap_{i \in I} M_i$.

Conversely, if $x \in \bigcap_{i \in I} M_i$ then $(x + M_i) \cap \{0\} \neq \emptyset$ for all $i \in I$.

(c) " \implies ": If M is separated, $\{0\}$ is a closed subset. By (b): $\{0\} = \bigcap_{i \in I} M_i$.

" \impliedby ": Let $x, y \in M$ with $x \neq y$. Since $x - y \neq 0$ there is an $i \in I$ with $x - y \notin M_i$. Since M_i is a submodule of M , $(x + M_i) \cap (y + M_i) = \emptyset$.

(10.3) Remark: Let N be a submodule of M .

(a) The linear topology on M/N defined by $\{M_i + N/N \mid i \in I\}$ is the quotient topology.

of the topology on M defined by $\{M_i \mid i \in I\}$.

$$(b) \overline{N} = \bigcap_{i \in I} (N + M_i).$$

Proof: (a) Let $\pi: M \rightarrow M/N$ be the natural map and let $U \subseteq M/N$ be a subset. Then U is open in the linear topology $\iff \forall x \in \pi^{-1}(U) \exists i \in I$ with $\pi(x) + \pi(M_i + N) \subseteq U \iff \forall x \in \pi^{-1}(U) \exists i \in I$ with $x + M_i \subseteq \pi^{-1}(U) \iff U$ is open in the quotient topology.

(b) By (a): $\overline{N}/N = \overline{N/N} = \bigcap_{i \in I} (M_i + N/N)$ where the last equation follows from (10.2)(b).

By (10.2)(b) the module $M/\bigcap_{i \in I} M_i$ is separated. It is called the separated module associated to M .

For all $i, j \in I$ with $i \leq j$ (and $M_j \subseteq M_i$) let $\pi_{ij}: M/M_j \rightarrow M/M_i$ denote the natural map. $\{M/M_i, \pi_{ij}\}$ is an inverse system and $\widehat{M} = \varprojlim M/M_i$ is called the completion of M w.r.t. \mathcal{F} .

By the universal property of inverse limits there is a unique R -linear map $\varphi: M \rightarrow \widehat{M}$

so that the diagram

$$M \xrightarrow{\varphi} \widehat{M}$$

$$\begin{array}{ccc} & & \\ \pi_i \searrow & & \swarrow \beta_i \\ & M/M_i & \end{array}$$

commutes. (π_i and β_i are the natural maps.)

Obviously, $\varphi(x) = (x + M_i \mid i \in I)$ and $\ker \varphi = \bigcap_{i \in I} M_i$. In particular, $\varphi: M \rightarrow \widehat{M}$ is injective if and only if M is separated. If $M = R$, then \widehat{R} is a ring and φ is a morphism of rings.

For all $i \in I$ the maps $\beta_i: \widehat{M} \rightarrow M/M_i$ are the restrictions of the projections:

$\rho_i: \prod_{j \in I} M/M_j \rightarrow M/M_i$. Let $M_i^* = \ker \beta_i$. Then $\mathcal{F}^* = \{M_i^* \mid i \in I\}$ is a family of submodules of \widehat{M} with $M_i^* \supseteq M_j^*$ for $i \leq j$. Thus \mathcal{F}^* defines a linear topology on \widehat{M} .

On the other hand, \mathcal{F} induces the discrete topology on M/M_i . Consider the product topology on $\prod_{i \in I} M/M_i$ and the topology induced by it on the submodule $\widehat{M} = \varprojlim M/M_i$.

A basic system of neighborhoods of 0 is given by $\{\bigcap_{\text{finite}} \beta_i^{-1}(0)\} = \{\bigcap_{\text{finite}} M_i^*\}$.

Since I is directed, this topology coincides with the topology defined by \mathcal{F}^* .

(10.4) Definition: An R -module M is called complete w.r.t. \mathcal{F} if $\varphi: M \xrightarrow{\cong} \widehat{M}$.

(10.5) Proposition: (a) $\varphi: M \rightarrow \widehat{M}$ is continuous.

(b) $\ker \varphi = \bigcap_{i \in I} M_i$

(c) $\text{im } \varphi$ is dense in \widehat{M}

(d) \widehat{M} is complete.

Proof: (a) Let $x \in M$. We have to show that for every $M_i^* \in \mathcal{F}^*$ there is an $M_j \in \mathcal{F}$ so that $\varphi(x + M_j) \subseteq \varphi(x) + M_i^*$. But $\varphi(M_i) \subseteq \ker \beta_i = M_i^*$.

(c) Let $y \in \widehat{M}$. We have to show that for every $M_i^* \in \mathcal{F}^*$ there is an $x \in M$ with $\varphi(x) \in y + M_i^*$. Since π_i is surjective there is an $x \in M$ with $\pi_i(x) = \beta_i(y)$, hence $\beta_i(\varphi(x) - y) = 0$ and $\varphi(x) - y \in M_i^*$.

(d) Since π_i are surjective, it follows that β_i are surjective and induce isomorphisms $\widehat{M}/M_i^* \cong M/M_i$ that are compatible with the projection maps. Thus there is a natural isomorphism $\widehat{\widehat{M}} \cong \widehat{M}$.

Let M and M' be R -modules with linear topologies given by $\mathcal{F} = \{M_i \mid i \in I\}$; $\mathcal{F}' = \{M'_j \mid j \in I'\}$ and let $f: M \rightarrow M'$ be an R -linear map which is continuous, that is, for all $j \in I'$ there is an $i(j) \in I$ with $f(M_{i(j)}) \subseteq M'_j$.

(10.6) Proposition: There is a unique continuous R -linear map $\widehat{f}: \widehat{M} \rightarrow \widehat{M}'$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \varphi \downarrow & & \downarrow \varphi' \\ \widehat{M} & \xrightarrow{\widehat{f}} & \widehat{M}' \end{array} \quad \text{commutes.}$$

Proof: Uniqueness follows since M is dense in \widehat{M} and \widehat{M}' is separated. For existence define $\widehat{f}: \varinjlim M/M_i \rightarrow \varinjlim M'/M'_j$ by sending $(x_i + M_i \mid i \in I)$ to $(f(x_{i(j)}) + M'_j \mid j \in I')$. This is a well defined R -linear map because $f(M_{i(j)}) \subseteq M'_j$ and because for $k \geq j$ in I' there is an $l \in I$ with $l \geq i(k)$ and $l \geq i(j)$. Hence $x_{i(k)} + M_{i(k)} = x_l + M_{i(k)}$ and $x_{i(j)} + M_{i(j)} = x_l + M_{i(j)}$. Thus $f(x_{i(k)}) + M'_k = f(x_{i(j)}) + M'_k$. Since $\widehat{f}(M_{i(j)}^*) \subseteq M'_j$, the map \widehat{f} is continuous.

(10.7) Remark: (a) The uniqueness of (10.6) implies that $\widehat{fg} = \widehat{f}\widehat{g}$ for \mathbb{R} -linear maps $f: M \rightarrow M'$ and $g: M'' \rightarrow M$ and that $\widehat{id_M} = id_{\widehat{M}}$.

(b) Let $M = M'$ and assume that \mathcal{F} and \mathcal{F}' define the same topology on M (i.e. for all $j \in I'$ there is an $i(j) \in I$ with $M_{i(j)} \subseteq M'_j$ and for all $i \in I$ there is an $j(i) \in I'$ with $M'_{j(i)} \subseteq M_i$). Then $f = id: M \xrightarrow{\cong} M'$ is a homeomorphism. Thus by (a), $\widehat{f}: \widehat{M} \xrightarrow{\cong} \widehat{M}'$ is an \mathbb{R} -linear homeomorphism. In particular, the topology on \widehat{M} only depends on the topology of M .

(10.8) Theorem: Let $I = \mathbb{N}$ and $N \subseteq M$ a submodule. Then the subspace topology on N coincides with the linear topology given by $\{M_i \cap N \mid i \in \mathbb{N}\}$. Moreover,

$$(*) \quad 0 \rightarrow \widehat{N} \xrightarrow{\widehat{i}} \widehat{M} \xrightarrow{\widehat{\pi}} \widehat{M/N} \rightarrow 0$$

is exact and $\widehat{M/N} \cong \widehat{M}/\widehat{N}$. Furthermore, $\widehat{i}(\widehat{N})$ is the closure of $\varphi(N)$ in \widehat{M} .

Proof: There are natural exact sequences $0 \rightarrow N/M_i \cap N \rightarrow N/M_i \rightarrow M/N + M_i \rightarrow 0$. Apply \varprojlim and use the fact that \varprojlim is left exact. Hence $(*)$ is left exact. The inverse systems $\{N/M_i \cap N \mid i \in \mathbb{N}\}$, $\{N/M_i \mid i \in \mathbb{N}\}$, and $\{M/N + M_i \mid i \in \mathbb{N}\}$ satisfy the assumptions of (6.104). In particular, for $j \geq i$: $N/M_j \cap N \rightarrow N/M_i \cap N$ is surjective. Thus $(*)$ is exact. To show that $\widehat{i}(\widehat{N}) = \overline{\varphi(N)}$ notice that $\varphi(N)$ is dense in \widehat{N} , hence $\varphi(N)$ is dense in $\widehat{i}(\widehat{N})$. On the other hand 0 is closed in $\widehat{M/N}$, hence $\widehat{i}(\widehat{N}) = \widehat{\pi}^{-1}(0)$ is closed in \widehat{M} .

Cauchy sequences: In the following let M be an \mathbb{R} -module and $\mathcal{F} = \{M_i \mid i \in \mathbb{N}\}$ a set of submodules of M with $M_j \subseteq M_i$ if $j \geq i$. For $x, y \in M$ define the distance of x and y by:

$$d(x, y) = \inf \{ 2^{-n} \mid x - y \in M_n \}.$$

Then for all $x, y, z \in M$ the following conditions are satisfied:

(a) $d(x, y) \in \mathbb{R}$, $d(x, y) \geq 0$, $d(x, x) = 0$

(b) $d(x, y) = d(y, x)$

(c) $d(x, y) \leq d(x, z) + d(z, y)$.

d defines a premetric on M . If additionally, $\bigcap_{i \in \mathbb{N}} M_i = 0$, then

(d) $d(x, y) = 0 \iff x = y$

and d defines a metric on M . The topology defined by d is the linear topology of M defined by \mathcal{F} .

An element $(x_n) \in \prod_{n \in \mathbb{N}} M$ is a Cauchy sequence of M (w.r.t. \mathcal{F}) if for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ with $x_i - x_j \in M_N$ for all $i, j \geq N$. Define

$$\mathcal{C}_{\mathcal{F}}(M) = \{(x_n) \in \prod_{n \in \mathbb{N}} M \mid (x_n) \text{ is a Cauchy sequence w.r.t. } \mathcal{F}\}.$$

$\mathcal{C}_{\mathcal{F}}(M)$ is an R -submodule of $\prod_{n \in \mathbb{N}} M$. (If M is a ring $\mathcal{C}_{\mathcal{F}}(M)$ is a subring of $\prod M$).

An element $(y_n) \in \prod_{n \in \mathbb{N}} M$ is called a zero sequence of M if for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ so that $y_i \in M_N$ for all $i \geq N$. Set

$$\mathcal{Z}_{\mathcal{F}}(M) = \{(y_n) \in \prod_{n \in \mathbb{N}} M \mid (y_n) \text{ is a zero sequence w.r.t. } \mathcal{F}\}.$$

$\mathcal{Z}_{\mathcal{F}}(M)$ is a submodule of $\mathcal{C}_{\mathcal{F}}(M)$. Define the *-completion of M w.r.t. \mathcal{F} by:

$$M^* = (M, \mathcal{F})^* = \mathcal{C}_{\mathcal{F}}(M) / \mathcal{Z}_{\mathcal{F}}(M).$$

(10.9) Properties of the *-completion

(a) Let $\varphi: M \rightarrow M^*$ be defined by $\varphi(x) = (x)_{n \in \mathbb{N}} + \mathcal{Z}_{\mathcal{F}}(M)$. Then φ is a R -linear map with $\ker(\varphi) = \bigcap_{n \in \mathbb{N}} M_n$.

(b) Let M and M' be R -modules with linear topologies $\mathcal{F} = \{M_i \mid i \in \mathbb{N}\}$ and $\mathcal{F}' = \{M'_i \mid i \in \mathbb{N}\}$. If $f: M \rightarrow M'$ is a continuous R -linear map (i.e. for all $j \in \mathbb{N}$ there is an $i(j) \in \mathbb{N}$ with $f(M_{i(j)}) \subseteq M'_j$) then f induces an R -linear map $f^*: M^* \rightarrow M'^*$.

Proof: Homework

(10.10) Theorem: There is an isomorphism $f: M^* = (M, \mathcal{F})^* \rightarrow (M, \mathcal{F})^{\wedge} = \widehat{M}$ so that the diagram

$$\begin{array}{ccc} & \varphi & \rightarrow M^* \\ M & & \downarrow f \\ & \psi & \rightarrow \widehat{M} \end{array}$$

commutes, where φ is defined as above and

$\psi: M \rightarrow \widehat{M} = \varinjlim M/M_i$ is the natural map defined by $\psi(x) = (x + M_i)$.

Proof: Define a map $\tilde{f}: \mathcal{C}_{\mathbb{Z}}(M) \rightarrow \widehat{M}$ as follows: If $x = (x_n) \in \mathcal{C}_{\mathbb{Z}}(M)$ choose for all $n \in \mathbb{N}$ an integer $N(n) \in \mathbb{N}$ so that $x_i - x_j \in M_n$ for all $i, j \geq N(n)$. Define $\tilde{f}(x) = (x_{N(n)} + M_n)$ and notice that $\tilde{f}(x)$ is independent of the choice of $N(n)$. In particular, every zero sequence is mapped into $0_{\widehat{M}} = (M_n)$. \tilde{f} is \mathbb{R} -linear and factors through an \mathbb{R} -linear map $f: M^* \rightarrow \widehat{M}$.

In order to show that f is an isomorphism we define an \mathbb{R} -linear map $g: \widehat{M} \rightarrow M^*$. For $y = (y_n + M_n) \in \widehat{M}$ let $y_n \in M$ denote a representative of $y_n + M_n$. Then $(y_n) \in \prod_{n \in \mathbb{N}} M$ is a Cauchy sequence. If y'_n denotes another representative of $y_n + M_n$ in M , then $(y_n) - (y'_n)$ is a zero sequence. Thus by defining $g(y) = (y_n) + \mathcal{C}_{\mathbb{Z}}(M)$ the \mathbb{R} -linear map g is well defined. Moreover, g is inverse to f .

§2: I-ADIC COMPLETIONS

Let R be a ring, $I \subseteq R$ an ideal, and M an R -module. The linear topology defined by $\{I^i M \mid i \in \mathbb{N}\}$ is called the I-adic topology on M . The I-adic completion $\widehat{M} = (M, I)^\wedge$ of M is the completion of M with respect to the I-adic topology: $\widehat{M} = \varprojlim M/I^i M$. The module M is called I-adically complete if $\widehat{M} \cong M$. The I-adic completion \widehat{R} of R is a ring and \widehat{M} is an \widehat{R} -module via $(a_i + I^i)(x_i + I^i M) = (a_i x_i + I^i M)$. If $f: M \rightarrow M'$ is an R -linear map, then f is continuous in the I-adic topology since $f(I^i M) \subseteq I^i M'$. Thus \widehat{f} exists and is defined by $\widehat{f}(x_i + I^i M) = (f(x_i) + I^i M')$. In particular, \widehat{f} is \widehat{R} -linear.

(10.11) Remark: Let R be a Noetherian ring, $I \subseteq R$ an ideal, M a finite R -module, and $N \subseteq M$ a submodule. By the Artin-Rees theorem there is an $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$: $I^n M \cap N = I^{n-n_0}(I^{n_0} M \cap N)$. The I-adic topology on N coincides with the induced topology on N given by $\{I^i M \cap N \mid i \in \mathbb{N}\}$. Thus (10.8) yields the following result:

(10.12) Proposition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ an exact sequence of finite R -modules. The induced sequence of the I-adic completions $0 \rightarrow \widehat{M}' \rightarrow \widehat{M} \rightarrow \widehat{M}'' \rightarrow 0$ is exact.

Let R be a ring, $I \subseteq R$ an ideal, and M an R -module. The map $\lambda: \widehat{R} \times M \rightarrow \widehat{M}$ defined by $\lambda((a_i + I^i), m) = (a_i m + I^i M)$ is R -bilinear. Thus λ induces an R -linear map $\tau: \widehat{R} \otimes_R M \rightarrow \widehat{M}$ defined by $\tau((a_i + I^i) \otimes m) = (a_i m + I^i M)$. τ is also \widehat{R} -linear.

(10.13) Theorem: Let R be a Noetherian ring and M a finite R -module. The \widehat{R} -linear map $\tau: \widehat{R} \otimes_R M \rightarrow \widehat{M}$ is an isomorphism.

Proof: A standard induction argument shows that for all $n \in \mathbb{N} - \{0\}$: $\widehat{R}^n \cong \widehat{R} \otimes_R R^n \cong \widehat{R}^n$. Let M be a finite R -module. Then there are $m, n \in \mathbb{N}$ with $R^m \rightarrow R^n \rightarrow M \rightarrow 0$ exact.

Since the tensor product is right exact, the sequence $\widehat{R} \otimes_R R^m \rightarrow \widehat{R} \otimes_R R^n \rightarrow \widehat{R} \otimes_R M \rightarrow 0$ is exact. By (10.12) the sequence $\widehat{R}^m \rightarrow \widehat{R}^n \rightarrow \widehat{M} \rightarrow 0$ is exact. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \widehat{R} \otimes_R R^m & \longrightarrow & \widehat{R} \otimes_R R^n & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \\ \downarrow \varphi_m & & \downarrow \varphi_n & & \downarrow \tau & & \\ \widehat{R}^m & \longrightarrow & \widehat{R}^n & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

φ_m and φ_n are isomorphisms. Thus τ is an isomorphism by the 5-lemma.

(10.14) Theorem: Let R be a Noetherian ring and $I \subseteq R$ an ideal. The I -adic completion $(R, I)^\wedge = \widehat{R}$ is flat over R .

Proof: By homework it suffices to show: If $\mathfrak{J} \subseteq R$ is an ideal, then the induced map $\mathfrak{J} \otimes_R \widehat{R} \rightarrow \widehat{\mathfrak{J}}$ is injective. This follows by (10.12) and (10.13) since $\mathfrak{J} \rightarrow \widehat{R}$ is injective.

(10.15) Proposition: Let R be a Noetherian ring, $I, \mathfrak{J}, K \subseteq R$ ideals, and let ${}^\wedge$ denote the I -adic completion. Then:

(a) $\widehat{\mathfrak{J}} \cong \mathfrak{J} \otimes_R \widehat{R} \cong \widehat{\mathfrak{J}}$

(b) $(\mathfrak{J}K)^\wedge = (\mathfrak{J}K, I)^\wedge \cong \widehat{\mathfrak{J}} \widehat{K}$.

Proof: (a) By (10.13) $\widehat{\mathfrak{J}} \cong \mathfrak{J} \otimes_R \widehat{R}$ and since \widehat{R} is flat over R , $\mathfrak{J} \otimes_R \widehat{R} \cong \widehat{\mathfrak{J}}$.

(b) By (a) $(\mathfrak{J}K)^\wedge \cong \mathfrak{J}K \widehat{R} = (\mathfrak{J} \widehat{R})(K \widehat{R}) \cong \widehat{\mathfrak{J}} \widehat{K}$.

In the following we identify $\widehat{\mathfrak{J}}$ and $\mathfrak{J} \widehat{R}$ and consider $\widehat{\mathfrak{J}}$ an ideal of \widehat{R} .

(10.16) Corollary: Under the assumptions of (10.15): $\widehat{R/\mathfrak{J}} \cong \widehat{R}/\widehat{\mathfrak{J}} = \widehat{R}/\mathfrak{J} \widehat{R}$.

Proof: Completing the exact sequence $0 \rightarrow \mathfrak{J} \rightarrow R \rightarrow R/\mathfrak{J} \rightarrow 0$ yields the exact sequence $0 \rightarrow \widehat{\mathfrak{J}} \rightarrow \widehat{R} \rightarrow \widehat{R/\mathfrak{J}} \rightarrow 0$.

For all $n \in \mathbb{N}$ there is a natural (R/I) -linear map $I^n/I^{n+1} \rightarrow \widehat{I}^n/\widehat{I}^{n+1}$ which extends to a homomorphism of graded rings: $\mathcal{J}: \mathcal{G}_I(R) = \bigoplus_{n \in \mathbb{N}} I^n/I^{n+1} \rightarrow \mathcal{G}_I(\widehat{R}) = \bigoplus_{n \in \mathbb{N}} \widehat{I}^n/\widehat{I}^{n+1}$.

(10.17) Proposition: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and let $\widehat{}$ denote the I -adic completions. The graded homomorphism $\mathcal{J}: \mathcal{G}_I(R) \rightarrow \mathcal{G}_I(\widehat{R})$ is an isomorphism.

Proof: Let $m < n$ and consider the exact sequence $0 \rightarrow I^n \rightarrow I^m \rightarrow I^m/I^n \rightarrow 0$. Completing yields an exact sequence $0 \rightarrow \widehat{I}^n \rightarrow \widehat{I}^m \rightarrow \widehat{I^m/I^n} \rightarrow 0$ and $\widehat{I^m/I^n} \cong \widehat{I}^m/\widehat{I}^n$. Since $I^k(I^m/I^n) = 0$ for all $k \geq m-n$, the R -module I^m/I^n is I -adically complete. Thus $\widehat{I^m/I^n} \cong I^m/I^n \cong \widehat{I}^m/\widehat{I}^n$.

(10.18) Corollary: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $\widehat{}$ the I -adic completion. Then for all $n \in \mathbb{N}$: $R/I^n \cong \widehat{R}/\widehat{I}^n \cong \widehat{R}/I^n \widehat{R}$.

Proof: By the proof of (10.17) with $m=0$.

(10.19) Corollary: Let R be a Noetherian ring, $I \subseteq R$ an ideal and \widehat{R} the I -adic completion of R . The natural morphism of rings: $\widehat{R} \rightarrow (\widehat{R}, I)^\wedge = (\widehat{R}, \widehat{I})^\wedge$ is an isomorphism.

Proof: By (10.18): $(\widehat{R}, \widehat{I})^\wedge = \varprojlim \widehat{R}/\widehat{I}^n = \varprojlim \widehat{R}/I^n \widehat{R} = (\widehat{R}, I)^\wedge = \varprojlim R/I^n = \widehat{R}$.

(10.20) Corollary: Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $\widehat{}$ the I -adic completion. Then $\widehat{I} \subseteq \text{rad}(\widehat{R})$.

Proof: We have to show that for all $\widehat{z} = (z_i + I^i) \in \widehat{I}$ the element $1 - \widehat{z} = (1 - z_i + I^i)$ is a unit in \widehat{R} . Note that $\widehat{z} \in \widehat{I}$ implies that $z_i \in I$ for all i . Consider $\widehat{y} = (1 + z_i + \dots + z_i^{l-1} + I^l)$. Then $(1 - \widehat{z})\widehat{y} = ((1 - z_i)(1 + z_i + \dots + z_i^{l-1}) + I^l) = (1 + I^l)$ and $1 - \widehat{z}$ is a unit.

(10.21) Corollary: Let R be a Noetherian ring, $m \subseteq R$ a maximal ideal of R , and $\widehat{R} = (R, m)^\wedge$ the m -adic completion of R . Then:

(a) \widehat{R} is a (quasi) local ring with maximal ideal $m\widehat{R} = \widehat{m}$.

(b) The mR_m -adic completion of R_m is isomorphic to \widehat{R} , i.e. $\widehat{R} \cong \widehat{R}_m = (R_m, mR_m)^\wedge$.

Proof: (a) By (10.18) $\widehat{R}/m\widehat{R} \cong R/m$, thus $m\widehat{R}$ is a maximal ideal of \widehat{R} . By (10.20) \widehat{R} is (quasi) local.

(10.22) Proposition: Let R be a Noetherian ring and $I \subseteq \text{rad}(R)$ an ideal. Then:

(a) The I -adic completion $\widehat{R} = (R, I)^\wedge$ is faithfully flat over R .

(b) The natural morphism $\varphi: R \rightarrow (R, I)^\wedge$ is injective.

Proof: (a) By (10.14) \widehat{R} is flat over R . Since $R/\mathfrak{m} \cong \widehat{R}/\mathfrak{m}\widehat{R}$ and $I \subseteq \text{rad}(R)$ for all maximal ideals $\mathfrak{m} \subseteq R$, $m\widehat{R} \neq \widehat{R}$. Thus \widehat{R} is faithfully flat over R .

(b) Since $I \subseteq \text{rad}(R)$: $\ker(\varphi) = \bigcap_{n \in \mathbb{N}} I^n = (0)$.

(10.23) Example: Let R be a ring, $S = R[x_1, \dots, x_n]$ the polynomial ring in n variables over R , and $I = (x_1, \dots, x_n) \subseteq S$. Then $\widehat{S} = (S, I)^\wedge = R[[x_1, \dots, x_n]]$.

Proof: Define $\varphi: R[x_1, \dots, x_n] \rightarrow \widehat{S} = \varprojlim S/I^i$ by $\varphi(\sum_{(i) \in \mathbb{N}^n} a_{(i)} x_1^{i_1} \dots x_n^{i_n}) = (\sum_{|(i)| < j} a_{(i)} x_1^{i_1} \dots x_n^{i_n} + I^j)$ where $|i| = i_1 + \dots + i_n$. φ is an isomorphism of rings.

(10.24) Remark: A similar argument as in the proof of Hilbert's basis theorem shows: If R is a Noetherian ring then the power series ring $R[[x_1, \dots, x_n]]$ is Noetherian.

(10.25) Theorem: Let R be a Noetherian ring, $I \subseteq R$ an ideal, \widehat{R} the I -adic completion. Then

(a) If $I = (a_1, \dots, a_n)$ then $\widehat{R} \cong R[[x_1, \dots, x_n]] / (x_1 - a_1, \dots, x_n - a_n)$.

(b) \widehat{R} is Noetherian.

Proof: Obviously, $R \cong R[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) = T$. Let $K = (x_1, \dots, x_n) \subseteq R[x_1, \dots, x_n]$, then $K + (x_1 - a_1, \dots, x_n - a_n) = I + (x_1 - a_1, \dots, x_n - a_n)$ and the K -adic topology of T equals the I -adic topology of T . Thus by (10.16) and (10.23): $(R, I)^\wedge \cong (T, K)^\wedge \cong (R[x_1, \dots, x_n], K)^\wedge / (x_1 - a_1, \dots, x_n - a_n) \cong R[x_1, \dots, x_n] / (x_1 - a_1, \dots, x_n - a_n)$.

(10.26) Corollary: Let (R, \mathfrak{m}, k) be a local Noetherian ring and $I \subseteq \mathfrak{m}$ an ideal. Then $\dim R = \dim (R, I)^\wedge$.

Proof: Let $\hat{R} = (R, I)^\wedge$. For all $n \in \mathbb{N}$: $\hat{R}/I^n \hat{R} \cong R/I^n$ and \hat{R} is a local Noetherian ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m} \hat{R}$. Thus for all $n \in \mathbb{N}$: $\hat{R}/\hat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$. R and \hat{R} have the same Hilbert-Samuel polynomial, thus $\dim R = \dim \hat{R}$.

(10.27) Corollary: Let (R, \mathfrak{m}, k) be a local Noetherian ring, $I \subseteq \mathfrak{m}$ an ideal, and $\hat{R} = (R, I)^\wedge$. R is regular if and only if \hat{R} is regular.

Proof: Since $R \rightarrow \hat{R}$ is faithfully flat, by (8.61)(a) if \hat{R} is regular so is R . The converse follows with (8.61)(b) since $\hat{R}/\mathfrak{m} \hat{R} = \hat{R}/\hat{\mathfrak{m}} \cong R/\mathfrak{m}$.

(10.28) Corollary: Let (R, \mathfrak{m}, k) be a local Noetherian ring, $I \subseteq \mathfrak{m}$ an ideal and $\hat{R} = (R, I)^\wedge$. Then $\text{depth } \hat{R} = \text{depth } R$. In particular, R is CM if and only if \hat{R} is CM.

Proof: Homework

(10.29) Theorem: Let R be a ring, $I \subseteq R$ an ideal, and M an R -module. Assume that R is I -adically complete and M is I -adically separated. Let $x_1, \dots, x_m \in M$ so that $x_1 + IM, \dots, x_m + IM$ generate the R/I -module M/IM . Then x_1, \dots, x_m generate M . In particular, M is finitely generated.

Proof: Let $y \in M$. For all $j \geq 0$ we construct $a_{ij} \in I^j$, $1 \leq i \leq m$, so that with $s_{ij} = \sum_{\nu=0}^j a_{i\nu}$ one has $y - \sum_{i=1}^m s_{ij} x_i \in I^{j+1} M$. Then $s_i = \lim_{j \rightarrow \infty} s_{ij} \in R$ and

$$y - \sum_{i=1}^m s_{ij} x_i \in \bigcap I^{j+1} M. \text{ Since } M \text{ is } I\text{-separated, } \bigcap I^j M = 0, \text{ and } y = \sum_{i=1}^m s_i x_i.$$

In order to construct a_{ij} , let $j=0$. Then there are $a_{i0} \in R$ so that

$$y - \sum_{i=1}^m a_{i0} x_i \in IM. \text{ Assume } a_{i\nu} \text{ for } 0 \leq \nu \leq j-1 \text{ have been constructed so that}$$

$$y - \sum_{i=1}^m s_{ij-1} x_i \in I^j M. \text{ Since } M = \sum_{i=1}^m R x_i + IM, \text{ there are } a_{ij} \in I^j \text{ so that}$$

$$y - \sum_{i=1}^m s_{ij-1} x_i - \sum_{i=1}^m a_{ij} x_i \in I^{j+1} M.$$

(10.30) Theorem: (Chevalley's Theorem) Let (R, \mathfrak{m}) be a complete Noetherian ring and let $\mathcal{F} = \{I_i \mid i \in \mathbb{N}\}$ be a filtration of R with $\bigcap_{i=0}^{\infty} I_i = 0$. Then for every integer j there is an $i(j) \in \mathbb{N}$ so that $I_{i(j)} \subseteq \mathfrak{m}^j$. (i.e. the \mathfrak{m} -adic topology is the unique weakest topology that is separated.)

Proof: Homework

§3: COHEN'S STRUCTURE THEOREMS

(10.31) Theorem: (Hensel's Lemma) Let (R, \mathfrak{m}, k) be a local ring that is \mathfrak{m} -adically complete and let $F \in R[x]$ be a monic polynomial. If $F + \mathfrak{m}R[x] = gh$ in $k[x]$ where g and h are monic polynomials in $k[x]$ with $\gcd(g, h) = 1$, then there are monic polynomials G and H in $R[x]$ with $F = GH$.

Proof: Let " $\bar{}$ " denote images in $k[x]$. We want to construct by induction on j monic polynomials G_j and H_j in $R[x]$ so that $F \equiv G_j H_j \pmod{\mathfrak{m}^{j+1}R[x]}$, $G_0 = g$, $H_0 = h$, and for $j > 0$ $G_j \equiv G_{j-1} \pmod{\mathfrak{m}^j R[x]}$ and $H_j \equiv H_{j-1} \pmod{\mathfrak{m}^j R[x]}$. Notice that all G_j have the same degree $\deg g$. Thus $G = \lim_{j \rightarrow \infty} G_j$ is a well defined polynomial in $R[x]$. Likewise for $H = \lim_{j \rightarrow \infty} H_j$. Then $F = GH$ and G, H are monic with $\bar{G} = g$ and $\bar{H} = h$.

In order to construct G_j and H_j for $j > 0$ let G_0 and H_0 be any monic polynomials with $\bar{G}_0 = g$ and $\bar{H}_0 = h$. For $j > 0$ assume that G_{j-1} and H_{j-1} have been constructed. Then $F - G_{j-1}H_{j-1} = \sum a_i U_i$ where $a_i \in \mathfrak{m}^j$ and $U_i \in R[x]$ with $\deg U_i = \deg F$. Since $\gcd(g, h) = 1$ there are $p_i, q_i \in k[x]$ with $\bar{U}_i = p_i g + q_i h$. Modifying p_i modulo h (division algorithm) we may assume that $\deg p_i < \deg h$. Then $\deg(q_i h) < \deg F = \deg g + \deg h$ and $\deg q_i < \deg g$. Let $P_i, Q_i \in R[x]$ with $\bar{P}_i = p_i$ and $\bar{Q}_i = q_i$ and with $\deg P_i = \deg p_i$ and $\deg Q_i = \deg q_i$. Then $U_i - P_i G_0 - Q_i H_0 \in \mathfrak{m}R[x]$. Setting $G_j = G_{j-1} + \sum a_i Q_i$ and $H_j = H_{j-1} + \sum a_i P_i$ we have $F - G_j H_j \equiv \sum a_i U_i - G_0(\sum a_i P_i) - (\sum a_i Q_i)H_0 \pmod{\mathfrak{m}^{j+1}R[x]}$ and hence $F - G_j H_j \in \mathfrak{m}^{j+1}R[x]$. Moreover, $G_j \equiv G_{j-1} \pmod{\mathfrak{m}^j R[x]}$ and G_j is monic since $\deg Q_i = \deg q_i < \deg g = \deg G_{j-1}$. Likewise for H_j .

(10.32) Proposition: Let $\varphi: R \rightarrow S$ be a homomorphism of rings and let $\mathfrak{J} \subseteq S$ be an ideal. Assume that S is \mathfrak{J} -adically complete and that a_1, \dots, a_n are elements of \mathfrak{J} .

(a) There is a unique homomorphism of rings $\psi: R[x_1, \dots, x_n] \rightarrow S$ with $\psi|_R = \varphi$ and $\psi(x_i) = a_i$ for $1 \leq i \leq n$.

(b) If the induced map $\pi\psi: R \rightarrow S/\mathfrak{J}$ is surjective and $\mathfrak{J} = (a_1, \dots, a_n)$, then ψ is surjective.

Proof: (a) Let $F = \sum b_{ij} x_1^i \dots x_n^j \in R[[x_1, \dots, x_n]]$. Define $\varphi(F) = \sum \varphi(b_{ij}) a_1^i \dots a_n^j$. Since S is \mathfrak{J} -adically complete, $\varphi(F)$ is a well defined element of S and φ is a homomorphism of rings with $\varphi|_R = \varphi$ and $\varphi(x_i) = a_i$ for all $1 \leq i \leq n$. In order to prove uniqueness notice $\varphi|_{R[[x_1, \dots, x_n]]}$ is uniquely determined and $R[[x_1, \dots, x_n]]$ is the $I = (x_1, \dots, x_n)$ -adic completion of $R[x_1, \dots, x_n]$. Since φ is continuous ($\varphi(I) \subseteq \mathfrak{J}$), φ extends uniquely to a homomorphism of the completions (10.6).
 (b) Let $T = R[[x_1, \dots, x_n]]$, $I = (x_1, \dots, x_n)T$, and consider S as a T -module via φ . T is I -adically complete, S is I -adically separated, and $1 + IS \in S/IS = S/\mathfrak{J}$ generates S/\mathfrak{J} over T/I . By (10.29) $1 \in S$ generates S as a T -module, that is, φ is surjective.

(10.33) Definition: Let R be a ring, $\varepsilon: \mathbb{Z} \rightarrow R$ the homomorphism of rings defined by $\varepsilon(1) = 1_R$. If $n \in \mathbb{N}$ with $\ker(\varepsilon) = (n)$, then R is called a ring of characteristic n . Notation: $\text{char}(R) = n$.

(10.34) Remark: (a) If R is a domain, $\text{char}(R) = 0$ or $\text{char}(R) = p > 0$, where p is prime.

(b) Let (R, \mathfrak{m}, k) be a local ring. If $\text{char}(k) = 0$, then $\text{char}(R) = 0$. In this case R contains a subfield which is isomorphic to \mathbb{Q} .

(c) Let (R, \mathfrak{m}, k) be a local ring. If $\text{char}(k) = p > 0$, then $\text{char}(R) = 0$ or $\text{char}(R) = p^n$ for some $n \geq 1$.

(d) For a local ring (R, \mathfrak{m}, k) the following four cases can occur:

(i) $\text{char}(R) = \text{char}(k) = 0$ ($\Leftrightarrow \mathbb{Q} \subseteq R$)

(ii) $\text{char}(R) = \text{char}(k) = p$, p prime ($\Leftrightarrow \mathbb{Z}/(p) \subseteq R$)

(iii) $\text{char}(R) = 0$ and $\text{char}(k) = p$, p prime.

(iv) $\text{char}(R) = p^n$, p prime and $K \cap \mathbb{Q} = 0$, $\text{char}(k) = p$. (cannot occur if R is reduced.)

Proof: (b) The composition of natural maps $\varphi = \pi \varepsilon: \mathbb{Z} \xrightarrow{\varepsilon} R \xrightarrow{\pi} k$ is injective. Thus $\varphi(\mathbb{Z} - (0)) \subseteq R - \mathfrak{m} = R^*$. φ extends to a map $\tilde{\varphi}: \mathbb{Q} \rightarrow R$.

(c) If $\text{char}(R) = r = p^n t$ with $n > 0$ and $p \nmid t$. Then $\varepsilon(t) \notin \mathfrak{m}$ and $\varepsilon(t)$ is a unit in R . Thus $t = 1$.

(10.35) Definition: A local ring (R, \mathfrak{m}, k) is called of equal characteristic if $\text{char}(R) = \text{char}(k)$ and of unequal characteristic if $\text{char}(R) \neq \text{char}(k)$.

(10.36) Remark: (a) A local ring (R, \mathfrak{m}, k) is of equal characteristic if and only if R contains a field.

(b) Let R be a ring which contains a field $k_0 \in R$. Then by Zorn's Lemma R contains a maximal subfield k_1 , with $k_0 \subseteq k_1$.

(10.37) Definition: Let (R, \mathfrak{m}, k) be a local ring with $p = \text{char}(k) \geq 0$. A coefficient ring of R is a subring $R_0 \subseteq R$ so that (R_0, \mathfrak{m}_0) is a complete Noetherian local ring with $\mathfrak{m}_0 = pR_0$ and $R_0/\mathfrak{m}_0 = k$ (i.e. $R = R_0 + \mathfrak{m}$).

Notice that a coefficient ring is a field if and only if $\text{char}(R) = \text{char}(k)$, in which case it is called a coefficient field.

(10.38) Theorem (Cohen): Let R be a complete local (Noetherian) ring. Then R has a coefficient ring. If $\text{char}(R) = p^n$, p prime and $1 \leq n < \infty$, then $R_0 \cong \mathbb{V}/(p^n)$ with $(\mathbb{V}, (p))$ a complete discrete valuation ring.

We only prove a special case:

(10.39) Theorem: Let (R, \mathfrak{m}, k) be a complete local (Noetherian) ring. If $\text{char}(k) = 0$, then R has a coefficient field.

Proof: Since R contains a field, by (10.36) R contains a maximal subfield. Let $\pi: R \rightarrow k$ be the natural map. We claim that $\pi(K) = k$. Suppose not and let $\alpha \in k - \pi(K)$.

Case 1: α is transcendental over $\pi(K)$. Let $a \in R$ with $\pi(a) = \alpha$. For every nonzero polynomial $F \in K[x]$, $\pi(F) \neq 0$, and $\pi(F)(\alpha) = \pi(F(a)) \neq 0$. Thus a is transcendental over K and for

all $F \in K[x] - (0)$, $F(a)$ is a unit in R . Therefore $K(a) \subseteq R$, contradicting that K is maximal.

Case 2: α is algebraic over $\pi(K)$. Since $\text{char}(k) = 0$, the element α is separable over $\pi(K)$. Thus there is a monic polynomial $F \in K[x]$ so that $\pi(F)$ is the minimal polynomial of α over $\pi(K)$. Note that F is irreducible in $K[x]$. Thus $\pi(F) = (x - \alpha)h$ in $k[x]$ with $\text{gcd}(x - \alpha, h) = 1$. By Hensel's Lemma (10.31) there are monic polynomials $G, H \in R[x]$ with $F = GH$ and $\pi(G) = x - \alpha$, $\pi(H) = h$. Then $G = x - a$ with $\pi(a) = \alpha$, in particular, $F(a) = 0$. The morphism $\partial: K[x] \rightarrow R$ with $\partial|_K = \text{id}_K$ and $\partial(x) = a$ factors through the field $K[x]/(F)$ which is isomorphic to a subfield of R that strictly contains K .

(10.40) Theorem: (Cohen's Structure Theorem) Let (R, \mathfrak{m}, k) be a complete local Noetherian ring.

(a) If $\text{char}(R) = \text{char}(k)$, then $R \cong k[[x_1, \dots, x_n]]/I$ for some n and some ideal I .

(b) If $\text{char}(R) \neq \text{char}(k)$, then $R \cong V[[x_1, \dots, x_n]]/I$ for some n and some ideal I , where $(V, (p))$ is a complete discrete valuation ring.

Proof: By (10.38) there is a homomorphism $k \rightarrow R$ or $V \rightarrow R$, respectively, so that the composition with $\pi: R \rightarrow R/\mathfrak{m}$ is surjective. If $\mathfrak{m} = (a_1, \dots, a_n)$ by (10.32) there is a surjective homomorphism $k[[x_1, \dots, x_n]] \rightarrow R$ or $V[[x_1, \dots, x_n]] \rightarrow R$, respectively.

(10.41) Proposition: Let (R, \mathfrak{m}, k) be a complete local Noetherian ring with $\text{char}(R) = \text{char}(k)$. If R is regular with $\dim R = d$, then $R \cong k[[x_1, \dots, x_d]]$.

Proof: Let a_1, \dots, a_d be a regular system of parameters of R . By the proof of (10.40) there is a surjective homomorphism $\varphi: k[[x_1, \dots, x_d]] \rightarrow R$ with $\varphi(x_i) = a_i$. Since $d = \dim(R) = \dim k[[x_1, \dots, x_d]]$ and since $k[[x_1, \dots, x_d]]$ is a domain, φ is injective.

(10.41) Remark: Assumptions as in (10.41) with $\text{char}(R) \neq \text{char}(k) = p$. If $p \cdot 1_R \notin \mathfrak{m}^2$ then $R \cong V[[x_1, \dots, x_n]]$ where $(V, (p))$ is a complete discrete valuation ring.

(10.42) Theorem: Let (R, \mathfrak{m}, k) be a complete local Noetherian ring of dimension d . If $\text{char}(R) = \text{char}(k)$, then there is a subring S of R such that R is a finite S -module and $S \cong k[[y_1, \dots, y_d]]$, the power series ring in d variables over k .

Proof: Let t_1, \dots, t_d be a SOP of R , that is, $\dim R = d$, $(t_1, \dots, t_d) \subseteq \mathfrak{m}$ and $\mathfrak{m}^r \subseteq (t_1, \dots, t_d)$ for some $r \in \mathbb{N}$. By (10.38) R contains a coefficient field K and there is an isomorphism $\varphi: k \xrightarrow{\cong} K$. Then by (10.32) there is a homomorphism of rings $\varphi: k[[y_1, \dots, y_d]] \rightarrow R$ with $\varphi(y_i) = t_i$ and $\varphi|_k = \varphi$. $R/(\varphi(t_1), \dots, \varphi(t_d)) = R/(t_1, \dots, t_d)$ is an Artinian ring and thus a finite k -vector space. By (10.29) R is a finite module over $k[[y_1, \dots, y_d]]$. By dimension reasons, φ is injective and R is a finite module over the subring $S = \text{im } \varphi \cong k[[y_1, \dots, y_d]]$.

(10.43) Remark: Let (R, \mathfrak{m}, k) be a complete local Noetherian ring of dimension d with $\text{char}(R) \neq \text{char}(k) = p > 0$. If $p \nmid R$ is a nonzero divisor of R , then there is a complete DVR V with maximal ideal $\mathfrak{m}_V = (p)$ so that (i) R contains a subring $S \cong V[[y_1, \dots, y_d]]$ and (ii) R is a finite S -module.

§4: ASCENT AND DESCENT

(10.44) Proposition: Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings, $Q \subseteq S$ a prime ideal and $P = \varphi^{-1}(Q) = Q \cap R \in \text{Spec}(R)$. Then

- (a) $\text{ht } Q \leq \text{ht } P + \dim S_Q/PS_Q$
 (b) If φ is flat, equality holds in (a).

Proof: We replace R by R_P , S by S_Q , and assume that $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a local homomorphism of local Noetherian rings.

(a) We have to show that $\dim S \leq \dim R + \dim S/\mathfrak{m}S$. Let x_1, \dots, x_r be an SOP of R and $y_1, \dots, y_s \in S$ so that $y_1 + \mathfrak{m}S, \dots, y_s + \mathfrak{m}S$ is an SOP of $S/\mathfrak{m}S$. Then there are $k, l \in \mathbb{N}$ so that $\mathfrak{m}^k \subseteq \mathfrak{m}S + \sum y_i S$ and $\mathfrak{m}^l \subseteq \sum x_i R$. Thus $\mathfrak{m}^{kl} \subseteq \sum x_i S + \sum y_i S$ and $\dim S \leq r + s$.

(b) Let $\dim S/\mathfrak{m}S = s$ and let $\mathfrak{n} = Q_0 \supseteq Q_1 \supseteq \dots \supseteq Q_s (\supseteq \mathfrak{m}S)$ be a strictly decreasing chain of prime ideals between \mathfrak{n} and $\mathfrak{m}S$. Then $Q_i \cap R = \varphi^{-1}(Q_i) = \mathfrak{m}$ for all $0 \leq i \leq s$. Let $\dim R = r$ and let $\mathfrak{m} = P_0 \supseteq P_1 \supseteq \dots \supseteq P_r$ be a strictly decreasing chain of primes of R . Since $\varphi: R \rightarrow S$ is flat, by (7.75) going down holds and there is a descending chain of prime ideals $Q_s \supseteq Q_{s+1} \supseteq \dots \supseteq Q_{s+r}$ in S with $Q_{s+i} \cap R = P_i$. Thus $\dim S \geq r + s$.

(10.45) Corollary: If $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a faithfully flat homomorphism of Noetherian rings then $\dim S = \dim R + \dim S/\mathfrak{m}S$.

For the remainder of this section let (R, \mathfrak{m}) be a local Noetherian ring and $(\hat{R}, \hat{\mathfrak{m}})$ its \mathfrak{m} -adic completion. The natural map $\varphi: R \rightarrow \hat{R}$ is faithfully flat with $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$.

This yields (see also (10.26)):

(10.46) Corollary: $\dim R = \dim \hat{R}$.

Corollary (10.27) shows that R is regular if and only if \hat{R} is regular. The goal is to

investigate if other properties of the local Noetherian ring R pass to completion \widehat{R} and vice versa.

(10.47) Theorem: Let $\varphi: R \rightarrow S$ be a homomorphism of Noetherian rings, M a finite R -module, and N an S -module which is flat over R .

(a) If $P \in \text{Spec}(R)$ with $N/PN \neq 0$, then $\varphi_*(\text{Ass}_S(N/PN)) = \text{Ass}_R(N/PN) = \{P\}$.

(b) If $Q \in \text{Ass}_S(M \otimes_R N)$, then $P = Q \cap R \in \text{Ass}_R(M)$. In particular, if $Q \in \text{Ass}_S(N)$ then $P = Q \cap R \in \text{Ass}_R(R)$.

(c) $\text{Ass}_S(M \otimes_R N) = \bigcup_{P \in \text{Ass}_R(M)} \text{Ass}_S(N/PN)$.

Proof: (a) Since R/P is an integral domain and $N/PN \cong N \otimes_R R/P$ is flat over R/P , every non zero element of R/P is regular on N/PN . Thus $\text{Ass}_R(N/PN) = \{P\}$. If $Q \in \text{Ass}_S(N/PN)$ then there is an element $y \in N/PN$ with $\text{ann}_S(y) = Q$. Then $\text{ann}_R(y) = Q \cap R \in \text{Ass}_R(N/PN) = \{P\}$.

(b) Let $Q \in \text{Ass}_S(M \otimes_R N)$. Since M is a finite R -module, $(0) \subseteq M$ has a shortest primary decomposition $(0) = L_1 \cap \dots \cap L_r$. The quotient modules $M_i = M/L_i$ are primary, that is, $\text{Ass}_R(M_i) = \{P_i\}$ and $\text{Ass}_R(M) = \{P_1, \dots, P_r\}$. Moreover, there is an injective R -linear map $M \rightarrow \bigoplus_{i=1}^r M_i$. Thus $M \otimes_R N \rightarrow \bigoplus_{i=1}^r M_i \otimes_R N$ is injective and $\text{Ass}_S(M \otimes_R N) \subseteq \bigcup_{i=1}^r \text{Ass}_S(M_i \otimes_R N)$. This implies that $Q \in \text{Ass}_S(M_i \otimes_R N)$ for some $1 \leq i \leq r$. Since $\text{Ass}_R(M_i) = \{P_i\}$, every element of $R - P_i$ is regular on M_i and thus, by flatness, regular on $M_i \otimes_R N$. This implies that $Q \cap R \subseteq P_i$. Conversely, since $\text{Ass}_R(M_i) = P_i$ and M_i finite, $P_i^t M_i = 0$ for some $t \in \mathbb{N}$. Hence $P_i^t (M_i \otimes_R N) = 0$ and $P_i \subseteq Q \cap R$.

(c) If $P \in \text{Ass}_R(M)$, then there is an exact sequence $0 \rightarrow R/P \rightarrow M$. Since N is flat over R , the sequence $0 \rightarrow N/PN \rightarrow M \otimes_R N$ is exact and $\text{Ass}_S(N/PN) \subseteq \text{Ass}_S(M \otimes_R N)$. Conversely, let $Q \in \text{Ass}_S(M \otimes_R N)$. As in the proof of (b), let $(0) = L_1 \cap \dots \cap L_r$ be a shortest primary decomposition and set $M_i = M/L_i$ for $1 \leq i \leq r$. As shown in (b), $Q \in \text{Ass}_S(M_i \otimes_R N)$ for some $1 \leq i \leq r$ and $Q \cap R = P_i$ where $\text{Ass}_R(M_i) = \{P_i\}$. Consider a chain of submodules $M_i = K_0 \supseteq K_1 \supseteq \dots \supseteq K_s = (0)$ with $K_j/K_{j+1} \cong R/W_j$ for some $W_j \in \text{Spec}(R)$. Then, by flatness of N , $M_i \otimes_R N \supseteq K_1 \otimes_R N \supseteq \dots \supseteq K_s \otimes_R N = (0)$ is a chain of submodules of $M_i \otimes_R N$ with

$K_j \otimes_R N / K_{j+1} \otimes_R N = (R/W_j) \otimes_R N = N/W_j N$. Moreover, $\text{Ass}_S(M_i \otimes_R N) \subseteq \bigcup_{i=1}^s \text{Ass}_S(N/W_j N)$ and $Q \in \text{Ass}_S(N/W_j N)$ for some $1 \leq j \leq s$. Thus, by (a), $Q \cap R = W_j = P_i$ and $Q \in \text{Ass}_S(N/P_i N)$.

(10.48) Proposition: Let (R, \mathfrak{m}, k) be a local Noetherian ring and M a finite R -module. M is flat over R if and only if $\text{Tor}_1^R(k, M) = 0$.

Proof: The forward direction is trivial. Conversely, if $\text{Tor}_1^R(k, M) = 0$, the first Betti number of M is 0 and M is free (see (7.53)).

(10.49) Remark: Proposition (10.48) is true for a larger class of R -modules M . By (7.86) if M is \mathfrak{m} -adically ideal separated then M is flat over R if and only if $\text{Tor}_1^R(k, M) = 0$. For example, if $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, k')$ is a local morphism of local Noetherian rings and M is a finite S -module, then M is \mathfrak{m} -adically ideal separated. Thus M is flat over R if and only if $\text{Tor}_1^R(k, M) = 0$. This result will be used in the following.

(10.50) Theorem: Let $\varphi: (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, k')$ be a local homomorphism of local Noetherian rings, and $u: M \rightarrow N$ a linear map of finite S -modules. If N is flat over R the following conditions are equivalent:

- (a) u is injective and $N/u(M)$ is flat over R .
- (b) $\bar{u}: M \otimes_R k \rightarrow N \otimes_R k$ is injective.

Proof: (a) \Rightarrow (b): Since $\text{Tor}_1^R(k, N/u(M)) = 0$, \bar{u} is injective.

(b) \Rightarrow (a): Let $x \in M$ with $u(x) = 0$. Then $\bar{u}(x) = 0$ and $\bar{x} = 0$, i.e. $x \in \mathfrak{m}M$. We will show by induction on n that $x \in \mathfrak{m}^n M$ for all $n \in \mathbb{N}$. Assume that $x \in \mathfrak{m}^n M$ and let a_1, \dots, a_r be a minimal system of generators of \mathfrak{m}^n . Let $y_i \in M$ with $x = \sum_{i=1}^r a_i y_i$. Then $0 = \sum_{i=1}^r a_i u(y_i)$. Since N is flat over R , by (7.79) there are $c_{ij} \in R$ and $z_j \in N$ so that $\sum_{i=1}^r a_i c_{ij} = 0$ for all j and $u(y_i) = \sum_j c_{ij} z_j$ for all i . Since a_1, \dots, a_r is a minimal system of generators of \mathfrak{m}^n , for all i, j : $c_{ij} \in \mathfrak{m}$ and $u(y_i) \in \mathfrak{m}N$. Thus $\bar{u}(y_i) = 0$ and

$y_i \in m M$ implying that $x \in m^{n+1} M$. This shows that $x \in \bigcap_{n \in \mathbb{N}} m^n M \subseteq \bigcap_{n \in \mathbb{N}} m^n M = 0$ and u is injective. From the exact sequence $0 \rightarrow M \xrightarrow{u} N \rightarrow N/u(M) \rightarrow 0$ we obtain that $\text{Tor}_i^R(k, N/u(M)) = 0$ (since N is R -flat). By (10.49) $N/u(M)$ is flat over R .

(10.51) Corollary: Let R and S be as in (10.50) and let M be a finite S -module. Set $\bar{S} = S \otimes_R k = S/m_S$ and for $x \in n$ set $\bar{x} = x + m_S$. Let $x_1, \dots, x_r \in n$. If M is a flat R -module and $\bar{x}_1, \dots, \bar{x}_r$ is an $M \otimes_R k$ -sequence, then x_1, \dots, x_r is an M -sequence and $M/(x_1, \dots, x_r)M$ is flat over R .

Proof: Set $M_0 = M$, $M_i = M/(x_1, \dots, x_i)M$ for $1 \leq i \leq r$ and $\bar{M}_i = M_i \otimes_R k$. Let $u_i: M_{i-1} \rightarrow M_i$ denote multiplication by x_i . By assumption \bar{u}_i is injective and $M_0 = M$ is R -flat. By (10.50) u_i is injective and M_i is R -flat. Continue.

(10.52) Remarks: The converse of (10.50) is also true, that is, if x_1, \dots, x_r is an M -sequence and $M_r = M/(x_1, \dots, x_r)M$ is R -flat, then $\bar{x}_1, \dots, \bar{x}_r$ is an \bar{M} -sequence and M is R -flat. (without proof).

(10.53) Theorem: Let $\varphi: (R, m, k) \rightarrow (S, n, k')$ be a local homomorphism of local Noetherian rings, M a finite R -module, and N a finite S -module. Assume that N is flat over R . Then:

$$\text{depth}_S(M \otimes_R N) = \text{depth}_R M + \text{depth}_S(N/m_S N).$$

Proof: Let $x_1, \dots, x_s \in m$ be a maximal M -sequence and $y_1, \dots, y_t \in n$ a maximal $N/m_S N$ -sequence. Setting $x'_i = \varphi(x_i)$ for $1 \leq i \leq s$, we want to show that $x'_1, \dots, x'_s, y_1, \dots, y_t$ is a maximal $M \otimes_R N$ -sequence. Since N is R -flat, x'_1, \dots, x'_s is an $M \otimes_R N$ -sequence. Set $M_s = M/(x_1, \dots, x_s)M$. Then $m \in \text{Ass}_R(M_s)$ and $M \otimes N/(x'_1, \dots, x'_s)(M \otimes N) \cong M_s \otimes_R N$. By (10.51) y_1 is N -regular and $N_1 = N/y_1 N$ is R -flat. Consider the exact sequence: $0 \rightarrow N \xrightarrow{y_1} N \rightarrow N_1 \rightarrow 0$. Since $\text{Tor}_i^R(M_s, N_1) = 0$, the sequence $0 \rightarrow M_s \otimes_R N \xrightarrow{y_1} M_s \otimes_R N \rightarrow M_s \otimes_R N_1 \rightarrow 0$ is exact. By (10.51) y_2 is N_1 -regular and $N_2 = N/(y_1, y_2)N$ is R -flat, etc. Continue. Thus y_1, \dots, y_t is an

$M_R \otimes_R N$ -sequence and $N_t = N/(y_1, \dots, y_t)N$ is R -flat. Moreover, $M_S \otimes_R N_t \cong M \otimes N/(x_1, \dots, x_s, y_1, \dots, y_t)M \otimes N$. It remains to show that $\text{depth}_S(M_S \otimes_R N_t) = 0$. By (10.47) $\text{Ass}_S(M_S \otimes_R N_t) = \bigcup_{P \in \text{Ass}(M_R)} \text{Ass}_S(N_t/PN_t) \supseteq \text{Ass}_S(N_t/mN_t)$. Since y_1, \dots, y_t is a maximal N/mN sequence, $\mathfrak{m} \in \text{Ass}_S(N_t/mN_t)$ and therefore $\mathfrak{m} \in \text{Ass}_S(M_S \otimes_R N_t)$. This implies that $\text{depth}_S(M_S \otimes_R N_t) = 0$.

(10.54) Corollary: Let $\varphi: (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, k')$ be a local homomorphism of local Noetherian rings and assume that S is flat over R . Then

- (a) $\text{depth } S = \text{depth } R + \text{depth } S/\mathfrak{m}S$.
 (b) S is CM if and only if R and $S/\mathfrak{m}S$ are CM.

Proof: (a) follows from (10.53).

(b) By (10.45) $\dim S = \dim R + \dim S/\mathfrak{m}S$. Thus if R and $S/\mathfrak{m}S$ are CM so is S . Since $\text{depth } R \leq \dim R$ and $\text{depth } S/\mathfrak{m}S \leq \dim S/\mathfrak{m}S$, the converse follows.

(10.55) Corollary: Let (R, \mathfrak{m}) be a local Noetherian ring and \widehat{R} its \mathfrak{m} -adic completion. Then

- (a) $\text{depth } R = \text{depth } \widehat{R}$
 (b) R is CM if and only if \widehat{R} is CM.

(10.56) Definition: Consider the following conditions (R_i) and (S_i) (for $i \in \mathbb{N}$) of a Noetherian ring R :

(R_i) R_P is regular for all $P \in \text{Spec}(R)$ with $\text{ht } P \leq i$.

(S_i) $\text{depth } R_P \geq \min(\text{ht } P, i)$ for all $P \in \text{Spec}(R)$.

Conditions (R_i) and (S_i) are called Serre conditions.

(10.57) Remark: (S_0) always holds. (S_1) says that all associated prime ideals of R are minimal, that is, R has no embedded prime ideals.

Recall that a Noetherian ring R is reduced if R has no nonzero nilpotent elements.

(10.58) Proposition: Let R be a Noetherian ring. R is reduced if and only if (R_0) and (S_1) hold.

Proof: " \Rightarrow ": Suppose that R is reduced. Thus for all $P \in \text{Spec}(R)$ with $\text{ht } P = 0$ the ring R_P is reduced and 0-dimensional. Hence R_P is a field and (R_0) holds. Let $P \in \text{Spec}(R)$ be any prime ideal. Since R is reduced, the zero ideal of R is the intersection of the minimal prime ideals of R and all prime ideals of $\text{Ass}(R)$ are minimal. Thus $\text{depth } R_P \geq \min(\text{ht } P, 1)$ and (S_1) holds.

" \Leftarrow ": Suppose that (R_0) and (S_1) hold. All associated prime ideals of R are minimal, that is, $\text{Ass}(R) = \{P_1, \dots, P_r\}$ is the set of minimal prime ideals of R . By (R_0) R_{P_i} is a field for all $1 \leq i \leq r$. Let $S = R - (P_1 \cup \dots \cup P_r)$. Then $S^{-1}R$ is a semilocal Noetherian ring of dimension 0. Thus $S^{-1}R$ is Artinian and by (1.104) $S^{-1}R \cong R_{P_1} \times \dots \times R_{P_r}$. By (R_0) $S^{-1}R$ is reduced. Let $a \in R$ with $a^N = 0$ for some N . Suppose that $a \neq 0$. Then $a^N = 0$ in $S^{-1}R$ and there is a $t \in S$ with $ta = 0$. This implies that $\text{ann}(a) \not\subseteq P_i$ for all $1 \leq i \leq r$. On the other hand, $\text{ann}(a) \subseteq \bigcup_{P \in \text{Ass } R} P = P_1 \cup \dots \cup P_r$, a contradiction.

(10.59) Definition: A Noetherian ring R is called normal if R_P is a normal domain for all $P \in \text{Spec}(R)$, that is, R_P is a domain and it is integrally closed in its field of quotients.

(10.60) Remark: Normal rings need not to be integral domains. For example, if K is a field, then $R = K \times K$ is a normal ring but not a domain.

(10.61) Theorem: A Noetherian ring R is normal if and only if R satisfies (R_1) and (S_2) .

Proof: " \Rightarrow ": Suppose that R is normal and let $P \in \text{Spec}(R)$ be a prime ideal of height one. Then R_P is a normal Noetherian local domain of dimension one. By (5.45) R_P is a DVR and (R_1) holds. Moreover, $\text{depth } R_P \geq \min(\text{ht } P, 2)$ for all prime ideals of height ≤ 1 . Let $P \in \text{Spec}(R)$ be a prime ideal of height ≥ 2 . Since R_P is normal, by (5.50) every prime ideal of $\text{Ass}_{R_P}(R_P/aR_P)$ is minimal for all $a \in P$ and $\text{depth } R_P \geq 2$. (S_2) holds.

" \Leftarrow ": We may assume that (R, \mathfrak{m}) is local. Since R satisfies (R_0) and (S_1) , R is reduced and $(0) = P_1 \cap \dots \cap P_r$ where P_i are the minimal primes of R . In particular, $\text{Ass}(R) = \{P_1, \dots, P_r\}$ and $S = R - (P_1 \cup \dots \cup P_r)$ is the set of NZD of R . Then $R \subseteq S^{-1}R = K_1 \times \dots \times K_r$ where $K_i = Q(R/P_i) = k(P_i)$ is the quotient field of R/P_i . We first show that R is integrally closed in $S^{-1}R$. Suppose that $a/b \in S^{-1}R$ with $a \in R, b \in S$, is integral over R . For all $P \in \text{Spec}(R)$ with $\text{ht } P = 1$, R_P is a RLR and $a/b \in b^{-1}R_P$. b is R -regular and by (S_2) all associated primes of $R/(b)$ have height one. Thus if $bR = Q_1 \cap \dots \cap Q_m$ is a shortest primary decomposition of bR with Q_i W_i -primary, then $\text{ht } W_i = 1$ for all $1 \leq i \leq m$. If $J = (b) :_R a$, then $J \subseteq W_i$ for some i or $J = R$. Since R_{W_i} is a DVR, $a/b \in b^{-1}R_{W_i}$ and $J \not\subseteq W_i$. Thus $J = R$ and $a \in bR$. R is integrally closed in $S^{-1}R$. Thus the idempotents $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in S^{-1}R$ are in R since $e_i^2 - e_i = 0$. Since $1 = \sum e_i$ and $e_i e_j = 0$ for $i \neq j$, $R = R e_1 \times \dots \times R e_r$. Since R is local, $r = 1$, and R is an integrally closed domain.

(10.62) Theorem: Let $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a local homomorphism of local Noetherian rings and suppose that S is flat over R . Then for all $i \in \mathbb{N}$:

- (a) If S satisfies (R_i) (or (S_i) , respectively) then R satisfies (R_i) (or (S_i) , respectively).
 (b) If R and all fiber rings $S \otimes_R k(P)$ for $P \in \text{Spec}(R)$ satisfy (R_i) (or (S_i) , respectively), then S satisfies (R_i) (or (S_i) , respectively).

Proof: (a) Let $P \in \text{Spec}(R)$. Since S is flat over R , there is a prime ideal $Q \subseteq S$ lying over P . Choose Q minimal, that is, $Q \in \text{Spec}(S)$ with $Q \cap R = P$ and $\text{ht}(Q/PS) = 0$. Then by (10.44) $\text{ht } P = \text{ht } Q$. Moreover, the induced map $\varphi_P: R_P \rightarrow S_Q$ is faithfully flat. By (8.61) if S_Q is regular so is R_P . This implies that if S satisfies (R_i) so does R . If S satisfies (S_i) then by (10.54) $\text{depth } S_Q = \text{depth } R_P$ and R satisfies (S_i) .

(b) Let $Q \in \text{Spec}(S)$ with $Q \cap R = P$. Suppose that R and the fiber rings $S \otimes_R k(P)$ satisfy (R_i) . If $\text{ht } Q \leq i$, then $\text{ht } P \leq i$ and R_P is regular. Moreover, $\text{ht}(Q/PS) \leq i$ and by assumption S_Q/PS_Q is regular. Apply (8.61) to the flat morphism $\varphi_P: R_P \rightarrow S_Q$ to obtain that S_Q is regular. Thus S satisfies (R_i) . Suppose now that R and the fiber

rings satisfy (S_i) . By (10.54) $\text{depth } S_Q = \text{depth } R_Q + \text{depth}(S_Q/PS_Q) \geq \min(\text{ht } P, i) + \min(\text{ht } Q/PS, i) \geq \min(\text{ht } P + \text{ht } Q/PS, i) = \min(\text{ht } Q, i)$ by (10.44). S satisfies (S_i) .

(10.63) Corollary: Let (R, \mathfrak{m}) be a local Noetherian ring and \widehat{R} its \mathfrak{m} -adic completion. Then:

- (a) If \widehat{R} is normal, R is normal.
 (b) If \widehat{R} is reduced, R is reduced.

(10.64) Remark: (a) In general, the converse of (10.63) is false. There are examples of local Noetherian normal rings R whose completion \widehat{R} is not reduced. On the other hand, the converse of (10.63) is true for a large class of local Noetherian rings. For example, if R is the localization of a finitely generated algebra over a field (or over \mathbb{Z}) the converse of (10.63) holds.

(b) So far we only discussed properties of rings which descent from the completion to the ring. There are, however, few properties which ascent, but do not necessarily descent. For example, if R is a local CM ring which has a canonical module, so does \widehat{R} . There are local CM-rings R which fail to have a canonical module but \widehat{R} admits a canonical module.